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Lower bounds for bootstrap percolation on Galton–Watson trees

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Abstract
Bootstrap percolation is a cellular automaton modelling the spread of an ‘infection’ on a graph. In this note, we prove a family of lower bounds on the critical probability for \( r \)-neighbour bootstrap percolation on Galton–Watson trees in terms of moments of the offspring distributions. With this result we confirm a conjecture of Bollobás, Gunderson, Holmgren, Janson and Przykucki. We also show that these bounds are best possible up to positive constants not depending on the offspring distribution.

Keywords: bootstrap percolation; Galton–Watson trees.
AMS MSC 2010: Primary 05C05; 60K35; 60C05; 60J80, Secondary 05C80.

1 Introduction
Bootstrap percolation, a type of cellular automaton, was introduced by Chalupa, Leath and Reich [1] and has been used to model a number of physical processes. Given a graph \( G \) and threshold \( r \geq 2 \), the \( r \)-neighbour bootstrap process on \( G \) is defined as follows: Given \( A \subseteq V(G) \), set \( A_0 = A \) and for each \( t \geq 1 \), define

\[
A_t = A_{t-1} \cup \{ v \in V(G) : |N(v) \cap A_{t-1}| \geq r \},
\]

where \( N(v) \) is the neighbourhood of \( v \) in \( G \). The closure of a set \( A \) is \( \langle A \rangle = \bigcup_{t \geq 0} A_t \).

Often the bootstrap process is thought of as the spread, in discrete time steps, of an ‘infection’ on a graph. Vertices are in one of two states: ‘infected’ or ‘healthy’ and a vertex with at least \( r \) infected neighbours becomes itself infected, if it was not already, at the next time step. For each \( t \), the set \( A_t \) is the set of infected vertices at time \( t \). A set \( A \subseteq V(G) \) of initially infected vertices is said to percolate if \( \langle A \rangle = V(G) \).

Usually, the behaviour of bootstrap processes is studied in the case where the initially infected vertices, i.e., the set \( A \), are chosen independently at random with a fixed probability \( p \). For an infinite graph \( G \) the critical probability is defined by

\[
p_c(G,r) = \inf\{ p : \mathbb{P}_p(\langle A \rangle = V(G)) > 0 \}.
\]

This is different from the usual definition of critical probability for finite graphs, which is generally defined as the infimum of the values of \( p \) for which percolation is more likely to occur than not.

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In this paper, we consider bootstrap percolation on Galton–Watson trees and answer a conjecture in [3] on lower bounds for their critical probabilities. For any offspring distribution $\xi$ on $\mathbb{N} \cup \{0\}$, let $T_\xi$ denote a random Galton–Watson tree (the family tree of a Galton–Watson branching process) with offspring distribution $\xi$ which we define as follows. Starting with a single root vertex in level 0, at each generation $n = 1, 2, 3, \ldots$ every vertex in level $n - 1$ gives birth to a random number of children in level $n$, where for every vertex the number of offspring is distributed according to the distribution $\xi$ and is independent of the number of children of any other vertex. For any fixed offspring distribution $\xi$, the critical probability $p_c(T_\xi, r)$ is almost surely a constant (see Lemma 3.2 in [3]) and we shall give lower bounds on the critical probability in terms of various moments of $\xi$.

Bootstrap processes on infinite regular trees were first considered by Chalupa, Leath and Reich [1]. Later, Balogh, Peres and Pete [2] studied bootstrap percolation on arbitrary infinite trees and one particular example of a random tree given by a Galton–Watson branching process. In [3], Galton–Watson branching processes were further considered, and it was shown that for every $r \geq 2$, there is a constant $c_r > 0$ so that

$$p_c(T_\xi, r) \geq \frac{c_r}{\mathbb{E}[\xi]} \exp \left(\frac{-\mathbb{E}[\xi]}{r - 1}\right)$$

and in addition, for every $\alpha \in (0, 1]$, there is a positive constant $c_{r, \alpha}$ so that,

$$p_c(T_\xi, r) \geq c_{r, \alpha} \left(\mathbb{E}[\xi^{1+\alpha}]\right)^{-1/\alpha}. \quad (1.1)$$

Additionally, in [3] it was conjectured that for any $r \geq 2$, inequality (1.1) holds for any $\alpha \in (0, r - 1]$. As our main result, we show that this conjecture is true. For the proofs to come, some notation from [3] is used. If an offspring distribution $\xi$ is such that $\mathbb{P}(\xi < r) > 0$, then one can easily show that $p_c(T_\xi, r) = 1$. With this in mind, for $r$-neighbour bootstrap percolation, we only consider offspring distributions with $\xi \geq r$ almost surely.

**Definition 1.1.** For every $r \geq 2$ and $k \geq r$, define

$$g_r^k(x) = \frac{\mathbb{P}(\text{Bin}(k, 1 - x) \leq r - 1)}{x} = \sum_{i=0}^{r-1} \binom{k}{i} x^{k-i-1}(1 - x)^i$$

and for any offspring distribution $\xi$ with $\xi \geq r$ almost surely, define

$$G_r^\xi(x) = \sum_{k \geq r} \mathbb{P}(\xi = k) g_r^k(x).$$

Some facts, which can be proved by induction, about these functions are used in the proofs to come. For any $r \geq 2$, we have $g_r^r(x) = \sum_{i=0}^{r-1} (1 - x)^i$ and for any $k > r$,

$$g_r^r(x) - g_r^k(x) = \sum_{i=r}^{k-1} \binom{i}{r-1} x^{i-r}(1 - x)^r. \quad (1.2)$$

Hence, for all distributions $\xi$ we have $G_r^\xi(x) \leq g_r^r(x)$ for $x \in [0, 1]$.

Developing a formulation given by Balogh, Peres and Pete [2], it was shown in [3] (see Theorem 3.6 in [3]) that if $\xi \geq r$, then

$$p_c(T_\xi, r) = 1 - \frac{1}{\max_{x \in [0, 1]} G_r^\xi(x)}. \quad (1.3)$$
2 Results

In this section, we shall prove a family of lower bounds on the critical probability \( p_c(T_\xi, r) \) based on the \((1+\alpha)\)-moments of the offspring distributions \( \xi \) for all \( \alpha \in (0, r-1) \), using a modification of the proofs of Lemmas 3.7 and 3.8 in [3] together with some properties of the gamma function and the beta function.

The gamma function is given, for \( z \) with \( \Re(z) > 0 \), by \( \Gamma(z) = \int_0^\infty t^{z-1} \exp(-t) \, dt \) and for all \( n \in \mathbb{N} \), satisfies \( \Gamma(n) = (n-1)! \). The beta function is given, for \( \Re(x), \Re(y) > 0 \), by \( B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, dt \) and satisfies \( B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \). We shall use the following bounds on the ratio of two values of the gamma function obtained by Gautschi [4]. For \( n \in \mathbb{N} \) and \( 0 \leq s \leq 1 \) we have

\[
\left( \frac{1}{n+1} \right)^{1-s} \leq \frac{\Gamma(n+s)}{\Gamma(n+1)} \leq \left( \frac{1}{n} \right)^{1-s}.
\]  

(2.1)

Let us now state our main result.

**Theorem 2.1.** For each \( r \geq 2 \) and \( \alpha \in (0, r-1) \), there exists a constant \( c_{r, \alpha} > 0 \) such that for any offspring distribution \( \xi \) with \( \mathbb{E}[\xi^{1+\alpha}] < \infty \), we have

\[
p_c(T_\xi, r) \geq c_{r, \alpha} \left( \mathbb{E}[\xi^{1+\alpha}] \right)^{-1/\alpha}.
\]

We prove Theorem 2.1 in two steps. First, in Lemma 2.2, we show that it holds for \( \alpha \in (0, r-1) \). Then, in Lemma 2.3, we consider the case \( \alpha = r-1 \).

**Lemma 2.2.** For all \( r \geq 2 \) and \( \alpha \in (0, r-1) \), there exists a positive constant \( c_{r, \alpha} \) such that for any distribution \( \xi \) with \( \mathbb{E}[\xi^{1+\alpha}] < \infty \), we have

\[
p_c(T_\xi, r) \geq c_{r, \alpha} \left( \mathbb{E}[\xi^{1+\alpha}] \right)^{-1/\alpha}.
\]

**Proof.** Fix \( r \geq 2 \), \( \alpha \in (0, r-1) \) with \( \alpha \notin \mathbb{Z} \) and an offspring distribution \( \xi \). Set \( t = [\alpha] \) and \( \varepsilon = \alpha - t \) so that \( \varepsilon \in (0, 1) \) and \( t \) is an integer with \( t \in [0, r-2] \). Set \( M = \max_{x \in [0,1]} G_\xi^r(x) \) and fix \( y \in [0,1] \) with the property that \( G_\xi^r(1-y) = M \). Such a \( y \) can always be found since \( G_\xi^r(x) \leq g_\xi^r(x) \) in \([0,1] \), \( G_\xi^r(1) = g_\xi^r(1) = 1 \) and \( g_\xi^r(x) \) is continuous. Thus, \( M = 1 + y + \ldots + y^{r-1} \) and so by equation (1.3)

\[
p_c(T_\xi, r) = 1 - \frac{1}{M} = \frac{y(1-y^{r-1})}{1-y^r} \geq \frac{y}{1-y^r} \geq \frac{r-1}{r-1-y^r}.
\]  

(2.2)

A lower bound on \( p_c(T_\xi, r) \) is given by considering upper and lower bounds for the integral \( \int_0^1 \frac{g_\xi^r(x) - g_\xi^r(x)}{(1-x)\alpha+2} \, dx \).

For the upper bound, using the definition of the beta function, for every \( k \geq r \)

\[
\int_0^1 \frac{g_\xi^r(x) - g_\xi^r(x)}{(1-x)\alpha+2} \, dx = \sum_{i=r}^{k-1} \frac{i}{(r-1)} \int_0^1 x^{i-r}(1-x)^{r-2-\alpha} \, dx \quad \text{(by eq. (1.2))}
\]

\[
= \sum_{i=r}^{k-1} \frac{i}{(r-1)} \frac{1}{(i-r+1)!} \frac{1}{(i-r)!} \frac{1}{(r-1)!} \Gamma(r-1) \Gamma(r-1-\alpha) \Gamma(i-r+1) \Gamma(i-r-\alpha)
\]

\[
= \sum_{i=r}^{k-1} \frac{i(i-1)\ldots(i-t)}{(i-r+1)!} \frac{1}{(i-r+1)!} \frac{1}{(r-1)!} \Gamma(r-1-t-\varepsilon) \Gamma(r-1-t-\varepsilon)
\]

\[
\frac{\Gamma(r-1-t-\varepsilon)}{(r-1)(r-2)\ldots(r-1-t)} \Gamma(r-1-t).
\]  

(2.3)
Let \( c_1 = c_1(r, \alpha) = \frac{\Gamma(r-1-t-\varepsilon)}{\Gamma(r-1-\varepsilon)} \). Note that by inequality (2.1), for \( t < r - 2 \),
\[
\frac{\Gamma(r-1-t-\varepsilon)}{\Gamma(r-1-t)} \geq \frac{1}{(r-1-t)} \quad \text{and so} \quad c_1 \geq \frac{1}{(r-1)^{t-1}} = (r-1)^{-\alpha}.\]
On the other hand, if \( t = r - 2 \), then \( c_1 = \frac{\Gamma(2-\varepsilon)}{\Gamma(1-\varepsilon)} \geq 2(r-1)^{(1-\varepsilon)} \).

Thus, continuing equation (2.3), applying inequality (2.1) again yields
\[
\sum_{i=r}^{k-1} \frac{i(i-1) \ldots (i-t) \Gamma(i-t)}{(i-r+1) \Gamma(i-t-\varepsilon) \Gamma(r-1) \Gamma(r-2) \ldots \Gamma(r-t) (r-1-t) \Gamma(r-1-t) } \leq c_1 \sum_{i=r}^{k-1} \frac{i}{i-r+1} (i-1)(i-2) \ldots (i-t)(i-t+\varepsilon)
\leq rc_1 \sum_{i=r}^{k-1} i^{t+\varepsilon}
\leq rc_1 k^{1+t+\varepsilon} = rc_1 k^{1+\alpha}.
\]

Thus, taking expectation over \( k \) with respect to \( \xi \),
\[
\int_0^1 \frac{g_r'(x) - G_r'(x)}{(1-x)^{2+\alpha}} \, dx \leq rc_1 E[\xi^{1+\alpha}]. \tag{2.4}
\]

Consider now a lower bound on the integral:
\[
\int_0^1 g_r'(x) - G_r'(x) \leq \int_0^{1-y} g_r'(x) - M \, dx
= \int_0^{1-y} \frac{(M-1)}{(1-x)^{2+\alpha}} + \sum_{i=0}^{r-2} \frac{1}{(1-x)^{\alpha+i+1}} \, dx
= \left( \frac{(M-1)}{(\alpha+1)(1-x)^{2+\alpha}} - \sum_{i=0}^{r-2} (\alpha+i)(1-x)^{\alpha+i+1} \right)
= \frac{(M-1)}{(\alpha+1)} \left( \frac{1}{y^{\alpha+1}} - 1 \right) + \sum_{i=0}^{r-2} \frac{y^{\alpha+i+1} - 1}{i+\alpha} + \sum_{i=t+1}^{r-2} \frac{y^{\alpha+i} - y^t}{i+\alpha}
= \frac{1}{y^\alpha} \left( \frac{M-1}{\alpha+1} \left( \frac{y^{\alpha+1} - 1}{y} \right) + \sum_{i=0}^{t} \frac{y^i - y^{i+1}}{i+\alpha} + \sum_{i=t+1}^{r-2} \frac{y^i - y^{i+1}}{i+\alpha} \right)
= \frac{1}{y^\alpha} \left( (1+y+y^2+\ldots+y^{t-2}) (y^{\alpha+1} - 1) + \sum_{i=0}^{t-2} \frac{y^i - y^{i+1}}{i+\alpha} + \sum_{i=t+1}^{r-2} \frac{y^i - y^{i+1}}{i+\alpha} \right)
= \frac{1}{y^\alpha} \left( \frac{-1}{\alpha+1} + \frac{1}{\alpha+1} + \sum_{i=1}^{t} \frac{y^i - y^{i+1}}{\alpha+i} + \sum_{i=t+1}^{r-2} \frac{y^{\alpha+i+1} - y^t}{\alpha+1} + \sum_{i=t+1}^{r-2} \frac{y^{\alpha+i} - y^t}{\alpha+1} - \sum_{i=0}^{t-2} \frac{y^i}{\alpha+i} - \sum_{i=0}^{r-2} \frac{y^{i+1}}{\alpha+i} \right)
\geq \frac{1}{y^\alpha} \left( \frac{1}{\alpha+1} + \frac{1}{\alpha+1} - \sum_{i=0}^{t} \frac{y^i}{\alpha+i} \right)
\geq \frac{1}{y^\alpha} \left( \frac{1}{\alpha+1} - y^{t+1} \sum_{i=0}^{t} \frac{1}{\alpha+i} \right).
\]

Set \( c_2 = c_2(\alpha) = \sum_{i=0}^{t+1} \frac{1}{\alpha+i+1} \) and consider separately two different cases. For the
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first, if \( y^\alpha c_2 \geq \frac{1}{2\alpha(\alpha + 1)} \) then since \( E[\xi^{\alpha + 1}] \geq 1 \),

\[
y^\alpha \geq \frac{1}{2\alpha(\alpha + 1)c_2} \geq \frac{1}{2\alpha(\alpha + 1)c_2} E[\xi^{\alpha + 1}]^{-1}.
\]

Thus, if \( c_2 = \left( \frac{1}{2\alpha(\alpha + 1)c_2} \right)^{1/\alpha} \), then \( y \geq c_2 E[\xi^{\alpha + 1}]^{-1/\alpha} \).

In the second case, if \( y^\alpha < \frac{1}{2\alpha(\alpha + 1)c_2} \), then

\[
\int_0^1 \frac{g^*_\epsilon(x) - G^*_\epsilon(x)}{(1 - x)^{2+\alpha}} \, dx \geq \frac{1}{y^\alpha} \frac{1}{2\alpha(\alpha + 1)}.
\]  

(2.5)

Combining equation (2.5) with equation (2.4) yields

\[
y^\alpha \geq \frac{1}{2\alpha(\alpha + 1)} \frac{1}{rc_1} E[\xi^{\alpha + 1}]^{-1}
\]

and setting \( c'_1 = (2\alpha(\alpha + 1)rc_1)^{-1/\alpha} \) gives \( y \geq c'_1 E[\xi^{\alpha + 1}]^{-1/\alpha} \).

Finally, set \( c_{r,\alpha} = \frac{r - 1}{r} \min\{c'_1, c_2\} \) so that by inequality (2.2) we obtain,

\[
p_c(T_\xi, r) \geq \frac{r - 1}{r} y \geq c_{r,\alpha} E[\xi^{\alpha + 1}]^{-1/\alpha}.
\]

For every natural number \( n \in [1, r - 2] \), note that \( \lim_{n \to n^+} c_{r,\alpha} = 0 \) and, by the monotone convergence theorem, there is a constant \( c_{r,n} > 0 \) so that

\[
p_c(T_\xi, r) \geq c_{r,n} E[\xi^{\alpha + 1}]^{-1/n}.
\]

This completes the proof of the lemma.

In the above proof, as \( \alpha \to (r - 1)^- \), \( c_1(r, \alpha) \to \infty \) and hence \( \lim_{\alpha \to (r-1)^-} c_{r,\alpha} = 0 \), so the proof of Lemma 2.2 does not directly extend to the case \( \alpha = r - 1 \). We deal with this problem in the next lemma. Using a different approach we prove an essentially best possible lower bound on \( p_c(T_\xi, r) \) based on the \( r \)-th moment of the distribution \( \xi \). The sharpness of our bound is demonstrated by the \( b \)-branching tree \( T_b \), a Galton-Watson tree with a constant offspring distribution, for which, as a function of \( b \), we have \( p_c(T_b, r) = (1 + o(1))(1 - 1/r) \left( \frac{r-1}{r} \right)^{1/(r-1)} \) (see Lemma 3.7 in [3]).

**Lemma 2.3.** For any \( r \geq 2 \) and any offspring distribution \( \xi \) with \( E[\xi^r] < \infty \),

\[
p_c(T_\xi, r) \geq \left( 1 - \frac{1}{r} \right) \left( \frac{r - 1}{E[\xi^r]} \right)^{1/(r-1)}.
\]

**Proof.** As in the proof of Lemma 3.7 of [3] note that for every \( k \geq r \) and \( t \in [0, 1] \),

\[
g^*_\epsilon(1-t) = \frac{\Pr(\text{Bin}(k, t) \leq r - 1)}{1-t} = \frac{1 - \Pr(\text{Bin}(k, t) \geq r)}{1-t} \\
\geq \frac{1 - \binom{k}{r} t^r}{1-t} \geq \frac{1 - \frac{1}{r!} k^r t^r}{1-t}.
\]  

(2.6)

Using the lower bound in inequality (2.6) for the function \( G^*_\epsilon(x) \) yields

\[
G^*_\epsilon(1-t) \geq \sum_{k \geq r} \Pr(\xi = k) \frac{1 - \frac{1}{r!} k^r t^r}{1-t} = \frac{1 - \frac{1}{r!} E[\xi^r]}{1-t}.
\]
In particular, it was shown that there are constants in Theorem 2.1 for any \( \eta \) distribution \( r \) in Theorem 2.1 is also best possible, up to constants. In [3], it was shown that for every \( d \) demonstrated, again, by the regular

\[ T \]

This completes the proof of the lemma.

Theorem 2.1 now follows immediately from Lemmas 2.2 and 2.3.

It is not possible to extend a result of the form of Theorem 2.1 to \( \alpha > r - 1 \), as demonstrated, again, by the regular \( b \)-branching tree. For every \( \alpha \), the \((1 + \alpha)\)-th moment of this distribution is \( b^{1+\alpha} \) and the critical probability for the constant distribution is

\[ P_c(T_b, r) = 1 - o(1) \left( \frac{r-1}{b} \right)^{r-1} \]

As we already noted, Lemma 2.3 is asymptotically sharp, giving the best possible constant in Theorem 2.1 for any \( r \geq 2 \) and \( \alpha = r - 1 \). We now show that for \( \alpha \in (0, r - 1) \), Theorem 2.1 is also best possible, up to constants. In [3], it was shown that for every \( r \geq 2 \), there is a constant \( C_r \) such that if \( b \geq (r-1)(\log(4r)+1) \), then there is an offspring distribution \( \eta_{r,b} \) with \( E[\eta_{r,b}] = b \) and \( P_c(T_{\eta_{r,b}}, r) \leq C_r \exp \left( - \frac{b}{r-1} \right) \) (see Lemma 3.10 in [3]).

In particular, it was shown that there are \( k_1 = k_1(r, b) \) \( \leq (r - 2) \exp \left( \frac{b}{r-1} + 1 \right) - 1 \) and \( A, \lambda \in (0, 1) \) so that the distribution \( \eta_{r,b} \) is given by

\[ P(\eta_{r,b} = k) = \begin{cases} \frac{r-1}{k(k-1)} & r < k \leq k_1, k \neq 2r + 1 \\ \frac{1}{2r+1} + \lambda A & k = r \\ \lambda A & k = 2r + 1. \end{cases} \]

For any \( \alpha > 0 \), the \((\alpha + 1)\)-th moment of \( \eta_{r,b} \) is bounded from above as follows,

\[ E[\eta_{r,b}^{\alpha+1}] \leq 2(r-1) \sum_{k=r}^{k_1} k^{\alpha-1} + 2(2r + 1)^{\alpha+1} \]

\[ \leq 2(r-1) \left( \int_{r}^{k_1+1} x^{\alpha-1} dx + r^{\alpha-1} \right) + 2(2r + 1)^{\alpha+1} \]

\[ \leq \frac{2(r-1)}{\alpha} (k_1 + 1)^{\alpha} + 3(2r + 1)^{\alpha+1} \]

\[ \leq \frac{2(r-1)}{\alpha} \left( (r - 2) \exp \left( \frac{b}{r-1} + 1 \right) \right)^{\alpha} + 3(2r + 1)^{\alpha+1}, \]
where the $r^{\alpha - 1}$ term makes the inequality hold for $\alpha < 1$. In particular, there is a constant $C_{r,\alpha}$ so that for $b$ sufficiently large, $E[\eta_{r,b}^{1+\alpha}]^{1/\alpha} \leq C_{r,\alpha} \exp\left(\frac{b}{r-1}\right)$. Thus, for some positive constant $C'_{r,\alpha}$,

$$p_c(T_{\eta_{r,b}}, r) \leq C_r \exp\left(\frac{-b}{r-1}\right) \leq C'_{r,\alpha} E[\eta_{r,b}^{1+\alpha}]^{1-1/\alpha}.$$

Hence the bounds in Theorem 2.1 are sharp up to a constant that does not depend on the offspring distribution $\xi$.

**References**


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