A NOTE ON COLOR-BIAS HAMILTON CYCLES IN DENSE GRAPHS

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Abstract. Balogh, Csaba, Jing, and Pluhár [Electron. J. Combin., 27 (2020)] recently determined the minimum degree threshold that ensures a 2-colored graph $G$ contains a Hamilton cycle of significant color bias (i.e., a Hamilton cycle that contains significantly more than half of its edges in one color). In this short note we extend this result, determining the corresponding threshold for $r$-colorings.

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1. Introduction. The study of color-biased structures in graphs concerns the following problem. Given graphs $H$ and $G$, what is the largest $t$ such that in any $r$-coloring of the edges of $G$, there is always a copy of $H$ in $G$ that has at least $t$ edges of the same color? Note if $H$ is a subgraph of $G$, one can trivially ensure a copy of $H$ with at least $|E(H)|/r$ edges of the same color, so one is interested in when one can achieve a color-bias significantly above this.

The topic was first raised by Erdős in the 1960s (see [4, 6]). Erdős et al. [5] proved the following: for some constant $c > 0$, given any 2-coloring of the edges of $K_n$ and any fixed spanning tree $T_n$ with maximum degree $\Delta$, $K_n$ contains a copy of $T_n$ such that at least $(n - 1)/2 + c(n - 1 - \Delta)$ edges of this copy of $T_n$ receive the same color. In [1], Balogh et al. investigated the color-bias problem in the case of spanning trees, paths, and Hamilton cycles for various classes of graphs $G$. Note all their results concern 2-colorings and therefore were expressed in the equivalent language of graph discrepancy. The following result determines the minimum degree threshold for forcing a Hamilton cycle of significant color-bias in a 2-edge-colored graph.

Theorem 1.1 (Balogh et al. [1]). Let $0 < c < 1/4$ and $n \in \mathbb{N}$ be sufficiently large. If $G$ is an $n$-vertex graph with

$$\delta(G) \geq (3/4 + c)n,$$

then given any 2-coloring of $E(G)$ there is a Hamilton cycle in $G$ with at least $(1/2 + c/64)n$ edges of the same color. Moreover, if $4$ divides $n$, there is an $n$-vertex graph $G'$ with $\delta(G') = 3n/4$ and a 2-coloring of $E(G')$ for which every Hamilton cycle in $G'$ has precisely $n/2$ edges in each color.

In [7], Gishboliner, Krivelevich, and Michaeli considered color-bias Hamilton cycles in the random graph $G(n, p)$. Roughly speaking, their result states that if $p$ is such that with high probability (w.h.p.) $G(n, p)$ has a Hamilton cycle, then in fact...
w.h.p., given any $r$-coloring of the edges of $G(n,p)$, one can guarantee a Hamilton cycle that is essentially as color-bias as possible (see [7, Theorem 1.1] for the precise statement). A discrepancy (therefore color-bias) version of Theorem 1.3 (i.e., for sufficiently large graphs) was proven in [2].

In this paper we give a very short proof of the following multicolor generalization of Theorem 1.1. We require the following definition to state it.

**Definition 1.2.** Let $t, r \in \mathbb{N}$ and $H$ be a graph. We say that an $r$-coloring of the edges of $H$ is $t$-unbalanced if at least $|E(H)|/r + t$ edges are colored with the same color.

**Theorem 1.3.** Let $n, r, d \in \mathbb{N}$ with $r \geq 2$. Let $G$ be an $n$-vertex graph with $\delta(G) \geq (\frac{1}{2} + \frac{1}{2r})n + 6dr^2$. Then for every $r$-coloring of $E(G)$ there exists a $d$-unbalanced Hamilton cycle in $G$.

Note that $n$, $r$, and $d$ may all be comparable in size. Further, Theorem 1.3 implies Theorem 1.1 with a slightly better bound on the color-bias. In the following section we give constructions that show Theorem 1.3 is best possible; that is, there are $n$-vertex graphs $G$ with minimum degree $\delta(G) = (1/2 + 1/2r)n$ such that for some $r$-coloring of $E(G)$, every Hamilton cycle in $G$ uses precisely $n/r$ edges of each color. The proof of Theorem 1.3 is constructive, producing the $d$-unbalanced Hamilton cycle in time polynomial in $n$.

Remark. After making our manuscript available online, we learned of simultaneous and independent work of Gishboliner, Krivelevich, and Michaeli [8]. They prove an asymptotic version of Theorem 1.3 (i.e., for sufficiently large graphs $G$) via Szemerédi’s regularity lemma. They also generalize a number of the results from [1].

### 2. The extremal constructions.

Our first extremal example is a generalization of a 2-color construction from [1].

**Extremal Example 1.** Let $r, n \in \mathbb{N}$ where $r \geq 2$ and such that $2r$ divides $n$. Then there exists a graph $G$ on $n$ vertices with $\delta(G) = (\frac{1}{2} + \frac{1}{2r})n$, and an $r$-coloring of $E(G)$, such that every Hamilton cycle uses precisely $n/r$ edges of each color.

**Proof.** The vertex set of $G$ is partitioned into $r$ sets $V_1, \ldots, V_r$ such that $|V_1| = \cdots = |V_{r-1}| = n/2r$, and $|V_r| = (r+1)n/2r$; the edge set of $G$ consists of all edges with at least one endpoint in $V_r$. Now color the edges of $G$ with colors $1, \ldots, r$ as follows:

- For each $i \in [r-1]$, color every edge with one endpoint in $V_i$ and one endpoint in $V_r$ with color $i$.
- Color every edge with both endpoints in $V_r$ with color $r$ (see Figure 1).

Observe that $\delta(G) = (\frac{1}{2} + \frac{1}{2r})n$, which is attained by every vertex in $V_1 \cup \cdots \cup V_{r-1}$. For each $i \in [r-1]$, every vertex in $V_i$ is only adjacent to edges of color $i$, $|V_i| = n/2r$ and $E(G[V_1 \cup \cdots \cup V_{r-1}]) = \emptyset$. Hence every Hamilton cycle in $G$ must contain precisely $n/r$ edges of each color $i \in [r-1]$. Since a Hamilton cycle has $n$ edges, every Hamilton cycle in $G$ must also contain $n/r$ edges of color $r$. Thus every Hamilton cycle in $G$ uses precisely $n/r$ edges of each color. \hfill $\square$

We also have an additional extremal example in the $r = 3$ case.

**Extremal Example 2.** Let $n \in \mathbb{N}$ such that $3$ divides $n$. Then there exists a graph $G$ on $n$ vertices with $\delta(G) = 2n/3$, and a $3$-coloring of $E(G)$, such that every Hamilton cycle uses precisely $n/3$ edges of each color and every vertex in $G$ is incident to precisely two colors.
Fig. 1. Extremal Example 1 for $r = 3$.

Proof. Let $G$ be the $n$-vertex 3-partite Turán graph. So $G$ consists of three vertex sets $V_1$, $V_2$, and $V_3$, such that $|V_1| = |V_2| = |V_3| = n/3$, and all possible edges that go between distinct $V_i$ and $V_j$. Color all edges between $V_1$ and $V_2$ red, all edges between $V_2$ and $V_3$ blue, and all edges between $V_3$ and $V_1$ green.

Clearly $\delta(G) = 2n/3$ and every vertex is incident to precisely two colors. Let $H$ be a Hamilton cycle in $G$ and let $r$, $b$, and $g$ be the number of red, blue, and green edges in $H$, respectively. Since all red and green edges in $H$ are incident to vertices in $V_1$, $|V_1| = n/3$ and $V_1$ is an independent set, we must have that $2n/3 = r + g$. Applying similar reasoning to $V_2$ and $V_3$, we have that $2n/3 = b + r$ and $2n/3 = g + b$. Hence $r = b = g = n/3$. Thus every Hamilton cycle in $G$ uses precisely $n/3$ edges of each color.

3. Proof of Theorem 1.3. As in [1], we require the following generalisation of Dirac’s theorem.

Lemma 3.1 (Pósa [9]). Let $1 \leq t \leq n/2$, $G$ be an $n$-vertex graph with $\delta(G) \geq \frac{n}{2} + t$ and $E'$ be a set of edges of a linear forest in $G$ with $|E'| \leq 2t$. Then there is a Hamilton cycle in $G$ containing $E'$.

Proof of Theorem 1.3. Recall that $G$ is a graph on $n$ vertices with $\delta(G) \geq (\frac{1}{2} + \frac{1}{r} + \frac{d}{r^2})n + 6dr^2$ for some integers $r \geq 2$ and $d \geq 1$. Consider any $r$-coloring of $E(G)$. Given a color $c$ we define the function $L_c : E(G) \to \{0, 1\}$ as follows:

$$L_c(e) := \begin{cases} 1 & \text{if } e \text{ is colored with } c, \\ 0 & \text{otherwise.} \end{cases}$$

Given a triangle $xyz$ and a color $c$, we define $\text{Net}_c(xyz, xy)$ as follows:

$$\text{Net}_c(xyz, xy) := L_c(xz) + L_c(yz) - L_c(xy).$$
This quantity comes from an operation we will perform later where we extend a cycle $H$ by a vertex $z$ via deleting the edge $xy$ from $H$ and adding the edges $xz$ and $yz$, to form a new cycle $H'$. One can see that $\text{Net}_c(xy, yz)$ is the change in the number of edges of color $c$ from $H$ to $H'$.

Since $\delta(G) \geq \frac{1}{2}n$, by Dirac’s theorem, $G$ contains a Hamilton cycle $C$. If $C$ is $d$-unbalanced we are done, so suppose it is not. Let $v \in V(G)$. Since $d(v) \geq (\frac{1}{2} + \frac{1}{r})n + 6dr^2$, there are at least $\frac{n}{r} + 12dr^2$ edges $e$ in $C$ such that $v$ and $e$ span a triangle.

This can be seen in the following way. Let $X$ be the set of neighbors of $v$ and $X^+$ be the set of vertices whose “predecessors” on $C$ are neighbors of $v$, having arbitrarily chosen an orientation for $C$. We have

$$n \geq |X \cup X^+| = |X| + |X^+| - |X \cap X^+| \geq n + \frac{n}{r} + 12dr^2 - |X \cap X^+|.$$ 

Hence $|X \cap X^+| \geq \frac{n}{r} + 12dr^2$. Clearly each element in $X \cap X^+$ yields a triangle containing $v$, thus giving the desired bound.

This property, together with the fact that $C$ is not $d$-unbalanced (so contains fewer than $n/r + d$ edges of each color) immediately implies the following.

**FACT 3.2.** Let $v \in V(G)$, $Y \subseteq V(G)$ with $|Y| \leq 5dr^2$, and $xy$ be any edge in $G$ that forms a triangle with $v$ and is disjoint to $Y$. Then there is an edge $zw$ on $C$ vertex-disjoint to $xy$, and distinct colors $c_1$ and $c_2$ such that $vzw$ induces a triangle, $xy$ has color $c_1$, $zw$ has color $c_2$, and $z, w \notin Y$.

Initially set $A := \emptyset$. Consider an arbitrary $v \in V(G)$ and let $x, y, z, w, c_1, c_2$ be as in Fact 3.2 (where $Y := \emptyset$), where $xy$ is chosen to be an edge of $C$ that forms a triangle with $v$.

If there exists a color $c$ such that $\text{Net}_c(vxy, xy) \neq \text{Net}_c(vzw, zw)$, then add the pair $(xy, zw)$ to the set $A$, and define $v_1 := v$. If there is no such color, then we must have that $\text{Net}_c(vxy, xy) = \text{Net}_c(vzw, zw)$ and so

$$L_{c_1}(vx) + L_{c_1}(vy) - L_{c_1}(xy) = L_{c_1}(vw) + L_{c_1}(vz) - L_{c_1}(wz),$$

$$L_{c_1}(vx) + L_{c_1}(vy) - 1 = L_{c_1}(vw) + L_{c_1}(vz) \geq 0,$$

as $xy$ has color $c_1$, $zw$ has color $c_2$ and $c_1 \neq c_2$. Hence $vx$ or $vy$ is colored with $c_1$. Without loss of generality, let $vx$ be colored with $c_1$. By the same argument with color $c_2$, we may assume that, without loss of generality, $vw$ is colored with $c_2$. Let $c_3$ be the color of $vy$. Then $\text{Net}_{c_3}(vxy, xy) = \text{Net}_{c_3}(vzw, zw)$ and so

$$L_{c_3}(vx) + L_{c_3}(vy) - L_{c_3}(xy) = L_{c_3}(vw) + L_{c_3}(vz) - L_{c_3}(wz),$$

$$1 = L_{c_3}(vz),$$

as $vx$ and $xy$ are both colored with $c_1$ and $vw$ and $wz$ are both colored with $c_2$. Hence $c_3$ is also the color of $vz$ (see Figure 2). Since $c_1 \neq c_2$, we may assume, without loss of generality, $c_1 \neq c_3$.

Now we apply Fact 3.2 with $x$ playing the role of $v$, $vy$ playing the role of $xy$, and $Y = \emptyset$. We thus obtain a color $c_4 \neq c_3$ and an edge $w'z'$ on $C$ that is vertex-disjoint from $vy$, so that $w'z'$ forms a triangle with $x$, and $w'z'$ is colored $c_4$. Note that by construction $\text{Net}_{c_3}(xvy, vy) = -1$ while, as $c_4 \neq c_3$, by definition $\text{Net}_{c_3}(xw'z', w'z') = L_{c_3}(xw') + L_{c_3}(z') - 0 \geq 0$. In this case we define $v_1 := x$ and add the pair $(vy, w'z')$ to $A$.

\footnote{Note sometimes in an application of this fact, $xy$ will be an edge of $C$, but other times not.}
Repeated applications of this argument thus yield sets $B := \{v_1, v_2, \ldots, v_{dr^2}\}$ and a set $A$ whose elements are pairs of edges from $G$ so that

- all vertices lying in $B$ and in edges in pairs from $A$ are vertex-disjoint,
- for each $u = v_i$ in $B$ there is a pair $(xy, zw) \in A$ associated with $u$, and a color $c_u$ so that (i) $uxy$ and $uzw$ are triangles in $G$, (ii) $\text{Net}_{c_u}(uxy, xy) \neq \text{Net}_{c_u}(uzw, zw)$. We call $c_u$ the color associated with $u$.

Note that it is for the first of these two conditions that we require the set $Y$ in Fact 3.2.

At a given step of our argument, $Y$ will be the set of vertices that have previously been added to $B$ or lie in an edge previously selected for inclusion in a pair from $A$.

There is some color $c^*$ for which $c^*$ is the color associated with (at least) $dr$ of the vertices in $B$. Let $B'$ denote the set of such vertices of $B$; without loss of generality we may assume $B' = \{v_1, v_2, \ldots, v_{dr}\}$. Let $A'$ denote the subset of $A$ that corresponds to $B'$. For each $i \in [dr]$, let $(x_iy_i, z_iw_i)$ denote the element of $A'$ associated with $v_i$. We may assume that for each $i \in [dr]$,

$$
\text{Net}_{c^*}(v_i x_i y_i, x_i y_i) > \text{Net}_{c^*}(v_i z_i w_i, z_i w_i).
$$

Consider the induced subgraph $G'$ of $G$ obtained from $G$ by removing the vertices from $B'$. Let $E'$ be the set of all edges which appear in some pair in $A'$. As $\delta(G') \geq n/2 + dr$, Lemma 3.1 implies that there exists a Hamilton cycle $C'$ in $G'$ which contains $E'$. Let $C_1$ be the Hamilton cycle of $G$ obtained from $C'$ by inserting each $v_i$ from $B'$ between $x_i$ and $y_i$; let $C_2$ be the Hamilton cycle of $G$ obtained from $C'$ by inserting each $v_i$ from $B'$ between $z_i$ and $w_i$. For $j = 1, 2$, write $E_j$ for the number of edges in $C_j$ of color $c^*$. Note that (1) implies that $E_1 - E_2 \geq dr$. It is easy to see that this implies one of $C_1$ and $C_2$ contains at least $n/r + d$ edges in the same color, thereby completing the proof. \(\square\)

4. **Concluding remarks.** As mentioned in [5, section 7] there are many possible directions for future research. One natural extension of our work is to seek an analogue of Theorem 1.3 in the setting of digraphs.

\[2\text{This color may not necessarily be } c^*.\]
QUESTION 4.1. Given any digraph \( G \) on \( n \) vertices with minimum in- and out-degree at least \( (1/2 + 1/2r + o(1))n \), and any \( r \)-coloring of \( E(G) \), can one always ensure a Hamilton cycle in \( G \) of significant color-bias?

Note that the natural digraph analogues of our extremal constructions for Theorem 1.3 show that one cannot lower the minimum degree condition in Question 4.1.

Given an \( r \)-colored \( n \)-vertex graph \( G \) and nonnegative integers \( d_1, \ldots, d_r \), we say that \( G \) contains a \((d_1, \ldots, d_r)\)-colored Hamilton cycle if there is a Hamilton cycle in \( G \) with precisely \( d_i \) edges of the \( i \)th color (for every \( i \in [r] \)). Note that the proof of Theorem 1.3 (more precisely (1)) ensures that given a graph \( G \) as in the theorem, one can obtain at least \( dr \) distinct vectors \((d_1, \ldots, d_r)\) such that \( G \) has a \((d_1, \ldots, d_r)\)-colored Hamilton cycle. It would be interesting to investigate this problem further.

That is, given an \( r \)-colored \( n \)-vertex graph \( G \) of a given minimum degree, how many distinct vectors \((d_1, \ldots, d_r)\) can we guarantee so that \( G \) contains a \((d_1, \ldots, d_r)\)-colored Hamilton cycle?

In [2], the question of determining the minimum degree threshold that ensures a color-bias \( k \)th power of a Hamilton cycle was raised; it would be interesting to establish whether a variant of the switching method from the proof of Theorem 1.3 can be used to resolve this problem (for all \( k \geq 2 \) and \( r \)-colorings where \( r \geq 2 \)).

Remark. Since a version of this paper first appeared online, Bradač [3] has used the regularity method to resolve this problem asymptotically for all \( k \geq 2 \) when \( r = 2 \).

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