

# All cartesian closed categories of quasicontinuous domains consist of domains

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# All Cartesian Closed Categories of Quasicontinuous Domains Consist of Domains

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## Abstract

Quasicontinuity is a generalisation of Scott's notion of continuous domain, introduced in the early 80s by Gierz, Lawson and Stralka. In this paper we ask which cartesian closed full subcategories exist in **qCONT**, the category of all quasicontinuous domains and Scott-continuous functions. The surprising, and perhaps disappointing, answer turns out to be that all such subcategories consist entirely of continuous domains. In other words, there are no new cartesian closed full subcategories in **qCONT** beyond those already known to exist in **CONT**.

To prove this, we reduce the notion of meet-continuity for dcpos to one which only involves well-ordered chains. This allows us to characterise meet-continuity by “forbidden substructures”. We then show that each forbidden substructure has a non-quasicontinuous function space.

**Keywords:** cartesian closed category; quasicontinuous domain; meet-continuity; meet\*-continuity

## 1 Introduction

Domain theory was introduced by Dana Scott in the late sixties as a mathematical universe within which to define the semantics of programming languages. In this programme, one seeks to identify the precise properties of

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domains that correspond to the language features of interest. Early on it became clear that the ability to define higher-order functions in programming has its counterpart in the formation of function spaces, more precisely, in the requirement that a category of domains be *cartesian closed*. In 1980, [17] Nasser Saheb-Djahromi considered programs with a probabilistic choice operator and in order to accommodate this in the semantics, introduced the probabilistic powerdomain construction. This was studied more deeply by Claire Jones and Gordon Plotkin, [9, 8], towards the end of that decade and it was shown that the behaviour of the probabilistic powerdomain construction is much more easily understood in the context of *continuous domains*. This confirmed the experience gained before, namely, that continuous domains allow one to say much more about the workings of domain constructions and their relationship with program constructs. In a nutshell, one can say that the nice properties of continuous domains result from the fact that they are *completions of finitary structures*.

However, while it is advantageous to have continuous domains as *inputs* to domain constructions, it is not always guaranteed that one obtains them in the *output*, too. The most prominent construction which causes a complication is the function space: given two continuous domains  $D$  and  $E$ , the space  $[D \rightarrow E]$  of Scott-continuous functions may not itself be continuous. For a detailed discussion of this phenomenon, see [10]. One way to overcome this problem is to restrict continuous domains even further and this indeed leads to various cartesian closed categories, such as the continuous Scott domains, RB-domains ([12]) and FS-domains ([11]). However, it remains an *open problem* to find such a category that is simultaneously closed under the probabilistic powerdomain construction.

In light of these difficulties, it is natural to ask whether the condition of continuity can be relaxed, which would allow us to cast our net wider. A natural candidate for a more liberal notion of approximation is that of *quasicontinuity*, a concept which was introduced in the early eighties by Gerhard Gierz, Jimmie Lawson and Albert Stralka, [3], as a generalization of classical continuous domains. Indeed, Jean Goubault-Larrecq, [4], was able to show that the category of *QRB-domains* (a special class of quasicontinuous domains) is closed under the probabilistic powerdomain construction, adding to what is a very small set of such closure results. This led many researchers to re-examine quasicontinuous domains [4, 5, 6, 14, 15, 19] and many pleasing properties were established. For example, it was proved by Goubault-Larrecq and the second author, [5], and independently by Jimmie Lawson and Xiaoyong Xi, [14], that QFS-domains and QRB-domains are the same class and that they can be characterised as being precisely

the Lawson-compact quasicontinuous domains, while, in the classical case, whether FS-domains and RB-domains are the same is one of the oldest and best-known open problems in domain theory.

However, unlike FS-domains or RB-domains, the category of QFS-domains (equivalently, QRB-domains) with Scott-continuous functions as morphisms is not cartesian closed (see Remark 3.9). This raises the question whether there are any new cartesian closed categories consisting of quasicontinuous domains at all. In this paper we show that this is not the case.

**A note on provenance.** The origin of the ideas for the present paper lies in the paper [19] published in Chinese by Haoran Zhao and Hui Kou. During 2014, the same two authors went on to show that a cartesian closed category of *countably based* quasicontinuous domains must consist of continuous domains entirely. Shortly afterwards and independently, the other three authors of the present paper also considered [19] and came up with the same result about ccc's of countably based quasicontinuous domains. They continued their investigation with the aim of removing the countability assumption and as we report below this was eventually successful. The parallels between the work of the two teams was discovered during the reviewing process of our respective journal submissions and we are grateful to our editor, Michael Mislove, for encouraging us to combine our efforts into a single paper.

## 2 Preliminaries

We use the standard definitions of domain theory as can be found in [1] or [2]. The following, taken from [2, Section III-3], may be less familiar: One says that a subset  $G$  of a dcpo is *way-below* a subset  $H$  if for every directed set  $D$ ,  $\sup D \in \uparrow H$  implies  $d \in \uparrow G$  for some  $d \in D$ . This generalises the usual way-below relation between elements which justifies writing  $G \ll H$  for it. If  $H$  consists of a single element  $x$  then one writes  $G \ll x$  instead of  $G \ll \{x\}$ . Consistent with this we define an order between subsets  $G, H$  by  $G \leq H \Leftrightarrow \uparrow H \subseteq \uparrow G$ . This implies that a family  $\mathcal{F}$  of subsets is *directed* if the corresponding family  $\{\uparrow G \mid G \in \mathcal{F}\}$  is a filter base.

**Definition 2.1.** A dcpo  $L$  is called *quasicontinuous* (or a *quasicontinuous domain*) if for each  $x \in L$  the family

$$\text{fin}(x) = \{F \mid F \text{ is finite, } F \ll x\}$$

is a directed family, and whenever  $x \not\leq y$ , then there exists  $F \in \text{fin}(x)$  with  $y \notin \uparrow F$ , i.e.,  $\uparrow x = \bigcap \{\uparrow F \mid F \in \text{fin}(x)\}$ .

The following key fact relies on the Axiom of Choice via Rudin's Lemma:

**Proposition 2.2.** [2, Proposition III-3.4] *Let  $\mathcal{F}$  be a directed family of non-empty finite sets in a dcpo. If  $G \ll H$  and  $\bigcap \{\uparrow F \mid F \in \mathcal{F}\} \subseteq \uparrow H$ , then  $F \subseteq \uparrow G$  for some  $F \in \mathcal{F}$ .*

We use this to prove the following convenient criterion for quasicontinuity:

**Proposition 2.3.** *A dcpo  $L$  is quasicontinuous, if for every  $x \in L$  the family  $\text{fin}(x)$  contains a directed subfamily  $\mathcal{G}$  such that  $\uparrow x = \bigcap \{\uparrow G \mid G \in \mathcal{G}\}$ .*

*Proof.* We only need to prove that the family  $\text{fin}(x)$  is directed. For  $F, H \in \text{fin}(x)$ , since  $F, H \ll x$  and  $\bigcap \{\uparrow G \mid G \in \mathcal{G}\} = \uparrow x$ , by Proposition 2.2, there exist  $G_1, G_2 \in \mathcal{G}$  such that  $G_1 \subseteq \uparrow F$  and  $G_2 \subseteq \uparrow H$ . Then some  $G \in \mathcal{G}$  is included in  $\uparrow F \cap \uparrow H$  since  $\mathcal{G}$  is directed.  $\square$

**Proposition 2.4.** [2, Proposition III-3.6] *Let  $P$  be a quasicontinuous domain. A subset  $U$  of  $P$  is Scott open iff for each  $x \in U$  there exists a finite  $F \ll x$  such that  $\uparrow F \subseteq U$ . The sets  $\uparrow F = \{x \mid F \ll x\}$  are Scott open and they form a basis for the Scott topology.*

Quasicontinuity is preserved by Scott-continuous retractions:

**Proposition 2.5.** *Let  $f$  be a Scott-continuous retraction from a quasicontinuous domain  $L$  to a dcpo  $M$ . Then  $M$  is quasicontinuous.*

*Proof.* Since  $f$  is a Scott-continuous retraction, then by definition there exists a Scott-continuous function  $g$  from  $M$  to  $L$  such that  $f \circ g = \text{id}_M$ . For every  $x \in M$  and finite set  $F \ll g(x)$ , we claim that  $f(F) \ll x$ . Indeed, let  $D$  be a directed set of  $M$  with  $x \leq \sup D$ ; then  $g(x) \leq g(\sup D) = \sup g(D)$  and we obtain an element  $d \in D$  such that  $g(d) \in \uparrow F$  because  $F \ll g(x)$ . So we get  $d = f(g(d)) \in f(\uparrow F) \subseteq \uparrow f(F)$ , and the claim is true.

Given  $x, y \in M$  with  $x \not\leq y$ , then  $x = f(g(x)) \in M \setminus \downarrow y$  and we get  $g(x) \in f^{-1}(M \setminus \downarrow y)$ . Since  $L$  is quasicontinuous and  $f^{-1}(M \setminus \downarrow y)$  is Scott open, we get from the preceding proposition that there exists  $G \in \text{fin}(g(x))$  such that  $G \subseteq f^{-1}(M \setminus \downarrow y)$ . This means that  $f(G) \subseteq M \setminus \downarrow y$  or  $y \notin \uparrow f(G)$ . By the claim above we know that  $f(G) \ll x$ , that is,  $f(G) \in \text{fin}(x)$ . So for every  $x \in M$ , we have  $\uparrow x = \bigcap \{\uparrow f(G) \mid G \in \text{fin}(g(x))\}$ .

Finally, we note that the family  $\{f(G) \mid G \in \text{fin}(g(x))\}$  is directed because  $\text{fin}(g(x))$  is and  $f$  preserves the order. Proposition 2.3 now allows us to conclude that  $M$  is quasicontinuous.  $\square$

Recall that a *well-ordered set* is a poset where every non-empty subset has a least element. Every well-ordered set is a chain and in fact order-isomorphic to an ordinal. In accordance with the theory of ordinal numbers we write  $c + 1$  to denote the least element of  $\{x \in C \mid c < x\}$ , provided this set is not empty. Note that every element of the form  $c + 1$  is compact in the sense of domain theory. Moreover we have the following:

**Proposition 2.6.** *Every well-ordered set  $C$  with a top element  $\top$  is an algebraic lattice, and every compact element of it is equal to the least element or of the form  $c + 1$  for some  $c \in C \setminus \{\top\}$ .*

*Proof.* Let  $C$  be a well-ordered chain with top element  $\top$  and least element  $\perp$ . By well-orderedness,  $C$  has infima for non-empty subsets and since we assume a top element, it follows that  $C$  is a complete lattice. For every non-compact element  $a$  one has  $a = \sup\{x \in C \mid x < a\} \leq \sup\{x + 1 \mid x \in C, x < a\} \leq a$ , so every element in  $C$  is the supremum of compact elements below it. Thus,  $C$  is an algebraic lattice.

If  $k \in C \setminus \{\perp\}$  is compact, then the set  $\{x \in C \mid x < k\}$  is non-empty and its supremum  $s$  strictly smaller than  $k$ . It is now easy to verify that  $k = s + 1$ .  $\square$

We will be concerned with well-ordered chains that are subsets of dcpos. We say that such a chain  $C$  is *limit embedded* in the dcpo  $L$ , if whenever  $x = \sup D$  in  $C$ , for  $D$  a non-empty subset of  $C$ , then  $x$  is also the supremum of  $D$  considered as a subset of  $L$ . (Equivalently, we could stipulate that the embedding of  $C$  into  $L$  preserves existing directed suprema.)

**Proposition 2.7.** *1. The image of a well-ordered set under a monotone function is well-ordered.*

*2. Let  $C$  be a bounded-complete chain and  $f: C \rightarrow L$  a Scott-continuous function into a dcpo  $L$ . Then the image of  $f$  is limit embedded in  $L$ .<sup>1</sup>*

*Proof.* (1) Assume  $f$  is a monotone function from a well-ordered set  $C$  to a poset  $Q$  and  $A$  is a non-empty subset of  $f(C)$ . Then  $f^{-1}(A)$  is a non-empty subset of  $C$  and hence contains a least element  $a$ . It is now easy to see that  $f(a)$  is minimal in  $A$ .

(2) Let  $D$  be a non-empty subset in the image of the Scott-continuous function  $f: C \rightarrow L$ . If  $f^{-1}(D)$  is bounded in  $C$  then it has a supremum  $c$  there. Since  $f^{-1}(D)$  is automatically directed we can use Scott-continuity

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<sup>1</sup>Note that this is not true for the image of a general dcpo under a Scott-continuous function.

of  $f$  to conclude that  $f(c) = \sup D$  which shows that the supremum of  $D$  lies in the image of  $f$ .

If  $f^{-1}(D)$  is unbounded in  $C$  then because  $C$  is a chain this means that  $f^{-1}(D)$  is cofinal in  $C$ . This implies that  $D$  is cofinal in  $f(C)$  and therefore if it has a supremum in  $f(C)$  then that is the largest element of  $D$  and clearly also the supremum of  $D$  in  $L$ .  $\square$

Note that a well-ordered chain is bounded-complete, so the second part of the preceding proposition applies to the subsets of interest in this paper.

### 3 Meet\*-continuity

Meet-continuity on dcpos was introduced by Kou, Liu, and Luo in [13]. It was found that quasicontinuity, meet-continuity, and continuity itself have a close relationship with each other. In this section, we define a restricted notion, called meet\*-continuity, and show that it is equivalent to meet-continuity. By using this new notion, we are able to give a characterisation of meet-continuous dcpos by “forbidden substructures” and this will prove essential for our main result.

**Definition 3.1.** A dcpo  $L$  is *meet-continuous* if for any  $x \in L$  and any directed set  $D \subseteq L$  with  $x \leq \sup D$ ,  $x$  is in the Scott closure of  $\downarrow x \cap \downarrow D$ .

**Proposition 3.2.** [2, Proposition III-3.10] *A continuous dcpo is also quasicontinuous, and a meet-continuous quasicontinuous dcpo is already continuous.*

**Definition 3.3.** A dcpo  $L$  is *meet\*-continuous* if for any  $x \in L$  and any well-ordered chain  $C$  limit embedded in  $L$ ,  $x \leq \sup C$  implies that  $x$  is in the Scott closure of  $\downarrow x \cap \downarrow C$ .

Although seemingly weaker than meet-continuity, we will now show that meet\*-continuity is in fact sufficient to establish the former. To this end we recall Iwamura’s decomposition of directed sets, [7], as presented by Markowsky:

**Theorem 3.4.** [16, Theorem 1] *If  $D$  is an infinite directed set, then there exists a transfinite sequence  $D_\alpha$ ,  $\alpha < |D|$ , of directed subsets of  $D$  having the following properties:*

1. *for each  $\alpha$ , if  $\alpha$  is finite, so is  $D_\alpha$ , while if  $\alpha$  is infinite  $|D_\alpha| = |\alpha|$  (thus for all  $\alpha$ ,  $|D_\alpha| < |D|$ );*

2. if  $\alpha < \beta < |D|$ ,  $D_\alpha \subset D_\beta$ ;
3. if  $\beta < |D|$  is a limit ordinal, then  $D_\beta = \bigcup \{D_\alpha \mid \alpha < \beta\}$ <sup>2</sup>;
4.  $D = \bigcup \{D_\alpha \mid \alpha < |D|\}$ .

**Remark 3.5.** Parts (2) and (3) imply that the mapping  $|D| \rightarrow \mathbb{P}D$ ,  $\alpha \mapsto D_\alpha$  preserves existing suprema (where we consider the powerset  $\mathbb{P}D$  as a poset ordered by subset inclusion). Thus the assumptions of Proposition 2.7(2) are satisfied and we may conclude that the chain  $\{D_\alpha \mid \alpha < |D|\}$  is well-ordered and limit embedded in  $\mathbb{P}D$ .

**Theorem 3.6.** *A dcpo  $L$  is meet-continuous if and only if it is meet\*-continuous.*

*Proof.* It is trivial that meet-continuity implies meet\*-continuity. Conversely, if  $L$  is meet\*-continuous, we use transfinite induction on the cardinality of the directed set  $D$  in the definition of meet-continuity.

If  $D$  is finite, and  $x \leq \sup D$ , then  $D$  has a greatest element and the fact that  $x$  is in the Scott closure of  $\downarrow x \cap \downarrow D$  is obvious.

Now suppose  $D$  is infinite and that  $y$  is in the Scott closure of  $\downarrow y \cap \downarrow G$  for any  $y \in L$  and any directed set  $G$  with cardinality smaller than  $|D|$  and  $y \leq \sup G$ . By Theorem 3.4  $D$  is the union of a chain  $\mathcal{C} = (D_\alpha)_{\alpha < |D|}$  of directed subsets of  $D$ , each of which has smaller cardinality than  $D$ . The chain  $(\sup D_\alpha)_{\alpha < |D|}$  of elements of  $L$  is well-ordered because it is a monotone image of the cardinal  $|D|$ . It is also limit embedded because of Remark 3.5 above and the fact that the supremum operation (from the set of directed subsets of  $D$ , ordered by inclusion, to  $L$ ) is Scott-continuous. Now, if  $x \leq \sup D = \sup \{\sup D_\alpha \mid \alpha < |D|\}$ , then  $x$  is in the Scott closure of  $\downarrow x \cap \downarrow \{\sup D_\alpha \mid \alpha < |D|\}$  since  $L$  is meet\*-continuous. For every Scott open set  $U$ , if  $x \in U$ , then  $U \cap \downarrow x \cap \downarrow \{\sup D_\alpha \mid \alpha < |D|\} \neq \emptyset$ , which means that there exists  $y \in U$  such that  $y \leq x$  and  $y \leq \sup D_\alpha$  for some  $\alpha < |D|$ . By the induction hypothesis,  $y$  is in the Scott closure of  $\downarrow y \cap \downarrow D_\alpha$ , whence  $U \cap \downarrow y \cap \downarrow D_\alpha \neq \emptyset$  and therefore  $U \cap \downarrow x \cap \downarrow D \neq \emptyset$ , so  $x$  is indeed in the Scott closure of  $\downarrow x \cap \downarrow D$ .  $\square$

**Corollary 3.7.** *For a dcpo  $L$  which has binary infima, the following statements are equivalent:*

1.  $L$  is meet-continuous (in the sense of Definition 3.1);

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<sup>2</sup>This is not stated in [16, Theorem 1] but appears in the proof.

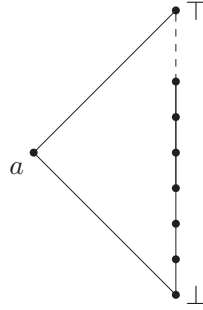


Figure 1:  $M(\mathbb{N})_\perp$

2. for every  $x \in L$  and every directed subset  $D$  of  $L$ ,  $x \wedge \sup\{d \mid d \in D\} = \sup\{x \wedge d \mid d \in D\}$ ;
3. for every  $x \in L$  and every well-ordered chain  $C$  limit embedded in  $L$ ,  $x \wedge \sup\{c \mid c \in C\} = \sup\{x \wedge c \mid c \in C\}$ .

*Proof.* The equivalence between (1) and (2) was shown in [13, Proposition 2.2]. The fact that (2) implies (3) is trivial. To prove (3) implies (1), from the preceding theorem one only needs to show that  $L$  is meet\*-continuous. So suppose  $x \in L$ ,  $C$  is a well-ordered chain and  $x \leq \sup C$ . From (3) one has  $\sup\{x \wedge c \mid c \in C\} = x \wedge \sup\{c \mid c \in C\} = x$  which shows that  $x$  is in the Scott closure of  $\{x \wedge c \mid c \in C\}$ . To conclude the proof it suffices to note that  $\{x \wedge c \mid c \in C\} \subseteq \downarrow x \cap \downarrow C$ .  $\square$

We give a general construction of non-continuous quasicontinuous dcpos of a special form:

**Definition 3.8.** For every well-ordered chain  $C$  without a top element, we define the poset  $M(C) = C \cup \{\top, a\}$ , where  $a$  and  $\top$  are not in  $C$  and the order on  $M(C)$  is:  $x \leq y$  iff  $x = y = a$  or  $y = \top$  or  $x, y \in C$ ,  $x \leq y$  in  $C$ . Define  $M(C)_\perp$  to be the lifting of  $M(C)$  by adding a least element  $\perp$ . Figure 1 shows  $M(\mathbb{N})_\perp$  (where  $\mathbb{N}$  is the ordered chain of natural numbers).

**Remark 3.9.** It is easy to show that  $M(\mathbb{N})_\perp$  is a QFS-domain in the sense of [5, 14, 15]. In [19] it was shown that the function space  $[M(\mathbb{N})_\perp \rightarrow M(\mathbb{N})_\perp]$  is not quasicontinuous, which implies that the category of QFS-domains is not cartesian closed. As we explained at the end of the Introduction, this result provided much of the inspiration for the current paper.

We come to our characterisation of meet-continuous dcpos:

**Theorem 3.10.** *Let  $L$  be a dcpo which is not meet-continuous. Then  $L$  has some  $M(C)$  or  $M(C)_\perp$  (as defined in 3.8) as a Scott-continuous retract, where  $C$  is a well-ordered chain without a top element.*

*Proof.* Let  $L$  be a dcpo which is not meet-continuous. By Theorem 3.6 this means that it is not meet\*-continuous either, and so there exist an element  $a$  and a well-ordered chain  $C'$  (limit embedded into  $L$ ) such that  $a < \sup C'$ , but  $a$  is not in  $\overline{\downarrow a \cap \downarrow C'}$ , the Scott closure of  $\downarrow a \cap \downarrow C'$ . Obviously, for every  $c \in C'$ ,  $a \not\leq c$  and therefore  $C'$  does not have a top element. Moreover, we can make every  $c \in C'$  incomparable to  $a$  by throwing away those elements of  $C'$  that are below  $a$ .

We now distinguish two cases:

Case 1,  $\downarrow a \cap \downarrow C' \neq \emptyset$ : Then there exist  $b \in L$  and  $c \in C'$  such that  $b \in \downarrow a \cap \downarrow c$ . Let  $C$  be the set  $C' \setminus \downarrow c$  and denote the set  $C \cup \{\sup C', a, b\}$  by  $M$  and order it by the induced order from  $L$ . Obviously,  $\sup C' = \sup C$  and  $M$  is isomorphic to  $M(C)_\perp$ . Define a function  $f$  from  $L$  to  $M$ :

$$f(x) = \begin{cases} b, & x \in \overline{\downarrow a \cap \downarrow C} \\ a, & x \in \downarrow a \setminus \overline{\downarrow a \cap \downarrow C} \\ \bigwedge \{c \mid x \leq c, c \in C\}, & x \in \downarrow C \setminus \downarrow a \\ \sup C, & x \notin \downarrow C \end{cases}$$

Since  $C$  is well-ordered,  $f$  is well-defined. We first prove that  $f$  is monotone, so let  $x, y \in L$  with  $x \leq y$ .

In case  $y \in \overline{\downarrow a \cap \downarrow C}$ ,  $x$  is in  $\overline{\downarrow a \cap \downarrow C}$  since  $\overline{\downarrow a \cap \downarrow C}$  is a lower set, and we see that  $f(x) = f(y) = b$ .

In case  $y \in \downarrow a \setminus \overline{\downarrow a \cap \downarrow C}$ , since  $x \leq y$ ,  $x$  must be in  $\downarrow a \setminus \overline{\downarrow a \cap \downarrow C}$  or in  $\overline{\downarrow a \cap \downarrow C}$ , and in both cases  $f(x) \leq f(y)$ .

In case  $y \notin \downarrow a$ , if  $\{c \mid y \leq c, c \in C\} = \emptyset$ , then  $f(y) = \sup C \geq f(x)$ . For  $\{c \mid y \leq c, c \in C\} \neq \emptyset$ , if  $x \in \downarrow a$ , then  $x \in \downarrow a \cap \downarrow y \subseteq \downarrow a \cap \downarrow C$ , and  $f(x) = b \leq f(y)$ . Otherwise  $x \notin \downarrow a$  and  $f(x) \leq f(y)$  follows immediately from the fact that  $\{c \mid y \leq c, c \in C\} \subseteq \{c \mid x \leq c, c \in C\}$ .

This covers all possible cases and we have established that  $f$  is monotone.

Now we show Scott continuity. To this end let  $D$  be a directed set in  $L$ .

In case  $f(\sup D) = b$ , for every  $x \in D$ ,  $b \leq f(x) \leq f(\sup D) = b$  since  $f$  is monotone, so  $f(\sup D) = \sup f(D) = b$ .

In case  $f(\sup D) = a$ , then  $\sup D \leq a$  and  $\sup D \notin \overline{\downarrow a \cap \downarrow C}$ . Since  $\downarrow a \cap \downarrow C$  is Scott closed and  $D$  is directed, there exists some  $x \in D$  such

that  $x \in \downarrow a \setminus \overline{\downarrow a \cap \downarrow C}$ , so  $a = f(x) \leq \sup f(D) \leq f(\sup D) = a$ .

In case  $f(\sup D) = \sup C$  we have that  $\sup D \not\leq a$  and for every  $c \in C$ ,  $\sup D \not\leq c$ . So given  $c \in C$  there exist  $x_1, x_2 \in D$  such that  $x_1 \not\leq a$  and  $x_2 \not\leq c$ , and by the directness of  $D$  there is some  $x \in D$  greater than  $x_1, x_2$ . For this element it holds that  $x \not\leq a$ ,  $x \not\leq c$ , and  $f(x) > c$ . So for every  $c \in C$  there is some  $x \in D$  such that  $f(x) > c$  which shows that  $\sup f(D)$  must equal  $\sup C$ .

The same deduction shows  $f(\sup D) = \sup f(D)$  for the case  $f(\sup D) \in C$ . Note though, that it is here where we need that the well-ordered chain  $C$  is limit embedded in  $L$ .

This covers all cases to be considered and we conclude that  $f$  is a Scott-continuous function from  $L$  to  $L$ . Inspecting the definition we see that the elements of  $M$  are fixed under  $f$ . Hence  $M$  is a Scott-continuous retract of  $L$ .

Case 2,  $\downarrow a \cap \downarrow C' = \emptyset$ : In this case let  $C = C'$  and  $N$  be the set  $C \cup \{\sup C, a\}$  with its order inherited from  $L$ . Obviously,  $N$  is isomorphic to  $M(C)$ . Define a function  $g$  from  $L$  to  $N$ :

$$g(x) = \begin{cases} a, & x \in \downarrow a \\ \bigwedge \{c \mid x \leq c, c \in C\}, & x \in \downarrow C \setminus \downarrow a \\ \sup C, & x \notin \downarrow C \end{cases}$$

The same deduction as in (1) shows that  $g$  is a Scott-continuous idempotent function on  $L$  with image  $N$ .  $\square$

## 4 Function spaces

The following is a generalisation of a result in [19].

**Proposition 4.1.** *Let  $C$  be an infinite well-ordered chain without a top element. Then neither the function space  $[M(C)_\perp \rightarrow M(C)_\perp]$  nor  $[M(C) \rightarrow M(C)]$  is quasicontinuous.*

*Proof.* We begin with  $D_\perp := [M(C)_\perp \rightarrow M(C)_\perp]$  and assume for the sake of a contradiction that it is quasicontinuous. Consider the function  $a \Rightarrow \perp$  that maps the element  $a$  to  $\perp$  and keeps everything else fixed. It is clearly strictly less than the identity on  $M(C)_\perp$ . By quasicontinuity this implies that we should have a finite subset  $F$  of  $D_\perp$  such that  $F \ll \text{id}_{M(C)_\perp}$  and  $a \Rightarrow \perp \notin \uparrow F$ . Consider  $F' := \{f \in F \mid f \leq \text{id}_{M(C)_\perp}\}$ . Clearly,  $F'$  is not empty (since  $\text{id}_{M(C)_\perp} \in \uparrow F$ ), and for each  $f \in F'$  we must have  $f(a) = a$

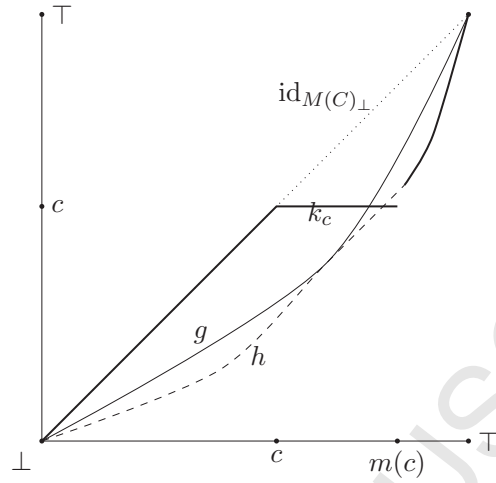


Figure 2: The situation in the proof of Proposition 4.1

as otherwise we would have  $f \leq a \Rightarrow \perp$ . Now  $\top$  can only be mapped to  $a$  or to itself by such an  $f$ . In the former case, some  $c \in C$  would also have to be mapped to  $a$  to ensure continuity but this would violate the condition  $f \leq \text{id}_{M(C)_\perp}$ ; so  $f(\top) = \top$  is the only possibility that remains. In other words, each such  $f$  continuously maps the infinite well-ordered chain  $C \cup \{\top\}$  into itself, keeping both  $\perp$  and  $\top$  fixed. We now show that such functions do not “isolate”  $\text{id}_{M(C)_\perp}$  against directed suprema from below. (For the argument that follows it may be useful to keep Figure 2 in mind.)

Consider the function  $g: M(C)_\perp \rightarrow M(C)_\perp$  defined by  $g(x) = \min\{f(x) \mid f \in F'\}$ . It is Scott-continuous because the chain  $C \cup \{\top\}$  is an algebraic lattice as we argued in Proposition 2.6, and hence meet-continuous. Furthermore, consider  $h: M(C)_\perp \rightarrow M(C)_\perp$  defined on  $a$  and the compact elements of  $C \cup \{\top\}$  by

$$h(x) = \begin{cases} a, & x = a \\ \perp, & x = \perp \\ g(c), & x = c + 1 \end{cases}$$

It follows that  $g$  and  $h$  agree for limit ordinals, but there are also many inputs where  $h$  is strictly less than  $g$ ; more precisely, for any  $e \in C$ , there exists a  $d \in C, d \geq e$  such that  $h(d+1) < g(d+1)$ . Indeed, suppose there exists some  $e \in C$  such that  $h(d+1) = g(d+1)$  for all  $d \geq e$ . Because

$h(d+1) = g(d)$ , it then follows that  $g(d) = g(d+1)$  when  $d \geq e$ . Using transfinite induction and the fact that  $g$  is Scott-continuous, we get that  $g(x) = g(y)$  for all  $x, y \geq e$ . In particular, we obtain  $g(e) = g(\top) = \top$ . However,  $g$  is below  $\text{id}_{M(C)_\perp}$  and this implies  $\top = g(e) \leq e$ , which is not possible since  $C$  does not have a top element.

From  $h(\top) = \top$  and Scott-continuity we get that for any  $c \in C$ , there exists  $m > c$  such that  $h(m) > c$ . Define  $m(c)$  be the least element of  $\{m \in C \mid h(m) > c\}$ . We use this to define a family  $\mathcal{K}$  of functions  $k_c: M(C)_\perp \rightarrow M(C)_\perp$  indexed by the elements of  $C$  and defined by

$$k_c(x) = \begin{cases} x, & x \leq c \\ c, & c < x \leq m(c) \\ h(x), & \text{otherwise} \end{cases}$$

It is clear that each  $k_c$  is Scott-continuous as it is pieced together from Scott-continuous functions on Scott-closed subsets. It is also clear that the supremum of  $\mathcal{K}$  is the identity on  $M(C)_\perp$ , but unfortunately,  $\mathcal{K}$  may not be directed.<sup>3</sup> This is only a small hindrance, however, because  $M(C)_\perp$  is a complete lattice and we can enrich  $\mathcal{K}$  with all finite suprema. This, then, yields a directed set with supremum  $\text{id}_{M(C)_\perp}$  no member of which is above  $g$  and therefore not above an element of  $F'$ . Since all of this takes place in  $\downarrow \text{id}_{M(C)_\perp}$ , none of them exceeds any of the other members of  $F$  either. Thus we have given a counterexample to the claim that  $F \ll \text{id}_{M(C)_\perp}$  and this contradiction shows that the assumption that the function space  $D_\perp$  is quasicontinuous must have been wrong.

The argument for  $D := [M(C) \rightarrow M(C)]$  is similar but easier because any order-preserving function below  $\text{id}_{M(C)}$  must map  $a$  to  $a$  and  $\top$  to  $\top$ .  $\square$

**Theorem 4.2.** *Let  $L$  be a dcpo which is not meet-continuous. Then the function space  $[L \rightarrow L]$  is not quasicontinuous.*

*Proof.* By Theorem 3.10 we know that  $L$  must have some  $M(C)$  or  $M(C)_\perp$  as a Scott-continuous retract. So either the function space  $[M(C) \rightarrow M(C)]$  or  $[M(C)_\perp \rightarrow M(C)_\perp]$  is a retract of  $[L \rightarrow L]$  (see for example [10, Proposition 1.22]), both of which we know from the preceding proposition not to be quasicontinuous. Proposition 2.5 now tells us that  $[L \rightarrow L]$  itself is not quasicontinuous.  $\square$

<sup>3</sup>One can avoid this problem by changing the second clause in the definition of  $k_c$  to  $k_c(x) = c$  if  $c < x < m(c)$ , but one then has to argue that the resulting function is Scott-continuous.

We come to the main result of our paper:

**Theorem 4.3.** *Let  $\mathbf{qCONT}$  be the category of quasicontinuous domains with Scott-continuous functions as morphisms and  $\mathbf{C}$  a cartesian closed full subcategory of  $\mathbf{qCONT}$ . Then every object in  $\mathbf{C}$  is continuous.*

*Proof.* Assume  $L$  is a quasicontinuous domain in  $\mathbf{C}$  which is not continuous. By Proposition 3.2  $L$  is not meet-continuous, so by the preceding theorem its function space is not quasicontinuous, so can't be an object of  $\mathbf{C}$ . However, it was shown in [18] that the function space is the exponential object in any cartesian closed full subcategory of  $\mathbf{DCPO}$ . This contradiction shows that  $L$  must be continuous.  $\square$

The maximal cartesian closed full subcategories of the category  $\mathbf{CONT}$  of continuous domains were fully classified by the second author in [10, 11]. In the pointed case, they consist of continuous L-domains or FS-domains. If no bottom element is present, then the objects are either disjoint unions of their pointed counterpart or finite gluings. In the countably based case, only (finite gluings of) FS-domains remain. The preceding theorem can be read as saying that these are also the maximal cartesian closed full subcategories of  $\mathbf{qCONT}$ .

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