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DOI:
10.1007/s10463-015-0547-4

Annals of the Institute of Statistical Mathematics manuscript No.
(will be inserted by the editor)

Measuring asymmetry and testing symmetry

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Received: date / Revised: date

Abstract In this paper we show that some of the most commonly used tests of symmetry do not have power which is reflective of the size of asymmetry. This is because the primary rationale for the test statistics that are proposed in the literature to test for symmetry is to detect the departure from symmetry, rather than the quantification of the asymmetry. As a result, tests of symmetry based upon these statistics do not necessarily generate power that is representative of the departure from the null hypothesis of symmetry. Recent research has produced new measures of asymmetry, which have been shown to do an admirable job of quantifying the amount of asymmetry. We propose several new tests based upon one such measure. We derive the asymptotic distribution of the test statistics and analyse the performance of these proposed tests through the use of a simulation study.

Keywords Symmetry · Asymmetry · Measure of asymmetry · Testing symmetry · Skewness

1 Introduction

The concept of symmetric random variables is important for the development and application of statistical theory. In particular, symmetry is an important assumption for many statistical models. For example, symmetry assumptions are essential in deriving many point or interval estimates of location parameters. In non-parametric statistics such as the Wilcoxon signed-rank test, proposed by Wilcoxon (1945) to test for differences between two samples with unknown distribution functions, the most crucial assumption is that the samples are from symmetric populations. Since very often the symmetry assumption does not hold in practice and the Wilcoxon signed-rank test is not robust against the assumption of symmetry, it is essential to check the assumption of symmetry before employing the Wilcoxon signed-rank procedure. The non-robustness of the Wilcoxon signed-rank procedure stems from the fact that the distribution of its test statistic is heavily dependent on the symmetry. To elaborate further, the distribution of the test statistic based on a sample from a population with a small departure from symmetry in the right-end is stochastically significantly larger than the Wilcoxon signed-rank test statistic based on a sample from a symmetric population. This means that the actual size of the Wilcoxon signed-rank test is very different

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The online version of this article contains supplementary material.

from the advertised size and the values of size and power that one obtains using the standard Wilcoxon table are simply meaningless. For details, see Kasuya (2010) and Voraprateep (2013).

Furthermore, as is the case with the Wilcoxon signed-rank test, a wide range of statistical techniques rely on the assumption of symmetry or somewhat indirectly on symmetry through the assumption of normality. For example, linear regression models assume that residuals are normally distributed, and assessing the symmetry of the residual distribution is an important precursor in assessing the normality of the residuals. Consequently, there are a wealth of options for testing the hypothesis of symmetry.

However, in this paper we show that some of the most commonly used tests of symmetry do not have power which is reflective of the size of asymmetry. This is because the primary rationale for the test statistics that are proposed in the literature to test for symmetry is to detect the departure from symmetry, rather than the quantification of the asymmetry. For example, a common procedure for testing for symmetry relies on using measures of skewness. Whilst these measures are equal to zero for symmetric random variables and non-zero for asymmetric random variables, these measures of skewness do not measure the underlying asymmetry. In section 2 we demonstrate this undesirable feature for a number of commonly used existing tests for symmetry using a combination of theoretical examples and a simulation study. In section 3 we introduce a recently proposed measure of asymmetry, which has been shown to do an admirable job of quantifying the amount of asymmetry. Using this new measure we construct several new tests and discuss the asymptotic properties of the new test statistics. We compare the power of the new tests with the existing tests using a simulation study. In particular we show that the new tests display an improvement in power and, moreover, have power which is more reflective of the size of asymmetry. In section 4 we discuss the advantages and limitations of the proposed tests.

2 Testing symmetry

2.1 Ordering distributions based on asymmetry

Consider samples taken from Normal, Cauchy, Normal mixtures, Log-Normal, Folded Normal, and Exponential populations. Figure 1 shows the density functions of these random variables. The Normal mixtures in Figure 1 are constructed using

$$pN(0,1) + (1-p)N(2,2),$$

for $p = 0.945, 0.872, 0.773$ and 0.606.

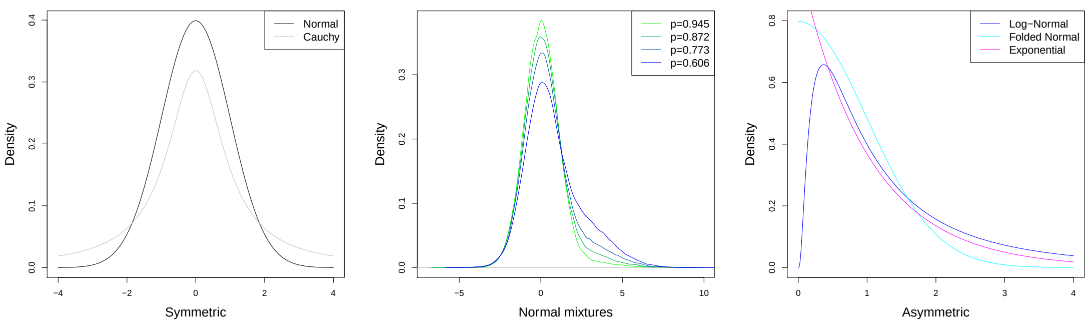

Fig. 1: The left figure shows the symmetric Normal and Cauchy density curves. The middle figure shows the density curves of Normal mixtures of the form $pN(0,1)+(1-p)N(2,2)$, for $0 < p < 1$. The rightmost figure shows the three 'highly' asymmetric densities which, in order of increasing asymmetry, are Log-Normal, Folded Normal and Exponential.

It is clear from the plot on the left of Figure 1 that the Normal and Cauchy densities are symmetric about zero, whilst the other density functions are clearly asymmetric. However, we are entitled to ask 'Which of these asymmetric densities, is the *most* asymmetric?' In this case it is possible to obtain a visual impression of the size of asymmetry present in the random variables. For example, consider the middle plot of Figure 1, which shows four Normal mixture densities. As p decreases the $N(2, 2)$ population has more of an effect on the mixture density and the curve becomes more asymmetric to the right. Thus, it is clear that as p decreases from near to 1 closer to 0.5, the resultant density becomes more asymmetric. The rightmost plot of Figure 1 exhibits several more extreme cases. For example, the Log-Normal density has a substantial proportion of its probability mass concentrated to the left and as a result, it is reasonable to say that it is even more asymmetric than the Normal mixtures. Further, the Folded Normal and the Exponential density represent an even more extreme example of asymmetry as they have no left tail whatsoever. Observe that the Folded Normal curve has a 'more even spread' of probability mass compared to the Exponential curve, hence one can reason that a Folded Normal random variable is not as asymmetric as an Exponential random variable.

Thus, for the random variables given above we can arrive at the following ordering of asymmetry, based on visual interpretation:

$$\text{Normal} <_a \text{Normal mixtures} <_a \text{Log-Normal} <_a \text{Folded Normal} <_a \text{Exponential},$$

where the binary operator $<_a$ represents the sentence "...appears to be less asymmetric than...".

This visual ordering is supported by the work of Patil et al. (2012) and Patil et al. (2014). An 'ideal' test statistic would have power which reflects this increasing departure from symmetry. In fact, it can be shown that many of the existing tests of symmetry do not exhibit this desirable property.

2.2 An oversight of some existing tests

2.2.1 Theoretical evidence

There are several tests in the literature to assess the symmetry of an unknown density $f(x)$ based on a random sample, see for example Hollander (2004) and references therein. However, these tests do not help to compare or quantify the asymmetry of the probability density function. For example, Butler (1969) propose a test of symmetry based on the sample version of

$$\eta_1(F) = \sup_{x \leq 0} |F(\theta + x) + F(\theta - x) - 1|,$$

where θ is the median. Alternatively, again with θ being the median, Boos (1982) proposes a test for symmetry using the sample version of

$$\eta_2(F) = \int_{\mathbb{R}} [F(\theta + x) + F(\theta - x) - 1]^2 \, dx,$$

and Rothman and Woodroofe (1972) propose using the sample version of

$$\eta_3(F) = \int_{\mathbb{R}} [F(\theta + x) + F(\theta - x) - 1]^2 \, dF(x).$$

However, with F_{FN} and F_{LN} respectively denoting the distribution functions of the Folded Normal and Log-Normal distribution, it is readily calculated that

$$\eta_1(F_{\mathrm{FN}}) = \operatorname{erf}\left(\frac{2\theta}{\sqrt{2}}\right) - 1 = \operatorname{erf}\left(2 \cdot \operatorname{erf}^{-1}(0.5)\right) - 1 \approx 0.177344,$$

for the Folded Normal distribution and $\eta_1(F_{\mathrm{LN}}) \approx 0.251508$ for the Log-Normal distribution, indicating that the Folded Normal density is less asymmetric than the Log-Normal density, which contradicts our earlier visual inspection.

To appraise η_2 and η_3 consider the following simple probability density function,

$$f_\epsilon(x) = \begin{cases} \frac{1}{2} + \epsilon \text{ if } -1 < x < 0 \\ \frac{1}{2} - \epsilon \text{ if } \quad 0 < x < 1 \\ 0 \qquad\quad \text{otherwise,} \end{cases}$$

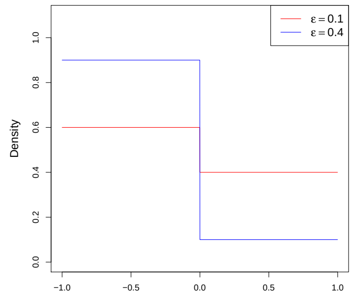

Fig. 2: The density functions, $f_{0.1}$ and $f_{0.4}$.

where $0 \leq \epsilon \leq \frac{1}{2}$ and let F_ϵ denote the corresponding distribution function. Figure 2 shows the density function f_ϵ for $\epsilon = 0.1$ and $\epsilon = 0.4$.

Observe that purely as a function (i.e. not as a 'probability density' function) the visual impression of $f_\epsilon(x)$ for all x where $f_\epsilon(x) > 0$ is that it looks and becomes a more and more symmetric function as ϵ approaches to zero and is exactly symmetric at $\epsilon = 0$. However, as ϵ increases towards 1/2, f_ϵ looks and becomes more and more asymmetric and is exactly symmetric at $\epsilon = 1/2$. That is, if there is a measure to quantify the asymmetry of f_ϵ as a function of ϵ say, $\eta^*(\epsilon)$, then we expect $\eta^*(\epsilon)$ to be a monotonically increasing continuous function of ϵ for $0 < \epsilon < 1/2$, $\eta^*(0) = \eta^*(1/2)$, discontinuous at 1/2 and continuous at 0.

If viewed as a probability density function though, the concept of asymmetry changes. Let θ be the median (of f_ϵ) and write the probability density function as $f_\epsilon(\theta + |u|)$. Then in the strict sense of the definition of symmetry, as u goes away from zero in either direction one expects every pair of intervals from 0 to u on either side to have equal probability content. If this is true for every u, the density function f_ϵ is symmetric. If this is true for every $u \in (-M, M)$ for small M, f_ϵ is more asymmetric and for large M it is less asymmetric. Thus, if there is a measure to quantify the asymmetry of the probability density function f_ϵ as a function of ϵ say, $\eta(\epsilon)$, then we expect $\eta(\epsilon)$ to be a monotonically decreasing continuous function of ϵ for $0 < \epsilon < 1/2$, $\eta(0) = \eta(1/2)$, discontinuous at 0 and continuous at 1/2.

For example, define

$$\eta(\epsilon) = P[X \in S_\epsilon],$$

where $S_\epsilon = \{x | f_\epsilon(\theta - x) \neq f_\epsilon(\theta + x)\}$. It is clear that $\eta(0) = \eta(1/2) = 0$, however, when ϵ is close to zero $\eta(\epsilon)$ is large. Indeed, if we consider a very small value for $\epsilon > 0$, the median $\theta = -\frac{2\epsilon}{1+2\epsilon}$ and it approaches zero from the left as ϵ approaches zero from the right. Hence, there is only a small interval about θ where the equality $f_\epsilon(\theta+x) = f_\epsilon(\theta-x)$ holds. This interval is given by $[2\theta, 0] \equiv \left[-\frac{4\epsilon}{1+2\epsilon}, 0\right]$ which clearly shrinks as ϵ approaches zero. Hence, as ϵ approaches 0, the set of values x such that $f_\epsilon(\theta+x) \neq f_\epsilon(\theta-x)$ consists of the entire support, with the exception of an increasingly small interval about the median θ. Therefore, f_0 is symmetric, but when ϵ is close to zero f_ϵ is very asymmetric and $\eta(\epsilon)$ is rightly discontinuous at 0.

Thus, to be appropriate measures of asymmetry, when $\eta_2(F)$ and $\eta_3(F)$ are applied to the probability density function f_ϵ one expect $\eta_i(F_\epsilon)$, $i = 2, 3$, to be monotonically decreasing continuous functions of ϵ for $0 < \epsilon < 1/2$, $\eta_i(F_0) = \eta_i\left(F_{1/2}\right)$, discontinuous at $\epsilon = 0$ and continuous at $\epsilon = 1/2$. It is readily calculated that

$$\eta_2(F_\epsilon) = \frac{32\epsilon^4 - 32\epsilon^3 + 8\epsilon^2}{12\epsilon^2 + 12\epsilon + 3},$$

and

$$\eta_3(F_\epsilon) = -\frac{16\epsilon^2 - 8\epsilon}{2\epsilon + 1}.$$

Clearly $\eta_2(F_0) = \eta_3(F_0) = 0$ and $\eta_2\left(F_{1/2}\right) = \eta_3\left(F_{1/2}\right) = 0$ as one would expect. However, it is clear that $\eta_2(F_\epsilon)$ and $\eta_3(F_\epsilon)$ are continuous at zero and, as a result, the power of the test based on the sample version of s_2 and s_3 will not reflect the magnitude of asymmetry. That is, the tests proposed by Boos

(1982) and Rothman and Woodroofe (1972) will both fail to have power which increases with the amount of asymmetry.

It is important to note that the measure of asymmetry $\eta(\epsilon)$ introduced above is only applicable to random variables which take two values, such as $X \sim f_\epsilon$ considered here. This simple measure is sufficient to demonstrate the subtle nature of asymmetry, however, in section 3 we revisit this idea and propose a more general measure asymmetry.

2.2.2 Other tests of symmetry

There are many other methods for testing symmetry and it is not possible to repeat the above argument for all such tests. As a result, we shall analyse the power of several other tests using a simulation study. Consider a random sample $X_1, \cdots, X_n$ identically drawn from a probability distribution. Then, Cabilio and Masaro (1996) propose a test based on sample skewness,

$$S_1 = \sqrt{n}\,\frac{\bar{x} - \tilde{\theta}}{s},$$

where $\bar{x}$ and $\tilde{\theta}$ are the sample mean and sample median respectively and s is the sample standard deviation. The simple rationale behind this statistic is the necessary condition that for a symmetric continuous population the mean is equal to the median. Thus, significantly large values of $|S_1|$ are indicative of departure from symmetry. As mentioned previously, the detection of departure from symmetry is the main focus of S_1 and and not the quantification of asymmetry.

Another test is suggested by Antille et al. (1982), who define the following test statistic based on ranks,

$$\mathcal{R}(\alpha) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} G_\alpha \left(\frac{R\left(|X_i - \tilde{\theta}|\right)}{2(n+1)} \right) \operatorname{sign}\left(X_i - \tilde{\theta}\right),$$

where $G_\alpha(x) = \min(x, \frac{1}{2} - \alpha)$ and $R(X_i)$ is defined as the rank of X_i among the X_is. Antille et al. (1982) propose a test based on $\mathcal{R}(\alpha)$, and determine the asymptotic properties of the test statistic. For simplicity we only consider $\alpha = 0$ and denote $S_2 = \mathcal{R}(0)$. Under the null hypothesis of symmetry S_2 is very close to zero and, hence, one rejects the null for large values of $|S_2|$.

Alternatively, Randles et al. (1980) define the following 'triples' test,

$$S_3 = \frac{1}{3}\binom{N}{3}^{-1} \sum_{i<j<k} \left[\operatorname{sign}(X_i + X_j - 2X_k) + \operatorname{sign}(X_i + X_k - 2X_j) + \operatorname{sign}(X_j + X_k - 2X_i)\right],$$

where $\operatorname{sign}(u) = -1, 0, \text{or } 1$ for $u <, =, \text{or } > 0$. A triple of observations (X_i, X_j, X_k) is defined as a right triple if the middle observation is closer to the smallest observation than it is to the largest observation, and vice-versa for a left triple. Thus, S_3 is a constant multiple of the difference between the proportion of right and left triples. As a result, $\mathrm{E}[S_3] = 0$ when the underlying distribution is symmetric. Suggesting that the class of asymmetric probability models for which $\mathrm{E}[S_3] = 0$ is small, Randles et al. (1980) use S_3 for testing symmetry. It is also worthy of note that the theoretical analogue of S_3, $\mathrm{E}[S_3]$, fails to measure the asymmetry of f_ϵ introduced in section 2.2.1. Indeed, one can show that as a function of ϵ it is continuous at $\epsilon = 1/2$ as required. However, it is also continuous at $\epsilon = 0$ and thus fails to quantify the asymmetry in f_ϵ when ϵ is is relatively close to zero.

Gupta (1967) details the classical test of skewness based on

$$S_4 = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^3}{\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right)^{\frac{3}{2}}}.$$

As with S_1, the rationale behind this test statistic is that a symmetric population has zero skewness. Thus, significantly large values of $|S_4|$ are indicative of departure from symmetry. Indeed, there are other tests for symmetry based on measures of skewness, as detailed by Ngatchou-Wandji (2006). However, one should be mindful that, as noted by Li and Morris (1991), measures of skewness do not correctly indicate the degree of asymmetry in a probability density function.

Finally, we consider the test proposed by McWilliams (1990), based on a runs statistic. To define the test statistic, let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ denote the sample values ordered from smallest to largest according to their absolute value, but retaining their sign, and let Δ_i indicate the sign of $X_{(i)}$, $i = 1, 2, \cdots, n$, by way of defining $\Delta_i = 1$ when $X_{(i)} > 0$ and zero otherwise. Then define

$$S_5 = 1 + I_2 + I_3 + \cdots + I_n,$$

where

$$I_k = \begin{cases} 0 \text{ if } \Delta_k = \Delta_{k-1} \\ 1 \text{ if } \Delta_k \neq \Delta_{k-1} \end{cases}, \quad k = 2, \cdots, n$$

which counts the number of runs in the sequence $\{\Delta_i\}$. Under the null hypothesis of symmetry, $S_5 - 1$ has a binomial distribution with parameters $n - 1$ and $\frac{1}{2}$. In this case, one rejects the null hypothesis if S_5 falls in the lower tail of the null distribution.

As we have mentioned, the tests discussed in this section share a common characteristic. Namely, that the rationale behind the test statistics is to detect departure from symmetry as opposed to the quantification of asymmetry. Ley and Paindaveine (2009), Cassart et al. (2008) and Cassart et al. (2011) propose tests that are optimal for a specific class of alternative distributions and, for these tests, the test statistics do quantify the asymmetry provided that the data are distributed according to the specified alternative. However, there is no guarantee that these test statistics quantify asymmetry in general.

2.3 Optimal tests

The tests proposed by Ley and Paindaveine (2009), Cassart et al. (2008) and Cassart et al. (2011) behave as one should expect for their specified alternatives. That is, the power of these tests increases as the asymmetry in the specified class of alternative probability probability density functions increases. However, as one expects there is no guarantee that these tests will have power which increases with the size of asymmetry for probability density functions outside the prescribed class of alternatives and, more importantly, the class of functions for which the tests are optimal is too restrictive.

For example, Cassart et al. (2008) propose a test which is locally and asymptotically optimal for Fechner-type asymmetry. Here for symmetric f_1 the class of asymmetric alternatives is of the form

$$f_{\theta,\sigma,\xi}(x) := \frac{1}{\sigma}\left[f_1\left(\frac{x-\theta}{(1+\xi)\sigma}\right)\mathrm{I}\left[x \leq \theta\right] + f_1\left(\frac{x-\theta}{(1-\xi)\sigma}\right)\mathrm{I}\left[x > \theta\right]\right], \tag{1}$$

where θ plays the role of a location parameter, σ is a scale parameter and $\xi \in (-1, 1)$ is a skewness parameter which quantifies the size of asymmetry. This class of two-piece distributions includes the Fernandez and Steel (1998) distribution. Indeed, setting $\sigma = \frac{1}{2}\left(\gamma + \frac{1}{\gamma}\right)$ and $\xi = \frac{\frac{1}{\gamma} - \gamma}{\gamma + \frac{1}{\gamma}}$ we obtain the Fernandez and Steel density function. Therefore, the test for symmetry in this case is to test $H_0 : \xi = 0$ against, for example, $H_1 : \xi > 0$. For this class of alternatives, the test statistic for the optimal test is

$$O_1 = \frac{\sum_i (X_i - \theta)\left(2m_1^* - |X_i - \theta|\right)}{\sqrt{n\left(m_4^{(n)} - 4m_1^{*(n)}m_3^{*(n)} + 4\left(m_1^{*(n)}\right)^2 m_2^{(n)}\right)}},$$

where

$$m_k^{*(n)} = \frac{1}{n}\sum_{i=1}^{n} |X_i - \theta|^k,$$

and

$$m_k^{(n)} = \frac{1}{n}\sum_{i=1}^{n} (X_i - \theta)^k.$$

Similarly, Cassart et al. (2011) propose optimal tests for a slightly modified class of asymmetric probability density functions

$$f(x) = \sigma^{-1} f_1(x) - \xi g_1(x) \left(x^2 - \kappa(f_1)\right) \mathrm{I}[|x| \leq |z^*|]$$
$$- \operatorname{sign}(\xi) f_1(x) \left\{ \mathrm{I}\left[|x| > \operatorname{sign}(-\xi)|z^*|\right] - \mathrm{I}\left[|x| \leq \operatorname{sign}(\xi)|z^*|\right] \right\},$$

where as before $\xi \in \mathbb{R}$ is a skewness parameter, κ is a generalized kurtosis coefficient ($\kappa = 3$ for $f_1 = \phi$) and z^* is the solution to

$$f_1(z^*) = \xi g_1(z^*)((z^*)^2 - \kappa),$$

where $g_1(x)$ satisfies

$$f_1(z_1) - f_1(z_2) = \int_{z_1}^{z_2} g_1(z) dz$$

and f_1 is a symmetric density function.

For this class of alternatives, the test statistic for the optimal test is

$$O_2 = \frac{1}{\sqrt{n\gamma^{(n)}}} \sum_{i=1}^{n} (X_i - \theta) \left((X_i - \theta)^2 - 3m_2^{(n)}\right),$$

where $\gamma^{(n)} = m_6^{(n)} - 6m_4^{(n)} + 9\left(m_2^{(n)}\right)^3$. The test based on O_2 is asymptotically equivalent to the classical test of symmetry S_4.

Ley and Paindaveine (2009) propose optimal tests for symmetry based on the general skewing mechanism proposed by Ferreira and Steel (2006),

$$f^L(x) = l(F(x))f(x),$$

where L is a distribution function over $[0, 1]$ and l is its respective probability density function. This is a very general class of functions and includes the Skew Normal random variables considered in the simulation study by using $l(x|\lambda) = 2F(\lambda F^{-1}(x))$, where $\lambda \in \mathbb{R}$.

The test statistic for the Skew Normal alternatives is given by

$$O_3 = \frac{\sqrt{\frac{2}{n\pi}} \sum_i S_i \varPhi^{-1}\left(\frac{1}{2}\left(1 + \frac{Ri}{n+1}\right)\right)}{\left(\varGamma_{22}\right)^{1/2}},$$

where $S_i = \operatorname{sign}(X_i)$, and R_i denotes the rank of $|X_i|$ among $|X_1|, \cdots, |X_n|$. Further, $\varPhi$ is the distribution function of the standard Normal distribution and

$$\varGamma_{22} = \int_0^1 \frac{2}{\pi}\left(\varPhi^{-1}(u)\right)^2.$$

Next, we conduct a simulation study to investigate the power behaviour of all of the tests discussed here in relation to the amount of asymmetry in the underlying distribution.

2.4 Simulation study

We now approximate the power (i.e. calculate the empirical power) of the tests proposed by Cabilio and Masaro (1996), Antille et al. (1982), Randles et al. (1980), Gupta (1967) and McWilliams (1990), as well as the optimal tests proposed by Ley and Paindaveine (2009), Cassart et al. (2008) and Cassart et al. (2011), for a range of different distributions. In particular, in addition to the probability distributions of section 2.1, we consider several other classes of asymmetric distributions, namely, the Skew Normal distribution proposed by Azzalini (1985); the Sinh-arcsinh distribution proposed by Jones and Pewsey (2009); and the skewed distribution introduced by Fernandez and Steel (1998). The Skew Normal distribution with parameter λ has density function

$$SN(z; \lambda) = 2\phi(z)\varPhi(\lambda z), \quad -\infty < z < \infty,$$

where ϕ and Φ are the standard Normal density and distribution functions respectively. When $\lambda = 0$ this reduces to the symmetric standard Normal distribution. When $\lambda > 0$ the distribution is skewed to the right and when $\lambda < 0$ the distribution is skewed to the left.

The Sinh-arcsinh distribution has density function

$$SAS(z;\epsilon,\delta) = \frac{1}{\sqrt{2\pi}} \frac{\delta C_{\epsilon,\delta}(z)}{\sqrt{1+z^2}} \exp\left\{-\frac{1}{2} S^2_{\epsilon,\delta}(z)\right\},$$

where

$$C_{\epsilon,\delta}(x) = \cosh\left[\epsilon + \delta \sinh^{-1}(x)\right],$$

and

$$S_{\epsilon,\delta}(x) = \sinh\left[\epsilon + \delta \sinh^{-1}(x)\right].$$

Here $\epsilon \in \mathbb{R}$ plays the role of a skewness parameter, while $\delta > 0$ controls the weight of the tails.

The skewed Fernandez and Steel distribution has density function

$$FAS(z;\gamma) = \frac{2}{\gamma + \frac{1}{\gamma}} \left\{ f\left(\frac{z}{\gamma}\right) \mathrm{I}[z \geq 0] + f(\gamma z)\, \mathrm{I}[z < 0] \right\},$$

for some $\gamma \in (0, \infty)$. This distribution will be symmetric when $\gamma = 1$ and is asymmetric whenever $\gamma \neq 1$.

In particular we consider the Skew Normal distributions with $\lambda = 1.214$, 1.795, 2.429, 3.221, 4.310, 5.970, 8.890, 15.570 respectively; the Sinh-arcsinh distribution with $\delta = 1$ and $\epsilon = 0.1$, 0.203, 0.311, 0.430, 0.565, 0.727, 0.939, 1.263; and the Fernandez and Steel distribution where f is the probability density function of the standard Normal distribution and $\gamma = 1.111$, 1.238, 1.385, 1.564, 1.791, 2.098, 2.557, 3.388.

We simulate samples of varying sizes ($n = 30, 50$ and 70) from each of the probability models. We simulate each sample $10,000$ times and calculate the test statistics each time to obtain a large sample from the sampling distributions of the test statistics. The null hypothesis of symmetry is accepted or rejected at the level $\alpha = 0.05$, based on the value of these statistics. The critical value, at which to reject symmetry, is determined from the asymptotic distribution of the sample statistics and then finally, the empirical powers (the proportion of rejections) of each of the tests are reported. We present the empirical powers of: the test based on sample skewness S_1 proposed by Cabilio and Masaro (1996); the test based on ranks S_2 suggested by Antille et al. (1982); the triples test S_3 proposed by Randles et al. (1980); the classical test of skewness S_4 presented by Gupta (1967); and runs test S_5 proposed by McWilliams (1990); as well as the optimal tests O_1, O_2 and O_3 proposed by Ley and Paindaveine (2009), Cassart et al. (2008) and Cassart et al. (2011), respectively.

Let NM1, NM2, NM3, NM4 denote the Normal mixtures with $p = 0.945$, 0.872, 0.773, 0.606 respectively, and let SN1-SN8, denote the Skew Normal distribution with $\lambda = 1.2135$, 1.795, 2.429, 3.221, 4.310, 5.970, 8.890, 15.570 respectively. Let SAS1-SAS8 denote the Sinh-arcsinh distribution with $\delta = 1$ and $\epsilon = 0.1$, 0.203, 0.311, 0.430, 0.565, 0.727, 0.939, 1.263 respectively. Let FAS1-FAS8 denote the Fernandez and Steel distribution with $\gamma = 1.111$, 1.238, 1.385, 1.564, 1.791, 2.098, 2.557, 3.388 respectively. The empirical powers are shown in Table 1. The table also includes a column entitled η, which is a measure of the asymmetry in the distribution and is formally defined in section 3.

We do not simulate from the Cauchy distribution for S_1 as this test requires the mean of the underlying distribution to exist. Observe that for S_1 the test has nominal empirical level for the symmetric Normal distribution, in keeping with the set level of 0.05. For $n = 30$ the empirical power is 0.036 rising to 0.04 for $n = 70$. As expected, the power steadily increases for the asymmetric families, however, the amount of power is not related to the amount of asymmetry as determined by our previous 'visual inspection'. For example, for $n = 30$ the test S_1 has power equal to 0.282 and 0.709 for the Folded Normal and Log-Normal distributions respectively. Although, S_1, identifies asymmetry in Log-Normal with a very good power of 0.709, it does a very poor job of identifying asymmetry in Folded Normal distribution, which is perceived to be more asymmetric than Log-Normal, with the power of 0.282 only.

The performance of the test S_2 is very similar to S_1. For the symmetric distributions the empirical level suggests the test is conservative, while the proportion of rejections slowly increases for the asymmetric families of distributions. However, when one considers the extremely asymmetric distributions, although the test has good rejection levels, its power does not reflect the size of asymmetry. For example, for

$n = 70$, although the Log-Normal density is less asymmetric than Exponential density, the power of S_2 does not reflect this with values of 0.704 and 0.603 respectively.

For the asymmetric distributions the test S_3 does achieve very high empirical power. Also, although it is not perfect, it does appear to capture the size of the asymmetry more accurately than S_1 and S_2. However, the test does not appear to be conservative. Indeed, the test has an estimated type-I error rate of 0.082 for a sample of size $n = 30$ from a Normal population and 0.075 for a substantial sample of size $n = 70$ from a Cauchy distribution.

For the classical test of skewness S_4, the test has nominal empirical level for the symmetric distributions, although the test appears to be overly conservative for the Cauchy case. Again, for the asymmetric distributions the test fails to capture the asymmetry present in the most asymmetric distributions. For example, when $n = 70$ the test has empirical power 0.792 for the Folded Normal distribution, but has much less power (0.677) to detect asymmetry for the Exponential distribution.

As with the previous tests, S_5 achieves nominal empirical level for the symmetric Normal and Cauchy distributions. There is also a steady increase in power through the increasingly asymmetric families of distributions. Again, like S_2, S_5 identifies asymmetry in the Log-Normal distribution with a very good power of 0.831, but does a very poor job of identifying asymmetry in Folded Normal distribution with the power of 0.454 only.

Table 2 shows the empirical level and power for the optimal tests O_1, O_2 and O_3. Firstly, it is apparent that the finite sample performance of these tests is generally poor. Indeed, for the symmetric Normal and Cauchy distributions all of the tests have empirical level which is much lower than the expected level of 0.05. For example, for a normally distributed sample of size $n = 70$ the test based on O_1 has empirical level of 0.029, but O_2 and O_3 only have empirical level of 0.013 and 0.002 respectively.

The test based on O_1 is locally and asymptotically optimal for the Fernandez and Steel distribution and achieves a reasonably good power for this family of distributions. Indeed, when $n = 70$ the test has empirical power ranging from 0.047 for FAS1 and 0.744 for FAS8. However, the test performs poorly outside of this class of densities, only achieving an empirical power of 0.400 for a Log-Normal sample of size $n = 70$.

The performance of the test based on O_2 is similar to O_1, although it generally achieves lower power than the first test. Again, the finite sample performance appears to be relatively poor and the empirical power is improved markedly as the sample size increases.

The test based on O_3 is particularly poor when there are only $n = 30$ observations, with very few rejections for any of the distributions. Indeed, even for the extremely asymmetric Exponential distribution the empirical power is just 0.016. Moreover, while the test is defined so as to be locally and asymptotically optimal for the Skew Normal distributions, the empirical power is still very low in these cases, although there is improvement as the sample size is increased.

2.5 Discussion

For the tests that are under investigation here, we demonstrated that they have either very poor power or the magnitude of the power does not reflect the size of asymmetry. The simulation study and theory proposed in section 2.2.1 validates the claim that existing tests of symmetry fail to capture the size of asymmetry in the underlying distribution. There are many other methodologies for testing symmetry and for further details on these different methods the authors recommend referring to Hollander (2004) and the references therein. It is not practical to demonstrate this point, through simulations or otherwise, for all other tests of symmetry. However, the theory in section 2.2.1 and the extensive simulation studies in section 2.4 suggest that existing tests of symmetry fail to generate power that is representative of the amount of asymmetry in the underlying distribution.

Recent research has led to the development of new measures aimed at quantifying the size of asymmetry and the main subject of the rest of this paper is to explore the use of one such measure to test symmetry.

3 Measuring asymmetry

3.1 A recently proposed measure of asymmetry

Intuitively it is believed that asymmetry is something that can be measured. When presented with two similar density curves, it is usually possible to provide some rationale on why one is more or less asymmetric than the other density curve (it was precisely this type of reasoning that generated our $<_a$ orderings in the previous section.) Despite this, it is a challenge to find a mathematical expression to effectively calibrate or quantify the amount of asymmetry.

Several measures of asymmetry have in fact been proposed. For example, see MacGillivray (1986) and Boshnakov (2007) and the references therein. However, each of these limits the class of density functions in one way or another. For a more general discussion on measuring asymmetry refer to Patil et al. (2012). Patil et al. (2012) propose measuring asymmetry using

$$\eta(X) = \eta(F) = \begin{cases} -\mathrm{Corr}(f(X), F(X)) & \text{if } 0 < \mathrm{Var}(f(X)) < \infty \\ 0 & \text{if } \mathrm{Var}(f(X)) = 0, \end{cases}$$

where X is a continuous random variable, with continuous probability density function f and distribution function F. This approach is founded on the fact that, for a symmetric random variable X with continuous probability density function f,

$$\mathrm{Cov}(f(X), F(X)) = 0.$$

A measure such as η is particularly desirable as, since it is based on $f(X)$ and $F(X)$, it utilises the maximum possible information available to quantify the asymmetry. Indeed, Patil et al. (2012) show that this user-friendly measure does a good job of quantifying the asymmetry of a number of different distributions.

For example, consider the f_ϵ density introduced in section 2.2.1. Technically the above measure cannot be applied to the density function f_ϵ since it is discontinuous, however, it is the limiting case of the following continuous probability density function,

$$f_{\epsilon,\delta}(x) = \begin{cases} \frac{1}{2} + \epsilon & \text{if } -1 < x < -\delta \\ \frac{1}{2} - \frac{\epsilon}{\delta}x & \text{if } \quad -\delta < x < \delta \\ \frac{1}{2} - \epsilon & \text{if } \quad \delta < x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

as $\delta \to 0$. One can apply the above measure to $f_{\epsilon,\delta}(x)$, for a very small value of δ. However, since the lessons learned there remain valid if we apply it to f_ϵ, for simplicity we evaluate $\eta(F_\epsilon)$. Note that if $\epsilon = 0$ or $\frac{1}{2}$ then $\mathrm{Var}(f_\epsilon(X)) = 0$ and therefore $\eta(F_\epsilon) = 0$ trivially. Therefore we concern ourselves only with $0 < \epsilon < \frac{1}{2}$. Note that in this case

$$\eta(F_\epsilon) = \frac{\sqrt{3}}{2}\sqrt{1 - 4\epsilon^2}.$$

Analysis of this function reveals that η is able to measure the amount of asymmetry in this distribution and that it is concurrent with our understanding of asymmetry for this special case. Note that $\eta\left(F_{1/2}\right)$ is zero and $\eta(F_\epsilon)$ increases as $\epsilon \to 0$. Furthermore, at ϵ equal to zero $\eta(F_\epsilon)$ is discontinuous and $\eta\left(F_0\right) = 0$ correctly identifying that f_0 is a symmetric density.

It may be worth mentioning that the quantification of asymmetry ξ provided by Cassart et al. (2008) is different from η. For example, if f_1 is taken to be a standard Normal density in equation (1) (with $\theta = 0$ and $\sigma = 1$) then as $|\xi| \to 1$, $|\eta| \to 0.95$.

In a recent article Patil et al. (2014) discuss a stronger measure η_s, where the condition $\eta_s = 0$ is a necessary and sufficient condition for symmetry. Unfortunately, a drawback of the stronger measure η_s is a loss of the 'user-friendly' aspect of η. Thus, we propose using η to devise a test for symmetry. But before that, the next subsection gives a brief description regarding the estimation of η.

3.2 Estimating η

Patil et al. (2012) construct three competing estimates of η. These are based upon calculating the sample correlation using different estimates for f and F. For example, $f(X_i)$ is estimated using kernel smoothing,

$$\hat{f}(X_i) = \frac{1}{n-1}\frac{1}{h}\sum_{j\neq i}^{n} K\left(\frac{X_j - X_i}{h}\right),$$

where K is a kernel density and h is the bandwidth. Now $F(X_i)$ is estimated by

$$\hat{F}(X_i) = \frac{1}{n-1}\sum_{j\neq i}^{n} \mathrm{I}[X_j < X_i].$$

We estimate η using

$$\hat{\eta} = -\frac{\sum_{i=1}^{n}\hat{f}(X_i)\hat{F}(X_i) - n\bar{\hat{f}}\bar{\hat{F}}}{\sqrt{\left(\sum_{i=1}^{n}(\hat{f}(X_i))^2 - n\bar{\hat{f}}^2\right)\left(\sum_{i=1}^{n}(\hat{F}(X_i))^2 - n\bar{\hat{F}}^2\right)}},$$

where $\bar{\hat{f}} = \frac{1}{n}\sum_i \hat{f}(X_i)$ and $\bar{\hat{F}} = \frac{1}{n}\sum_i \hat{F}(X_i)$. It was shown via simulation that $\hat{\eta}$ is the most effective estimator of η considered by Patil et al. (2012). Furthermore, they state that standard methods can be used to show the consistency of this estimate.

3.3 Test statistics and asymptotic analysis

The simplest suggestion for a test statistic based on $\hat{\eta}$ to test for symmetry is to use $\hat{\eta}$ directly. For example, the standardised test statistic would be

$$T_1 := \sqrt{n}\frac{\hat{\eta}}{\sqrt{\hat{\sigma}_1^2}},$$

where $\hat{\sigma}_1^2$ is the estimate of the variance of $\sqrt{n}\hat{\eta}$.

Observe that $\hat{\eta}$ is effectively a sample correlation coefficient, but Tjostheim (1996) notes that the estimation of the sample correlation coefficient r is somewhat problematic. Indeed, Tjostheim states that, “It is well established that the sampling distribution of the sample correlation coefficient is appreciably skewed for quite substantial sample sizes”. This presents a problem when using T_1 as a test statistic to test for symmetry. However, the Fisher Z-transform of r

$$Z(r) = \frac{1}{2}\log\left(\frac{1+r}{1-r}\right),$$

is known to be a better approximation to normality. Indeed, simulations appear to suggest that the finite sample behaviour of $Z(\hat{\eta})$ is better than $\hat{\eta}$, that is, $Z(\hat{\eta})$ appears to follow a Normal distribution more closely than $\hat{\eta}$ for small samples. This motivates a second test statistic

$$T_2 := \sqrt{n}\frac{Z(\hat{\eta})}{\sqrt{\hat{\sigma}_2^2}},$$

where $\hat{\sigma}_2^2$ is the estimate of the variance of $\sqrt{n}Z(\hat{\eta})$.

An alternative way to avoid dealing with the asymptotic behaviour of T_1 is to simply ignore the denominator terms in $\hat{\eta}$. Indeed, if we are only interested in testing for symmetry (and not providing a scaled measure of the asymmetry in the sample) then we can simply base our test statistic on $\nu_{fF} = \mathrm{Cov}(f(X), F(X))$. That leads to

$$T_3 := \sqrt{n}\frac{\hat{\nu}_{fF}}{\sqrt{\hat{\sigma}_3^2}},$$

where

$$\hat{\nu}_{fF} = \frac{1}{n}\sum_{i=1}^{n} \hat{f}(X_i)\left(\hat{F}(X_i) - \frac{1}{2}\right),$$

and $\hat{\sigma}_3^2$ is the estimate of the variance of $\sqrt{n}\hat{\nu}_{fF}$.

The asymptotic distributions of $\hat{\eta}$, $Z(\hat{\eta})$ and $\hat{\nu}_{fF}$ are established in Theorem 1 below. For that, we require the following assumptions:

A1 Assume that $E[f^2(X)] < \infty$.
A2 The kernel function K is smooth, has bounded support and is of bounded variation.
A3 The bandwidth $h \sim n^{-\gamma}$ for $\frac{1}{4} \le \gamma < \frac{1}{2}$.

Theorem 1 *Let $X_1, \cdots, X_n$ be a random sample from a continuous probability density function $f(x)$ and distribution function $F(x)$ and further suppose that assumptions* **A1**, **A2** *and* **A3** *all hold. Then as $n \to \infty$,*

(i)

$$\sqrt{n}\,[\hat{\eta} - \eta] \xrightarrow{L} N(0, \sigma^2),$$

(ii)

$$\sqrt{n}\,[Z(\hat{\eta}) - Z(\eta)] \xrightarrow{L} N(0, \tau^2),$$

(iii)

$$\sqrt{n}\,[\hat{\nu}_{fF} - \nu_{fF}] \xrightarrow{L} N(0, \upsilon^2),$$

where

$$\begin{aligned}
\sigma^2 &= \mathrm{Var}\Bigg[\frac{2}{\sqrt{\nu_f\nu_F}}\left(f(X)F(X) - \frac{1}{2}f(X)\right) + \int_X^\infty \frac{f(y)^2}{\sqrt{\nu_f\nu_F}}dy \\
&\quad + \eta\left\{\frac{\left(F(X)-\frac{1}{2}\right)^2}{2\nu_F} + \frac{(f(X)-\mu_f)^2}{2\nu_f} + \int_X^\infty \frac{\left(F(y)-\frac{1}{2}\right)}{\nu_F}f(y)dy + \frac{(f(X)-\mu_f)f(X)}{\nu_f}\right\}\Bigg], \qquad (2) \\
\tau^2 &= \frac{\sigma^2}{(1-\eta^2)^2}, \\
\upsilon^2 &= \mathrm{Var}\left[2\left(f(X)F(X) - \frac{1}{2}f(X)\right) + \int_X^\infty f(y)^2 dy\right],
\end{aligned}$$

where ν_f and ν_F denote $\mathrm{Var}(f(X))$ *and* $\mathrm{Var}(F(X))$ $\left(= \frac{1}{12}\right)$ *respectively, and* $\mu_f = \mathrm{E}[f(X)]$.

We present the proof of (i) and (iii) below using Theorem 1 from Giné and Mason (2008). More details regarding this theorem are given in the appendix. Further, a simple application of the delta method to (i) gives (ii).

Proof Recall that

$$\hat{\eta} = -\widehat{\mathrm{Corr}}\left(\hat{f}, \hat{F}\right) = -\frac{\sum_i \left(\hat{f}_i - \bar{\hat{f}}\right)\left(\hat{F}_i - \bar{\hat{F}}\right)}{\sqrt{\sum_i \left(\hat{f}_i - \bar{\hat{f}}_i\right)^2}\sqrt{\sum_i \left(\hat{F}_i - \bar{\hat{F}}_i\right)^2}},$$

where $\hat{f}_i = \hat{f}(X_i)$ and $\hat{F}_i = \hat{F}(X_i)$, for $i = 1, \cdots, n$. This is an estimate of the population correlation coefficient

$$\eta = -\mathrm{Corr}\left(f(X), F(X)\right) = -\frac{\mathrm{E}[f(X)F(X)] - \mathrm{E}[f(X)]\mathrm{E}[F(X)]}{\sqrt{\mathrm{Var}[f(X)]\mathrm{Var}[F(X)]}}.$$

To ease the notation let

$$\begin{aligned}
\nu_{fF} &= \mathrm{E}[f(X)F(X)] - \mathrm{E}[f(X)]\mathrm{E}[F(X)], \\
\hat{\nu}_{fF} &= \frac{1}{n}\sum_i \left(\hat{f}_i - \bar{\hat{f}}\right)\left(\hat{F}_i - \bar{\hat{F}}\right) = \frac{1}{n}\sum_i \left(\hat{f}_i\right)\left(\hat{F}_i - \bar{\hat{F}}\right) = \frac{1}{n}\sum_i \hat{f}_i \left(\hat{F}_i - \frac{1}{2}\right), \\
\nu_f &= \mathrm{Var}[f(X)], \\
\hat{\nu}_f &= \frac{1}{n}\sum_i \left(\hat{f}_i - \bar{\hat{f}}\right)^2, \\
\nu_F &= \mathrm{Var}[F(X)] = \frac{1}{12}, \\
\hat{\nu}_F &= \frac{1}{n}\sum_i \left(\hat{F}_i - \bar{\hat{F}}\right)^2 = \frac{1}{n}\sum_i \left(\hat{F}_i - \frac{1}{2}\right)^2.
\end{aligned}$$

Then

$$\hat{\eta} = -\frac{\hat{\nu}_{fF}}{\sqrt{\hat{\nu}_f\hat{\nu}_F}} \quad \text{and} \quad \eta = -\frac{\nu_{fF}}{\sqrt{\nu_f\nu_F}}.$$

Firstly, observe that

$$\begin{aligned}
\sqrt{n}(\hat{\eta} - \eta) &= -\sqrt{n}\left(\frac{\hat{\nu}_{fF}}{\sqrt{\hat{\nu}_f\hat{\nu}_F}} - \frac{\nu_{fF}}{\sqrt{\nu_f\nu_F}}\right) \\
&= -\sqrt{n}\left(\frac{\hat{\nu}_{fF}}{\sqrt{\hat{\nu}_f\hat{\nu}_F}} - \frac{\nu_{fF}}{\sqrt{\hat{\nu}_f\hat{\nu}_F}} + \frac{\nu_{fF}}{\sqrt{\hat{\nu}_f\hat{\nu}_F}} - \frac{\nu_{fF}}{\sqrt{\nu_f\nu_F}}\right) \\
&= -\sqrt{n}\frac{1}{\sqrt{\hat{\nu}_f\hat{\nu}_F}}\left(\hat{\nu}_{fF} - \nu_{fF}\right) + \sqrt{n}\frac{\nu_{fF}}{\sqrt{\hat{\nu}_f\hat{\nu}_F}\sqrt{\nu_f\nu_F}}\left(\sqrt{\hat{\nu}_f\hat{\nu}_F} - \sqrt{\nu_f\nu_F}\right). \qquad (3)
\end{aligned}$$

Ignoring the sign of the first term on the right-hand side of equation (3) rewrite

$$\begin{aligned}
\sqrt{n}\frac{1}{\sqrt{\hat{\nu}_f\hat{\nu}_F}}\left(\hat{\nu}_{fF} - \nu_{fF}\right) &= \sqrt{n}\frac{1}{\sqrt{\nu_f\nu_F}}\left(\hat{\nu}_{fF} - \nu_{fF}\right) + \sqrt{n}\frac{1}{\sqrt{\hat{\nu}_f\hat{\nu}_F}}\left(\hat{\nu}_{fF} - \nu_{fF}\right) - \sqrt{n}\frac{1}{\sqrt{\nu_f\nu_F}}\left(\hat{\nu}_{fF} - \nu_{fF}\right) \\
&= \sqrt{n}\frac{1}{\sqrt{\nu_f\nu_F}}\left(\hat{\nu}_{fF} - \nu_{fF}\right) + \sqrt{n}\frac{1}{\sqrt{\hat{\nu}_f\hat{\nu}_F}\sqrt{\nu_f\nu_F}}\left(\hat{\nu}_{fF} - \nu_{fF}\right)\left(\sqrt{\nu_f\nu_F} - \sqrt{\hat{\nu}_f\hat{\nu}_F}\right).
\end{aligned}$$

Claim 1: $\sqrt{n}\left(\hat{\nu}_{fF} - \nu_{fF}\right)$ converges in law to a Normal distribution with finite variance.

From Hall and Marron (1987) it follows that $\hat{\nu}_f$ and $\hat{\nu}_F$ converge in probability to ν_f and ν_F. Therefore by this fact and Claim 1 we have

$$\sqrt{n}\frac{1}{\sqrt{\hat{\nu}_f\hat{\nu}_F}}\left(\hat{\nu}_{fF} - \nu_{fF}\right) = \sqrt{n}\frac{1}{\sqrt{\nu_f\nu_F}}\left(\hat{\nu}_{fF} - \nu_{fF}\right) + o_p(1). \qquad (4)$$

Write the second term in equation (3) as

$$\begin{aligned}
&\sqrt{n}\frac{\nu_{fF}}{\sqrt{\hat{\nu}_f\hat{\nu}_F}\sqrt{\nu_f\nu_F}}\left(\sqrt{\hat{\nu}_f\hat{\nu}_F} - \sqrt{\nu_f\nu_F}\right) \\
&\quad = \sqrt{n}\frac{\nu_{fF}\left(\hat{\nu}_f\hat{\nu}_F - \nu_f\nu_F\right)}{\sqrt{\hat{\nu}_f\hat{\nu}_F}\sqrt{\nu_f\nu_F}\left(\sqrt{\hat{\nu}_f\hat{\nu}_F} + \sqrt{\nu_f\nu_F}\right)} \\
&\quad = \sqrt{n}\nu_{fF}\frac{\hat{\nu}_f\hat{\nu}_F - \nu_f\nu_F}{2\nu_f\nu_F\sqrt{\nu_f\nu_F}} \\
&\qquad\qquad + \sqrt{n}\nu_{fF}\left(\hat{\nu}_f\hat{\nu}_F - \nu_f\nu_F\right)\frac{2\nu_f\nu_F - \sqrt{\hat{\nu}_f\hat{\nu}_F}\left(\sqrt{\hat{\nu}_f\hat{\nu}_F} + \sqrt{\nu_f\nu_F}\right)}{2\nu_f\nu_F\sqrt{\nu_f\nu_F}\sqrt{\hat{\nu}_f\hat{\nu}_F}\left(\sqrt{\hat{\nu}_f\hat{\nu}_F} + \sqrt{\nu_f\nu_F}\right)} \\
&\quad = \sqrt{n}\nu_{fF}\frac{\hat{\nu}_f\hat{\nu}_F - \nu_f\nu_F}{2\nu_f\nu_F\sqrt{\nu_f\nu_F}} + o_p(1), \qquad (5)
\end{aligned}$$

again, using the fact that $\hat{\nu}_f$ and $\hat{\nu}_F$ converge in probability to ν_f and ν_F. Furthermore,

$$\begin{aligned}\sqrt{n}\left(\hat{\nu}_f\hat{\nu}_F - \nu_f\nu_F\right) &= \sqrt{n}\left(\nu_f(\hat{\nu}_F - \nu_F) + \nu_F(\hat{\nu}_f - \nu_f)\right) + \sqrt{n}\left(\hat{\nu}_f - \nu_f\right)\left(\hat{\nu}_F - \nu_F\right) \\ &= \sqrt{n}\left(\nu_f(\hat{\nu}_F - \nu_F) + \nu_F(\hat{\nu}_f - \nu_f)\right) + o_p(1). \qquad (6)\end{aligned}$$

Hence, using equations (4), (5), and (6), rewrite (3) as

$$\begin{aligned}\sqrt{n}(\hat{\eta} - \eta) &= -\sqrt{n}\left(\frac{1}{\sqrt{\nu_f\nu_F}}\left(\hat{\nu}_{fF} - \nu_{fF}\right) - \frac{\nu_{fF}}{2\nu_f\nu_F\sqrt{\nu_f\nu_F}}\left(\nu_f(\hat{\nu}_F - \nu_F) + \nu_F(\hat{\nu}_f - \nu_f)\right)\right) + o_p(1) \\ &= -\sqrt{n}\left(\frac{1}{\sqrt{\nu_f\nu_F}}\left(\hat{\nu}_{fF} - \nu_{fF}\right) - \frac{\nu_{fF}}{2\nu_F\sqrt{\nu_f\nu_F}}\left(\hat{\nu}_F - \nu_F\right) - \frac{\nu_{fF}}{2\nu_f\sqrt{\nu_f\nu_F}}\left(\hat{\nu}_f - \nu_f\right)\right) + o_p(1).\end{aligned}$$

The leading order term in the expansion for $\sqrt{n}(\hat{\eta} - \eta)$ is composed of three parts. We show that the linear combination is asymptotically Normal by applying Theorem 1 given by Giné and Mason (2008). Firstly, observe that

$$\begin{aligned}\sqrt{n}\hat{\nu}_f &= \sqrt{n}\frac{1}{n}\sum_{i=1}^{n}\left(\hat{f}_i - \bar{\hat{f}}\right)^2 \\ &= \sqrt{n}\frac{1}{n}\sum_{i=1}^{n}\left(\hat{f}_i - \mu_f\right)^2 + o_p(1),\end{aligned}$$

where $\mu_f = \mathrm{E}[f(X)]$. Hence, defining

$$\tilde{\nu}_f = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{f}_i - \mu_f\right)^2,$$

it is clear that

$$\begin{aligned}\sqrt{n}(\hat{\eta} - \eta) &= -\sqrt{n}\left(\frac{1}{\sqrt{\nu_f\nu_F}}\left(\hat{\nu}_{fF} - \nu_{fF}\right) - \frac{\nu_{fF}}{2\nu_F\sqrt{\nu_f\nu_F}}\left(\hat{\nu}_F - \nu_F\right) - \frac{\nu_{fF}}{2\nu_f\sqrt{\nu_f\nu_F}}\left(\tilde{\nu}_f - \nu_f\right)\right) + o_p(1) \\ &= -\sqrt{n}\hat{\Theta} + o_p(1),\end{aligned}$$

where

$$\hat{\Theta} = \frac{1}{\sqrt{\nu_f\nu_F}}\left(\hat{\nu}_{fF}\right) - \frac{\nu_{fF}}{2\nu_F\sqrt{\nu_f\nu_F}}\left(\hat{\nu}_F\right) - \frac{\nu_{fF}}{2\nu_f\sqrt{\nu_f\nu_F}}\left(\tilde{\nu}_f\right).$$

Claim 2: $\sqrt{n}\hat{\Theta} \xrightarrow{\mathrm{L}} N(0, \sigma^2)$.

The proof of the Theorem will be complete if we prove Claim 1 and Claim 2. Since the proof of Claim 1 and 2 are similar we prove Claim 2, whilst Claim 1 follows similarly. Observe that $\hat{\Theta}$ is in the form

$$\frac{1}{n}\sum_{i=1}^{n}\hat{\phi}\left(\hat{f}(X_i), \hat{F}(X_i)\right),$$

and is an estimator of

$$\Theta = \int_{-\infty}^{\infty}\left\{\frac{f(x)F(x) - \frac{1}{2}f(x)}{\sqrt{\nu_f\nu_F}} - \frac{\nu_{fF}}{2\nu_F\sqrt{\nu_f\nu_F}}\left(F(x) - \frac{1}{2}\right)^2 - \frac{\nu_{fF}}{2\nu_f\sqrt{\nu_f\nu_F}}(f(x) - \mu_f)^2\right\}f(x)dx = 0.$$

Therefore we can apply Theorem 1 of Giné and Mason (2008) to show that $\hat{\Theta}$ is asymptotically Normal once we have verified the conditions *I-VIII* of the theorem. These conditions are described in detail in the appendix. Conditions *I, VI, VII* and *VIII* hold directly from the assumptions **A1**, **A2** and **A3**. Also, under the assumption **A3**, condition *II* holds with $H = f$ as the suitable measurable function. To verify the remaining conditions note that, in Giné and Mason's notation, we have that

$$\begin{aligned}\psi(x, F(x), f(x)) &= \frac{f(x)F(x) - \frac{1}{2}f(x)}{\sqrt{\nu_f\nu_F}} - \frac{\nu_{fF}}{2\nu_F\sqrt{\nu_f\nu_F}}\left(F(x) - \frac{1}{2}\right)^2 - \frac{\nu_{fF}}{2\nu_f\sqrt{\nu_f\nu_F}}(f(x) - \mu_f)^2, \\ \psi(x, y_0, y_1) &= \frac{y_0y_1 - \frac{1}{2}y_1}{\sqrt{\nu_f\nu_F}} - \frac{\nu_{fF}}{2\nu_F\sqrt{\nu_f\nu_F}}\left(y_0 - \frac{1}{2}\right)^2 - \frac{\nu_{fF}}{2\nu_f\sqrt{\nu_f\nu_F}}(y_1 - \mu_f)^2.\end{aligned}$$

Further,

$$\psi_m(x) = \frac{\partial}{\partial y_m}\psi(x, y_0, y_1)\big|_{(x,F(x),f(x))}.$$

Hence,

$$\begin{aligned}
\psi_0(x) = \frac{\partial}{\partial y_0}\psi(x, y_0, y_1) &= \frac{y_1}{\sqrt{\nu_f\nu_F}} - \frac{\nu_{fF}}{\nu_F\sqrt{\nu_f\nu_F}}\left(y_0 - \frac{1}{2}\right) \\
&= \frac{f(x)}{\sqrt{\nu_f\nu_F}} - \frac{\nu_{fF}}{\nu_F\sqrt{\nu_f\nu_F}}\left(F(x) - \frac{1}{2}\right), \\
\psi_1(x) = \frac{\partial}{\partial y_1}\psi(x, y_0, y_1) &= \frac{y_0 - \frac{1}{2}}{\sqrt{\nu_f\nu_F}} - \frac{\nu_{fF}}{\nu_f\sqrt{\nu_f\nu_F}}(y_1 - \mu_f) \\
&= \frac{F(x) - \frac{1}{2}}{\sqrt{\nu_f\nu_F}} - \frac{\nu_{fF}}{\nu_f\sqrt{\nu_f\nu_F}}(f(x) - \mu_f).
\end{aligned}$$

Therefore *III* and *IV* hold under the assumption **A1**.

Further define,

$$\begin{aligned}
\xi(X) &= \psi(X) - \mathrm{E}\left[\psi(X)\right] \\
&= \frac{f(X)F(X) - \frac{1}{2}f(X)}{\sqrt{\nu_f\nu_F}} - \frac{\nu_{fF}}{2\nu_F\sqrt{\nu_f\nu_F}}\left(F(X) - \frac{1}{2}\right)^2 - \frac{\nu_{fF}}{2\nu_f\sqrt{\nu_f\nu_F}}(f(X) - \mu_f)^2, \\
\xi_0(X) &= \int_X^\infty \left\{\frac{f(y)}{\sqrt{\nu_f\nu_F}} - \frac{\nu_{fF}}{\nu_F\sqrt{\nu_f\nu_F}}\left(F(y) - \frac{1}{2}\right)\right\} f(y)dy - \frac{1}{2}\frac{\mu_f}{\sqrt{\nu_f\nu_F}}, \\
\chi_1(y) &= \psi_1(y)f(y) \\
&= \frac{F(y)f(y) - \frac{1}{2}f(y)}{\sqrt{\nu_f\nu_F}} - \frac{\nu_{fF}}{\nu_f\sqrt{\nu_f\nu_F}}(f(y) - \mu_f)f(y).
\end{aligned}$$

Hence, condition *V* is also satisfied. Define

$$\begin{aligned}
\xi_1(X) &= \chi_1(X) - \mathrm{E}\left[\chi_1(X)\right] \\
&= \frac{F(X)f(X) - \frac{1}{2}f(X)}{\sqrt{\nu_f\nu_F}} - \frac{\nu_{fF}}{\nu_f\sqrt{\nu_f\nu_F}}(f(X) - \mu_f)f(X).
\end{aligned}$$

Finally, define

$$\begin{aligned}
Y &= \xi(X) + \xi_0(X) + \xi_1(X) \\
&= \frac{f(X)F(X) - \frac{1}{2}f(X)}{\sqrt{\nu_f\nu_F}} - \frac{\nu_{fF}}{2\nu_F\sqrt{\nu_f\nu_F}}\left(F(X) - \frac{1}{2}\right)^2 - \frac{\nu_{fF}}{2\nu_f\sqrt{\nu_f\nu_F}}(f(X) - \mu_f)^2 \\
&\quad + \int_{X_i}^\infty \left\{\frac{f(y)}{\sqrt{\nu_f\nu_F}} - \frac{\nu_{fF}}{\nu_F\sqrt{\nu_f\nu_F}}\left(F(y) - \frac{1}{2}\right)\right\} f(y)dy - \frac{1}{2}\frac{\mu_f}{\sqrt{\nu_f\nu_F}} \\
&\quad + \frac{F(X)f(X) - \frac{1}{2}f(X)}{\sqrt{\nu_f\nu_F}} - \frac{\nu_{fF}}{\nu_f\sqrt{\nu_f\nu_F}}(f(X) - \mu_f)f(X).
\end{aligned}$$

Hence, we can conclude, under the assumptions **A1-A3**, that $\sqrt{n}\left(\hat{\Theta}-\Theta\right)$ is asymptotically normally distributed with mean 0 and variance

$$
\begin{aligned}
\sigma^2 := \mathrm{Var}(Y) = \mathrm{Var}\Bigg[&\frac{2}{\sqrt{\nu_f\nu_F}}\left(f(X)F(X)-\frac{1}{2}f(X)\right)+\int_{X_i}^{\infty}\frac{f(y)^2}{\sqrt{\nu_f\nu_F}}dy-\frac{1}{2}\frac{\mu_f}{\sqrt{\nu_f\nu_F}} \\
&-\nu_{fF}\left\{\frac{\left(F(X)-\frac{1}{2}\right)^2}{2\nu_F\sqrt{\nu_f\nu_F}}+\frac{(f(X)-\mu_f)^2}{2\nu_f\sqrt{\nu_f\nu_F}}+\int_{X_i}^{\infty}\frac{\left(F(y)-\frac{1}{2}\right)}{\nu_F\sqrt{\nu_f\nu_F}}f(y)dy+\frac{(f(X)-\mu_f)f(X)}{\nu_f\sqrt{\nu_f\nu_F}}\right\}\Bigg] \\
= \mathrm{Var}\Bigg[&\frac{2}{\sqrt{\nu_f\nu_F}}\left(f(X)F(X)-\frac{1}{2}f(X)\right)+\int_{X_i}^{\infty}\frac{f(y)^2}{\sqrt{\nu_f\nu_F}}dy \\
&-\frac{\nu_{fF}}{\sqrt{\nu_f\nu_F}}\left\{\frac{\left(F(X)-\frac{1}{2}\right)^2}{2\nu_F}+\frac{(f(X)-\mu_f)^2}{2\nu_f}+\int_{X_i}^{\infty}\frac{\left(F(y)-\frac{1}{2}\right)}{\nu_F}f(y)dy+\frac{(f(X)-\mu_f)f(X)}{\nu_f}\right\}\Bigg] \\
= \mathrm{Var}\Bigg[&\frac{2}{\sqrt{\nu_f\nu_F}}\left(f(X)F(X)-\frac{1}{2}f(X)\right)+\int_{X_i}^{\infty}\frac{f(y)^2}{\sqrt{\nu_f\nu_F}}dy \\
&+\eta\left\{\frac{\left(F(X)-\frac{1}{2}\right)^2}{2\nu_F}+\frac{(f(X)-\mu_f)^2}{2\nu_f}+\int_{X_i}^{\infty}\frac{\left(F(y)-\frac{1}{2}\right)}{\nu_F}f(y)dy+\frac{(f(X)-\mu_f)f(X)}{\nu_f}\right\}\Bigg].
\end{aligned}
$$

The expression for the variance given in equation (2) is complicated, as well as difficult to estimate, and thus the next subsection is devoted to discussing a variety of methods to estimate the variance σ^2.

3.4 Estimating the variance

We now provide details of how to estimate the variance σ^2 in equation (2), which is associated with test statistic T_1. The variances associated with the other test statistics can be estimated by similar methods. One approach is to estimate σ^2 using the Monte Carlo method replacing f and F with $\hat{f}$ and $\hat{F}$ respectively, and evaluating the integrals using a numerical method. Under the null hypothesis of symmetry $\eta = 0$ and therefore, in this case, σ^2 reduces to

$$\sigma_0^2 = \frac{1}{\nu_f\nu_F}\mathrm{Var}\left(2\left\{f(X)F(X)-\frac{1}{2}f(X)\right\}+\int_X^{\infty}f^2(y)dy\right).$$

To ease the notation somewhat let

$$\Phi_1(X) = \int_X^{\infty}f^2(y)dy.$$

Hence, for symmetric random variables we have

$$
\begin{aligned}
\sigma_0^2 &= \frac{1}{\nu_f\nu_F}\mathrm{Var}\left[2\left\{f(X)F(X)-\frac{1}{2}f(X)\right\}+\Phi_1(X)\right] \\
&= \frac{1}{\nu_f\nu_F}\Bigg(\mathrm{Var}\left[2\left\{f(X)F(X)-\frac{1}{2}f(X)\right\}\right]+\mathrm{Var}\left[\Phi_1(X)\right] \\
&\qquad +2\mathrm{Cov}\left(2\left\{f(X)F(X)-\frac{1}{2}f(X)\right\},\Phi_1(X)\right)\Bigg) \\
&= \frac{1}{\nu_f\nu_F}\Bigg(\mathrm{E}\left[4\left\{f(X)F(X)-\frac{1}{2}f(X)\right\}^2\right]+\mathrm{Var}\left[\Phi_1(X)\right] \\
&\qquad +4\mathrm{E}\left[\left\{f(X)F(X)-\frac{1}{2}f(X)\right\}\Phi_1(X)\right]\Bigg) \\
&= \frac{1}{\nu_f\nu_F}\Big(4\mathrm{E}\left[f(X)^2F(X)^2\right]+\mathrm{E}\left[f(X)^2\right]-4\mathrm{E}\left[f(X)^2F(X)\right]+\mathrm{E}\left[\Phi_1(X)^2\right]-\left[\mathrm{E}\Phi_1(X)\right]^2 \\
&\qquad +4\mathrm{E}\left[f(X)F(X)\Phi_1(X)\right]-2\mathrm{E}\left[f(X)\Phi_1(X)\right]\Big).
\end{aligned}
$$

Further, observe that by changing the order of integration

$$\begin{aligned}
\mathrm{E}\left[\varPhi_1(X)\right] &= \mathrm{E}\left[\int_X^\infty f^2(u)du\right] = \int_{-\infty}^\infty f(y)\int_y^\infty f^2(u)dudy = \int_{-\infty}^\infty f^2(u)\int_{-\infty}^u f(y)dydu \\
&= \int_{-\infty}^\infty f^2(u)F(u)du = \mathrm{E}\left[f(X)F(X)\right],
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}\left[f(X)\varPhi_1(X)\right] &= \mathrm{E}\left[f(X)\int_X^\infty f^2(u)du\right] = \int_{-\infty}^\infty f^2(y)\int_y^\infty f^2(u)dudy = \int_{-\infty}^\infty f^2(u)\int_{-\infty}^u f^2(y)dydu \\
&= \int_{-\infty}^\infty f^2(u)\left\{\mathrm{E}[f(X)] - \int_u^\infty f^2(y)dy\right\}du = \mathrm{E}[f(X)]^2 - \mathrm{E}\left[f(X)\varPhi_1(X)\right].
\end{aligned}$$

Thus, under the null hypothesis

$$\begin{aligned}
\mathrm{E}\left[\varPhi_1(X)\right] &= \frac{1}{2}\mathrm{E}\left[f(X)\right] = \mathrm{E}\left[f(X)F(X)\right], \\
2\mathrm{E}\left[f(X)\varPhi_1(X)\right] &= \left[\mathrm{E}f(X)\right]^2 = 4\left[\mathrm{E}f(X)F(X)\right]^2.
\end{aligned}$$

Hence,

$$\sigma_0^2 = \frac{1}{\nu_f\nu_F}\left\{4m_{22} + m_{20} - 4m_{21}\right\} + 4\left\{\mathrm{E}\left[f(X)F(X)\varPhi_1(X)\right] - m_{11}^2\right\} + \left\{\mathrm{E}[\varPhi_1(X)^2] - m_{11}^2\right\},$$

where

$$m_{ij} = \mathrm{E}\left[f(X)^iF(X)^j\right], \qquad i = 1,2,\ j = 0,1,2.$$

For a random sample we can readily estimate m_{ij} using

$$\hat{m}_{ij} = \frac{1}{n}\sum_{k=1}^n \hat{f}(X_k)^i\hat{F}(X_k)^j \qquad i = 1,2,\ j = 0,1,2.$$

One can readily generalise the results of Hall and Marron (1987) to verify the consistency of $\hat{m}_{ij}$. Even in this greatly reduced form, the presence of the terms involving $\varPhi_1(X)$ means that the expression for the variance is a complex one to evaluate in practice. In general, one could carry out a numerical integration technique using the estimated density function $\hat{f}(x)$ in place of $f(x)$. Alternatively, in most situations one is primarily interested in whether samples are taken from a Normal population. If we add the additional assumption (under the null hypothesis) that X is a normally distributed random variable we obtain

$$\int_x^\infty f^2(y)dy = \frac{1}{4\sqrt{\pi}\sigma_X}\mathrm{cerf}\left(\frac{x-\mu_X}{\sigma_X}\right),$$

where μ_X and σ_X are the mean and variance of the random variable X respectively and

$$\mathrm{cerf}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}dt,$$

is the complementary error function. This greatly simplifies the expression for the variance under the null hypothesis and removes the need to carry out a computationally expensive numerical integration technique.

More generally, one can define

$$\begin{aligned}
Y_i = \Bigg[&\frac{2}{\sqrt{\hat{\nu}_f\hat{\nu}_F}}\left(\hat{f}(X_i)\hat{F}(X_i) - \frac{1}{2}f(X_i)\right) + \frac{1}{\sqrt{\hat{\nu}_f\hat{\nu}_F}}\hat{\varPhi}_1(X_i) \\
&+ \hat{\eta}\left\{\frac{\left(\hat{F}(X_i)-\frac{1}{2}\right)^2}{2\hat{\nu}_F} + \frac{\left(\hat{f}(X_i)-\bar{\hat{f}}\right)^2}{2\hat{\nu}_f} + \hat{\varPhi}_2(X_i) + \frac{\left(\hat{f}(X_i)-\bar{\hat{f}}\right)\hat{f}(X_i)}{\hat{\nu}_f}\right\}\Bigg],
\end{aligned}$$

where $\hat{\Phi}_1(X)$ is a numerical approximation of the integral $\Phi_1(X)$ using $\hat{f}(X)$ as an estimate of the curve $f(x)$, and $\hat{\Phi}_2(X)$ is a numerical approximation of

$$\int_{X_i}^{\infty} \frac{\left(F(y) - \frac{1}{2}\right)}{\nu_F} f(y)dy,$$

estimating f and F by $\hat{f}$ and $\hat{F}$, respectively. It is then possible to estimate σ^2 using

$$\hat{\sigma}^2 = \widehat{\mathrm{Var}}(Y) = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2. \tag{7}$$

The R code for calculating the test statistics T_1, T_2, and T_3 using the variance estimate in equation (7) is available in the appendix.

It is also important to note that, the asymmetry measure η is based on the functionals of f and the test statistics are based on the estimation of η, that is, estimation of functionals of f. We estimate these functionals by replacing f by its nonparametric kernel-based estimator. As is the case for kernel-based estimators of the density function, the choice of kernel has no bearing on the convergence rate of the estimators of these functionals. Also following the results of Hall and Marron (1987) it is easy to note that for a reasonable range of bandwidths, these estimators converge to the true value with mean squared error rate n^{-1}.

3.5 Power analysis of new tests

We subject the three new test statistics to a similar simulation study as in section 2. Once again, the test statistic is generated $m = 10,000$ times from samples of size $n = 30, 50$ and 70. In each case the density function is estimated using a Normal kernel and the bandwidth is estimated using the simple rule of thumb given by Silverman (1986), based on normality. Table 3 shows the results where σ^2 is estimated from the data (for example, the variance of T_1 is estimated from the sample using $\hat{\sigma}^2$ defined in equation (7) and similarly for the other tests).

For all three tests we observe a steady increase in power for the asymmetric families. For the remaining asymmetric distributions (Log-Normal, Folded Normal, Exponential) the test achieves a much higher level of power. Furthermore, whilst not perfect, the amount of power is closely related to the amount of asymmetry. This is to be expected since the tests are based on η, which has previously been identified as a more effective measure of the magnitude of asymmetry. For example, for $n = 50$ the empirical power of T_1 for the Log-Normal and Folded Normal distribution is 0.994 and 0.786 respectively. This is considerably better than the empirical power for S_1, S_2 and S_4, and comparable to that achieved by S_3.

It is clear that, of the newly proposed tests, the test based on T_1 performs best in terms of power. Indeed, in Table 3 the empirical powers of the test based on T_1 are uniformly larger than T_2 and T_3 for the asymmetric distributions. In fact, T_1 outperforms the existing tests S_1, S_2 and S_4 in terms of power and, whilst S_3 has marginally higher power than T_1, recall that S_3 is not conservative. Indeed, for a Normal sample T_1 has estimated type-I error of 0.040 for $n = 30$ compared to 0.082 for S_3. For a sample of $n = 30$ from the Cauchy distribution this difference is even more stark with a type-I error estimate of 0.037 for T_1 compared to 0.108 for S_3. Hence, the additional power achieved by S_3 is somewhat artificial if the test is not conservative under the null hypothesis (i.e. it is unable maintain a maximum level of 0.05 for the symmetric distributions.) The test based on T_1 also outperforms the optimal tests O_1, O_2 and O_3 for all of the distributions under consideration here.

In fact, all of the proposed tests are competitive in terms of power, with the exception of T_2 for $n = 30$. Indeed, the sample-based estimate of the variance of the Z-transformed statistic S_2,

$$\hat{\tau}^2 = \frac{\hat{\sigma}^2}{\left(1 - \hat{\eta}^2\right)^2},$$

is somewhat unstable for small samples. Further investigation, not included here, suggests that the bootstrap provides a more effective procedure to estimate the variance of T_2.

4 Conclusion

In this paper some of the existing tests of symmetry have been appraised and shown to perform well at detecting departure from symmetry. However, an undesirable feature was identified, which seems to have been overlooked in the existing tests of symmetry. Namely, that the tests failed to reject the symmetry hypothesis with greater power for the most asymmetric distributions. This trait was exhibited using a combination of theoretical examples and a simulation study. The reason for this is principally because, until recently, there was no measure of asymmetry which adequately quantified the amount of asymmetry. However, a recently proposed measure η, which has been shown to do a good job of measuring the size of asymmetry, was introduced and discussed. By considering sample estimates of η, several new tests for symmetry were proposed. Furthermore, the asymptotic properties of these tests were determined and the tests were compared with the existing tests in a simulation study.

In conclusion, it was shown that η provides a useful starting point for a test for symmetry. The great advantage of a test based upon η is that it is an effective and easy to understand measure of the amount of asymmetry in the underlying distribution. As a result, the new tests have the desirable property that, for the most part, the higher the amount of asymmetry in the underlying distribution, the greater the rejection power of the test. This means that the tests based on η provide a valid alternative to the existing tests. Finally, of these new tests it is identified that the test based directly on $\hat{\eta}$ has greatest power for the distributions under consideration here.

Table 1: Empirical power/level of the tests based on S_1, S_2, S_3, S_4 and S_5 for a variety of density functions and sample sizes.

	S_1			S_2			S_3			S_4			S_5		
η Dist.	$n=30$	$n=50$	$n=70$	$n=30$	$n=50$	$n=70$	$n=30$	$n=50$	$n=70$	$n=30$	$n=50$	$n=70$	$n=30$	$n=50$	$n=70$
0 N	0.036	0.042	0.040	0.016	0.030	0.035	0.082	0.068	0.065	0.027	0.034	0.041	0.016	0.030	0.031
0 C	-	-	-	0.004	0.011	0.012	0.108	0.079	0.075	0.024	0.013	0.009	0.014	0.025	0.029
0.1 NM1	0.040	0.055	0.064	0.015	0.027	0.031	0.113	0.125	0.134	0.033	0.045	0.061	0.015	0.030	0.037
0.2 NM2	0.078	0.110	0.159	0.017	0.031	0.041	0.200	0.284	0.353	0.088	0.160	0.244	0.017	0.046	0.051
0.3 NM3	0.128	0.238	0.338	0.023	0.050	0.074	0.369	0.519	0.645	0.187	0.354	0.523	0.027	0.061	0.085
0.4 NM4	0.212	0.368	0.509	0.047	0.109	0.173	0.483	0.693	0.835	0.275	0.499	0.669	0.042	0.100	0.137
0.1 SN1	0.043	0.046	0.052	0.018	0.032	0.036	0.100	0.100	0.106	0.039	0.052	0.068	0.017	0.030	0.035
0.2 SN2	0.062	0.075	0.100	0.026	0.039	0.058	0.152	0.186	0.229	0.058	0.102	0.159	0.020	0.035	0.045
0.3 SN3	0.086	0.135	0.185	0.031	0.062	0.092	0.224	0.327	0.427	0.098	0.198	0.307	0.026	0.048	0.065
0.4 SN4	0.130	0.209	0.286	0.042	0.087	0.139	0.332	0.494	0.638	0.146	0.298	0.467	0.036	0.076	0.103
0.5 SN5	0.184	0.295	0.394	0.066	0.133	0.198	0.438	0.629	0.770	0.198	0.407	0.603	0.052	0.116	0.149
0.6 SN6	0.224	0.353	0.471	0.082	0.165	0.243	0.524	0.734	0.867	0.247	0.500	0.693	0.074	0.152	0.217
0.7 SN7	0.251	0.393	0.523	0.097	0.190	0.275	0.588	0.790	0.903	0.284	0.553	0.750	0.092	0.194	0.289
0.8 SN8	0.267	0.407	0.548	0.106	0.207	0.306	0.617	0.826	0.932	0.305	0.593	0.779	0.122	0.255	0.367
0.1 SAS1	0.042	0.053	0.053	0.020	0.033	0.040	0.101	0.098	0.100	0.038	0.048	0.067	0.015	0.029	0.038
0.2 SAS2	0.057	0.084	0.097	0.023	0.047	0.059	0.149	0.178	0.216	0.054	0.099	0.143	0.019	0.033	0.042
0.3 SAS3	0.091	0.133	0.180	0.034	0.062	0.095	0.227	0.312	0.413	0.093	0.186	0.284	0.027	0.052	0.069
0.4 SAS4	0.141	0.226	0.310	0.050	0.100	0.151	0.332	0.487	0.627	0.154	0.310	0.467	0.036	0.077	0.100
0.5 SAS5	0.211	0.344	0.459	0.071	0.158	0.226	0.476	0.677	0.815	0.223	0.449	0.635	0.052	0.113	0.166
0.6 SAS6	0.292	0.471	0.615	0.097	0.215	0.322	0.615	0.826	0.923	0.304	0.571	0.761	0.080	0.180	0.262
0.7 SAS7	0.398	0.623	0.761	0.141	0.309	0.440	0.742	0.920	0.978	0.383	0.662	0.823	0.128	0.285	0.392
0.8 SAS8	0.516	0.732	0.863	0.198	0.394	0.562	0.843	0.967	0.995	0.446	0.722	0.844	0.204	0.432	0.591
0.1 FAS1	0.044	0.048	0.054	0.022	0.030	0.044	0.099	0.090	0.096	0.037	0.045	0.063	0.015	0.032	0.033
0.2 FAS2	0.056	0.079	0.099	0.024	0.051	0.064	0.141	0.173	0.207	0.057	0.095	0.144	0.018	0.039	0.046
0.3 FAS3	0.090	0.127	0.167	0.034	0.070	0.096	0.208	0.293	0.376	0.085	0.164	0.262	0.024	0.051	0.068
0.4 FAS4	0.120	0.185	0.244	0.046	0.092	0.129	0.292	0.432	0.552	0.131	0.270	0.422	0.037	0.071	0.092
0.5 FAS5	0.154	0.242	0.326	0.058	0.125	0.176	0.390	0.571	0.696	0.180	0.366	0.556	0.050	0.102	0.140
0.6 FAS6	0.191	0.305	0.399	0.074	0.150	0.221	0.472	0.662	0.814	0.226	0.466	0.659	0.061	0.142	0.196
0.7 FAS7	0.223	0.356	0.457	0.087	0.177	0.255	0.536	0.745	0.871	0.260	0.525	0.728	0.091	0.193	0.274
0.8 FAS8	0.245	0.387	0.511	0.099	0.196	0.281	0.590	0.799	0.908	0.294	0.576	0.770	0.114	0.250	0.355
0.91 LN	0.709	0.916	0.976	0.263	0.514	0.704	0.973	0.999	1.000	0.303	0.353	0.392	0.329	0.663	0.831
0.95 FN	0.282	0.426	0.552	0.114	0.213	0.309	0.633	0.837	0.936	0.321	0.609	0.792	0.138	0.314	0.454
1 EXP	0.610	0.841	0.940	0.217	0.450	0.603	0.917	0.992	0.999	0.409	0.587	0.677	0.306	0.629	0.808

Table 2: Empirical power/level of the tests O_1, O_2 and O_3 for a variety of density functions and sample sizes.

	O_1			O_2			O_3		
η Dist.	$n=30$	$n=50$	$n=70$	$n=30$	$n=50$	$n=70$	$n=30$	$n=50$	$n=70$
0 N	0.013	0.021	0.029	0.004	0.011	0.013	0.000	0.002	0.002
0 C	0.003	0.003	0.003	0.002	0.002	0.002	0.000	0.000	0.000
0.1 NM1	0.012	0.021	0.038	0.005	0.010	0.018	0.000	0.002	0.003
0.2 NM2	0.022	0.083	0.174	0.010	0.042	0.096	0.000	0.002	0.006
0.3 NM3	0.070	0.241	0.453	0.030	0.112	0.263	0.000	0.005	0.012
0.4 NM4	0.130	0.402	0.657	0.040	0.167	0.370	0.001	0.015	0.050
0.1 SN1	0.014	0.036	0.050	0.006	0.015	0.028	0.000	0.002	0.004
0.2 SN2	0.025	0.068	0.122	0.009	0.030	0.070	0.000	0.004	0.009
0.3 SN3	0.042	0.130	0.254	0.014	0.052	0.130	0.000	0.009	0.020
0.4 SN4	0.064	0.221	0.415	0.018	0.087	0.216	0.001	0.011	0.040
0.5 SN5	0.092	0.319	0.553	0.025	0.125	0.303	0.001	0.023	0.073
0.6 SN6	0.116	0.398	0.668	0.030	0.159	0.369	0.003	0.040	0.109
0.7 SN7	0.142	0.450	0.729	0.037	0.182	0.411	0.003	0.054	0.140
0.8 SN8	0.152	0.505	0.772	0.041	0.196	0.442	0.004	0.059	0.156
0.1 SAS1	0.018	0.031	0.044	0.005	0.015	0.025	0.000	0.002	0.004
0.2 SAS2	0.026	0.066	0.111	0.008	0.026	0.057	0.000	0.003	0.010
0.3 SAS3	0.045	0.129	0.249	0.014	0.052	0.129	0.000	0.006	0.021
0.4 SAS4	0.066	0.230	0.419	0.018	0.091	0.219	0.001	0.013	0.044
0.5 SAS5	0.102	0.343	0.603	0.027	0.127	0.317	0.002	0.028	0.088
0.6 SAS6	0.143	0.477	0.745	0.033	0.171	0.389	0.002	0.057	0.161
0.7 SAS7	0.186	0.585	0.843	0.042	0.199	0.428	0.005	0.107	0.260
0.8 SAS8	0.228	0.647	0.883	0.045	0.198	0.420	0.011	0.162	0.379
0.1 FAS1	0.015	0.034	0.047	0.004	0.014	0.023	0.000	0.003	0.004
0.2 FAS2	0.023	0.070	0.114	0.008	0.026	0.057	0.000	0.005	0.010
0.3 FAS3	0.041	0.114	0.229	0.010	0.043	0.119	0.001	0.007	0.023
0.4 FAS4	0.056	0.192	0.369	0.017	0.081	0.194	0.001	0.014	0.040
0.5 FAS5	0.078	0.284	0.515	0.021	0.114	0.280	0.002	0.025	0.064
0.6 FAS6	0.106	0.364	0.617	0.027	0.148	0.359	0.002	0.039	0.095
0.7 FAS7	0.127	0.432	0.698	0.034	0.183	0.404	0.003	0.045	0.124
0.8 FAS8	0.140	0.482	0.744	0.038	0.205	0.431	0.004	0.059	0.143
0.91 LN	0.106	0.260	0.400	0.015	0.038	0.070	0.019	0.252	0.548
0.95 FN	0.157	0.509	0.790	0.041	0.213	0.456	0.004	0.070	0.173
1 EXP	0.181	0.495	0.711	0.032	0.116	0.232	0.016	0.201	0.441

Table 3: Empirical power/level of the tests based on T_1, T_2 and T_3 for a variety of density functions and sample sizes.

η	Dist.	T_1 $n=30$	T_1 $n=50$	T_1 $n=70$	T_2 $n=30$	T_2 $n=50$	T_2 $n=70$	T_3 $n=30$	T_3 $n=50$	T_3 $n=70$
0	N	0.040	0.037	0.038	0.004	0.017	0.023	0.012	0.016	0.020
0	C	0.037	0.036	0.035	0.006	0.018	0.026	0.020	0.023	0.028
0.1	NM1	0.049	0.055	0.059	0.006	0.025	0.040	0.020	0.028	0.038
0.2	NM2	0.075	0.113	0.147	0.008	0.056	0.098	0.044	0.074	0.117
0.3	NM3	0.128	0.229	0.321	0.020	0.131	0.251	0.088	0.188	0.303
0.4	NM4	0.202	0.392	0.573	0.031	0.257	0.460	0.136	0.330	0.528
0.1	SN1	0.046	0.054	0.060	0.005	0.025	0.036	0.017	0.026	0.033
0.2	SN2	0.068	0.100	0.127	0.006	0.049	0.087	0.028	0.049	0.088
0.3	SN3	0.105	0.179	0.261	0.014	0.097	0.184	0.055	0.113	0.199
0.4	SN4	0.167	0.296	0.425	0.021	0.175	0.328	0.091	0.223	0.366
0.5	SN5	0.251	0.446	0.591	0.033	0.277	0.468	0.154	0.351	0.531
0.6	SN6	0.359	0.574	0.737	0.045	0.367	0.607	0.218	0.501	0.701
0.7	SN7	0.443	0.681	0.841	0.051	0.406	0.670	0.300	0.610	0.791
0.8	SN8	0.506	0.757	0.883	0.043	0.370	0.679	0.361	0.690	0.868
0.1	SAS1	0.044	0.050	0.061	0.005	0.025	0.037	0.017	0.025	0.030
0.2	SAS2	0.069	0.096	0.130	0.008	0.050	0.084	0.028	0.052	0.082
0.3	SAS3	0.109	0.186	0.257	0.011	0.096	0.174	0.047	0.110	0.183
0.4	SAS4	0.170	0.311	0.436	0.023	0.185	0.329	0.090	0.221	0.358
0.5	SAS5	0.270	0.477	0.655	0.036	0.309	0.543	0.158	0.387	0.592
0.6	SAS6	0.408	0.671	0.833	0.055	0.455	0.729	0.265	0.579	0.792
0.7	SAS7	0.581	0.838	0.948	0.080	0.613	0.878	0.424	0.779	0.931
0.8	SAS8	0.738	0.936	0.987	0.079	0.656	0.934	0.607	0.917	0.983
0.1	FAS1	0.047	0.046	0.063	0.005	0.023	0.034	0.017	0.022	0.029
0.2	FAS2	0.070	0.098	0.128	0.006	0.044	0.080	0.025	0.049	0.079
0.3	FAS3	0.103	0.172	0.247	0.012	0.092	0.174	0.050	0.108	0.175
0.4	FAS4	0.170	0.281	0.402	0.021	0.161	0.292	0.079	0.190	0.310
0.5	FAS5	0.242	0.405	0.556	0.030	0.241	0.429	0.134	0.309	0.477
0.6	FAS6	0.326	0.541	0.687	0.036	0.317	0.542	0.198	0.436	0.636
0.7	FAS7	0.409	0.645	0.794	0.043	0.357	0.629	0.276	0.566	0.773
0.8	FAS8	0.482	0.729	0.866	0.040	0.343	0.641	0.333	0.658	0.851
0.91	LN	0.908	0.991	0.999	0.074	0.540	0.906	0.882	0.994	1.000
0.95	FN	0.550	0.790	0.910	0.035	0.305	0.618	0.392	0.744	0.903
1	EXP	0.866	0.981	0.998	0.037	0.296	0.676	0.784	0.976	0.998

Acknowledgements We are grateful to the two reviewers, whose constructive comments lead to the current version of the paper. This paper is based on work carried out as part of the PhD thesis of the first author, who gratefully acknowledges the funding received by the EPSRC. We are also thankful to Dr. Dimitrios Bagkavos for bringing the Kasuya (2010) reference to our notice.

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