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A NOTE ON COLOR-BIAS HAMILTON CYCLES IN DENSE GRAPHS*

ANDREA FRESCHI[†], JOSEPH HYDE[†], JOANNA LADA[‡], AND ANDREW TREGLOWN[†]

Abstract. Balogh, Csaba, Jing, and Pluhár [*Electron. J. Combin.*, 27 (2020)] recently determined the minimum degree threshold that ensures a 2-colored graph G contains a Hamilton cycle of significant color bias (i.e., a Hamilton cycle that contains significantly more than half of its edges in one color). In this short note we extend this result, determining the corresponding threshold for r -colorings.

Key words. Hamilton cycles, color-bias, discrepancy

AMS subject classifications. 05C35, 05C45, 05C15, 05C55

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1. Introduction. The study of color-biased structures in graphs concerns the following problem. Given graphs H and G , what is the largest t such that in any r -coloring of the edges of G , there is always a copy of H in G that has at least t edges of the same color? Note if H is a subgraph of G , one can trivially ensure a copy of H with at least $|E(H)|/r$ edges of the same color, so one is interested in when one can achieve a color-bias significantly above this.

The topic was first raised by Erdős in the 1960s (see [4, 6]). Erdős et al. [5] proved the following: for some constant $c > 0$, given any 2-coloring of the edges of K_n and any fixed spanning tree T_n with maximum degree Δ , K_n contains a copy of T_n such that at least $(n-1)/2 + c(n-1-\Delta)$ edges of this copy of T_n receive the same color. In [1], Balogh et al. investigated the color-bias problem in the case of spanning trees, paths, and Hamilton cycles for various classes of graphs G . Note all their results concern 2-colorings and therefore were expressed in the equivalent language of *graph discrepancy*. The following result determines the minimum degree threshold for forcing a Hamilton cycle of significant color-bias in a 2-edge-colored graph.

THEOREM 1.1 (Balogh et al. [1]). *Let $0 < c < 1/4$ and $n \in \mathbb{N}$ be sufficiently large. If G is an n -vertex graph with*

$$\delta(G) \geq (3/4 + c)n,$$

then given any 2-coloring of $E(G)$ there is a Hamilton cycle in G with at least $(1/2 + c/64)n$ edges of the same color. Moreover, if 4 divides n , there is an n -vertex graph G' with $\delta(G') = 3n/4$ and a 2-coloring of $E(G')$ for which every Hamilton cycle in G' has precisely $n/2$ edges in each color.

In [7], Gishboliner, Krivelevich, and Michaeli considered color-bias Hamilton cycles in the random graph $G(n, p)$. Roughly speaking, their result states that if p is such that with high probability (w.h.p.) $G(n, p)$ has a Hamilton cycle, then in fact

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w.h.p., given any r -coloring of the edges of $G(n, p)$, one can guarantee a Hamilton cycle that is essentially as color-bias as possible (see [7, Theorem 1.1] for the precise statement). A discrepancy (therefore color-bias) version of the Hajnal–Szemerédi theorem was proven in [2].

In this paper we give a very short proof of the following multicolor generalization of Theorem 1.1. We require the following definition to state it.

DEFINITION 1.2. *Let $t, r \in \mathbb{N}$ and H be a graph. We say that an r -coloring of the edges of H is t -unbalanced if at least $|E(H)|/r + t$ edges are colored with the same color.*

THEOREM 1.3. *Let $n, r, d \in \mathbb{N}$ with $r \geq 2$. Let G be an n -vertex graph with $\delta(G) \geq (\frac{1}{2} + \frac{1}{2r})n + 6dr^2$. Then for every r -coloring of $E(G)$ there exists a d -unbalanced Hamilton cycle in G .*

Note that n , r , and d may all be comparable in size. Further, Theorem 1.3 implies Theorem 1.1 with a slightly better bound on the color-bias. In the following section we give constructions that show Theorem 1.3 is best possible; that is, there are n -vertex graphs G with minimum degree $\delta(G) = (\frac{1}{2} + \frac{1}{2r})n$ such that for some r -coloring of $E(G)$, every Hamilton cycle in G uses precisely n/r edges of each color. The proof of Theorem 1.3 is constructive, producing the d -unbalanced Hamilton cycle in time polynomial in n .

Remark. After making our manuscript available online, we learned of simultaneous and independent work of Gishboliner, Krivelevich, and Michaeli [8]. They prove an asymptotic version of Theorem 1.3 (i.e., for sufficiently large graphs G) via Szemerédi’s regularity lemma. They also generalize a number of the results from [1].

2. The extremal constructions. Our first extremal example is a generalization of a 2-color construction from [1].

EXTREMAL EXAMPLE 1. *Let $r, n \in \mathbb{N}$ where $r \geq 2$ and such that $2r$ divides n . Then there exists a graph G on n vertices with $\delta(G) = (\frac{1}{2} + \frac{1}{2r})n$, and an r -coloring of $E(G)$, such that every Hamilton cycle uses precisely n/r edges of each color.*

Proof. The vertex set of G is partitioned into r sets $V_1, \dots, V_r$ such that $|V_1| = \dots = |V_{r-1}| = n/2r$, and $|V_r| = (r+1)n/2r$; the edge set of G consists of all edges with at least one endpoint in V_r . Now color the edges of G with colors $1, \dots, r$ as follows:

- For each $i \in [r-1]$, color every edge with one endpoint in V_i and one endpoint in V_r with color i .
- Color every edge with both endpoints in V_r with color r (see Figure 1).

Observe that $\delta(G) = (\frac{1}{2} + \frac{1}{2r})n$, which is attained by every vertex in $V_1 \cup \dots \cup V_{r-1}$. For each $i \in [r-1]$, every vertex in V_i is only adjacent to edges of color i , $|V_i| = n/2r$ and $E(G[V_1 \cup \dots \cup V_{r-1}]) = \emptyset$. Hence every Hamilton cycle in G must contain precisely n/r edges of each color $i \in [r-1]$. Since a Hamilton cycle has n edges, every Hamilton cycle in G must also contain n/r edges of color r . Thus every Hamilton cycle in G uses precisely n/r edges of each color. $\square$

We also have an additional extremal example in the $r = 3$ case.

EXTREMAL EXAMPLE 2. *Let $n \in \mathbb{N}$ such that 3 divides n . Then there exists a graph G on n vertices with $\delta(G) = 2n/3$, and a 3-coloring of $E(G)$, such that every Hamilton cycle uses precisely $n/3$ edges of each color and every vertex in G is incident to precisely two colors.*

FIG. 1. *Extremal Example 1 for $r = 3$.*

Proof. Let G be the n -vertex 3-partite Turán graph. So G consists of three vertex sets V_1 , V_2 , and V_3 , such that $|V_1| = |V_2| = |V_3| = n/3$, and all possible edges that go between distinct V_i and V_j . Color all edges between V_1 and V_2 red, all edges between V_2 and V_3 blue, and all edges between V_3 and V_1 green.

Clearly $\delta(G) = 2n/3$ and every vertex is incident to precisely two colors. Let H be a Hamilton cycle in G and let r , b , and g be the number of red, blue, and green edges in H , respectively. Since all red and green edges in H are incident to vertices in V_1 , $|V_1| = n/3$ and V_1 is an independent set, we must have that $2n/3 = r + g$. Applying similar reasoning to V_2 and V_3 , we have that $2n/3 = b + r$ and $2n/3 = g + b$. Hence $r = b = g = n/3$. Thus every Hamilton cycle in G uses precisely $n/3$ edges of each color. $\square$

3. Proof of Theorem 1.3. As in [1], we require the following generalisation of Dirac's theorem.

LEMMA 3.1 (Pósa [9]). *Let $1 \leq t \leq n/2$, G be an n -vertex graph with $\delta(G) \geq \frac{n}{2} + t$ and E' be a set of edges of a linear forest in G with $|E'| \leq 2t$. Then there is a Hamilton cycle in G containing E' .*

Proof of Theorem 1.3. Recall that G is a graph on n vertices with $\delta(G) \geq (\frac{1}{2} + \frac{1}{2r})n + 6dr^2$ for some integers $r \geq 2$ and $d \geq 1$. Consider any r -coloring of $E(G)$. Given a color c we define the function $L_c : E(G) \rightarrow \{0, 1\}$ as follows:

$$L_c(e) := \begin{cases} 1 & \text{if } e \text{ is colored with } c, \\ 0 & \text{otherwise.} \end{cases}$$

Given a triangle xyz and a color c , we define $\text{Net}_c(xyz, xy)$ as follows:

$$\text{Net}_c(xyz, xy) := L_c(xz) + L_c(yz) - L_c(xy).$$

This quantity comes from an operation we will perform later where we extend a cycle H by a vertex z via deleting the edge xy from H and adding the edges xz and yz , to form a new cycle H' . One can see that $\text{Net}_c(xyz, xy)$ is the change in the number of edges of color c from H to H' .

Since $\delta(G) \geq \frac{1}{2}n$, by Dirac's theorem, G contains a Hamilton cycle C . If C is d -unbalanced we are done, so suppose it is not. Let $v \in V(G)$. Since $d(v) \geq (\frac{1}{2} + \frac{1}{2r})n + 6dr^2$, there are at least $\frac{n}{r} + 12dr^2$ edges e in C such that v and e span a triangle.

This can be seen in the following way. Let X be the set of neighbors of v and X^+ be the set of vertices whose "predecessors" on C are neighbors of v , having arbitrarily chosen an orientation for C . We have

$$n \geq |X \cup X^+| = |X| + |X^+| - |X \cap X^+| \geq n + \frac{n}{r} + 12dr^2 - |X \cap X^+|.$$

Hence $|X \cap X^+| \geq \frac{n}{r} + 12dr^2$. Clearly each element in $X \cap X^+$ yields a triangle containing v , thus giving the desired bound.

This property, together with the fact that C is not d -unbalanced (so contains fewer than $n/r + d$ edges of each color) immediately implies the following.

FACT 3.2. *Let $v \in V(G)$, $Y \subseteq V(G)$ with $|Y| \leq 5dr^2$, and xy be any edge in G that forms a triangle with v and is disjoint to Y .¹ Then there is an edge zw on C vertex-disjoint to xy , and distinct colors c_1 and c_2 such that vzw induces a triangle, xy has color c_1 , zw has color c_2 , and $z, w \notin Y$.*

Initially set $A := \emptyset$. Consider an arbitrary $v \in V(G)$ and let x, y, z, w, c_1, c_2 be as in Fact 3.2 (where $Y := \emptyset$), where xy is chosen to be an edge of C that forms a triangle with v .

If there exists a color c such that $\text{Net}_c(vxy, xy) \neq \text{Net}_c(vzw, zw)$, then add the pair (xy, zw) to the set A , and define $v_1 := v$. If there is no such color, then we must have that $\text{Net}_{c_1}(vxy, xy) = \text{Net}_{c_1}(vzw, zw)$ and so

$$\begin{aligned} L_{c_1}(vx) + L_{c_1}(vy) - L_{c_1}(xy) &= L_{c_1}(vw) + L_{c_1}(vz) - L_{c_1}(wz), \\ L_{c_1}(vx) + L_{c_1}(vy) - 1 &= L_{c_1}(vw) + L_{c_1}(vz) \geq 0, \end{aligned}$$

as xy has color c_1 , wz has color c_2 and $c_1 \neq c_2$. Hence vx or vy is colored with c_1 . Without loss of generality, let vx be colored with c_1 . By the same argument with color c_2 , we may assume that, without loss of generality, vw is colored c_2 . Let c_3 be the color of vy . Then $\text{Net}_{c_3}(vxy, xy) = \text{Net}_{c_3}(vzw, zw)$ and so

$$\begin{aligned} L_{c_3}(vx) + L_{c_3}(vy) - L_{c_3}(xy) &= L_{c_3}(vw) + L_{c_3}(vz) - L_{c_3}(wz), \\ 1 &= L_{c_3}(vz), \end{aligned}$$

as vx and xy are both colored with c_1 and vw and wz are both colored with c_2 . Hence c_3 is also the color of vz (see Figure 2). Since $c_1 \neq c_2$, we may assume, without loss of generality, $c_1 \neq c_3$.

Now we apply Fact 3.2 with x playing the role of v , vy playing the role of xy , and $Y = \emptyset$. We thus obtain a color $c_4 \neq c_3$ and an edge $w'z'$ on C that is vertex-disjoint from vy , so that $w'z'$ forms a triangle with x , and $w'z'$ is colored c_4 . Note that by construction $\text{Net}_{c_3}(xvy, vy) = -1$ while, as $c_4 \neq c_3$, by definition $\text{Net}_{c_3}(xw'z', w'z') = L_{c_3}(xw') + L_{c_3}(xz') - 0 \geq 0$. In this case we define $v_1 := x$ and add the pair $(vy, w'z')$ to A .

¹Note sometimes in an application of this fact, xy will be an edge of C , but other times not.

FIG. 2. A Hamilton cycle C for G . There is no color c with $\text{Net}_c(vxy, xy) \neq \text{Net}_c(vzw, zw)$ implying the color arrangement above.

Repeated applications of this argument thus yield sets $B := \{v_1, v_2, \dots, v_{dr^2}\}$ and a set A whose elements are pairs of edges from G so that

- all vertices lying in B and in edges in pairs from A are vertex-disjoint,
- for each $u = v_i$ in B there is a pair $(xy, zw) \in A$ associated with u , and a color c_u so that (i) uxy and uzw are triangles in G , (ii) $\text{Net}_{c_u}(uxy, xy) \neq \text{Net}_{c_u}(uzw, zw)$. We call c_u the color associated with u .

Note that it is for the first of these two conditions that we require the set Y in Fact 3.2. At a given step of our argument, Y will be the set of vertices that have previously been added to B or lie in an edge previously selected for inclusion in a pair from A .

There is some color c^* for which c^* is the color associated with (at least) dr of the vertices in B . Let B' denote the set of such vertices of B ; without loss of generality we may assume $B' = \{v_1, v_2, \dots, v_{dr}\}$. Let A' denote the subset of A that corresponds to B' . For each $i \in [dr]$, let $(x_i y_i, z_i w_i)$ denote the element of A' associated with v_i . We may assume that for each $i \in [dr]$,

$$(1) \quad \text{Net}_{c^*}(v_i x_i y_i, x_i y_i) > \text{Net}_{c^*}(v_i z_i w_i, z_i w_i).$$

Consider the induced subgraph G' of G obtained from G by removing the vertices from B' . Let E' be the set of all edges which appear in some pair in A' . As $\delta(G') \geq n/2 + dr$, Lemma 3.1 implies that there exists a Hamilton cycle C' in G' which contains E' . Let C_1 be the Hamilton cycle of G obtained from C' by inserting each v_i from B' between x_i and y_i ; let C_2 be the Hamilton cycle of G obtained from C' by inserting each v_i from B' between z_i and w_i . For $j = 1, 2$, write E_j for the number of edges in C_j of color c^* . Note that (1) implies that $E_1 - E_2 \geq dr$. It is easy to see that this implies one of C_1 and C_2 contains at least $n/r + d$ edges in the same color,² thereby completing the proof. $\square$

4. Concluding remarks. As mentioned in [5, section 7] there are many possible directions for future research. One natural extension of our work is to seek an analogue of Theorem 1.3 in the setting of digraphs.

²This color may not necessarily be c^* .

QUESTION 4.1. *Given any digraph G on n vertices with minimum in- and out-degree at least $(1/2 + 1/2r + o(1))n$, and any r -coloring of $E(G)$, can one always ensure a Hamilton cycle in G of significant color-bias?*

Note that the natural digraph analogues of our extremal constructions for Theorem 1.3 show that one cannot lower the minimum degree condition in Question 4.1.

Given an r -colored n -vertex graph G and nonnegative integers $d_1, \dots, d_r$, we say that G contains a $(d_1, \dots, d_r)$ -colored Hamilton cycle if there is a Hamilton cycle in G with precisely d_i edges of the i th color (for every $i \in [r]$). Note that the proof of Theorem 1.3 (more precisely (1)) ensures that given a graph G as in the theorem, one can obtain at least dr distinct vectors $(d_1, \dots, d_r)$ such that G has a $(d_1, \dots, d_r)$ -colored Hamilton cycle. It would be interesting to investigate this problem further. That is, given an r -colored n -vertex graph G of a given minimum degree, how many distinct vectors $(d_1, \dots, d_r)$ can we guarantee so that G contains a $(d_1, \dots, d_r)$ -colored Hamilton cycle?

In [2], the question of determining the minimum degree threshold that ensures a color-bias k th power of a Hamilton cycle was raised; it would be interesting to establish whether a variant of the switching method from the proof of Theorem 1.3 can be used to resolve this problem (for all $k \geq 2$ and r -colorings where $r \geq 2$).

Remark. Since a version of this paper first appeared online, Bradač [3] has used the regularity method to resolve this problem asymptotically for all $k \geq 2$ when $r = 2$.

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