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Evseev, Anton; Tsuchioka, Shunsuke

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ON GRADED CARTAN INVARIANTS OF SYMMETRIC GROUPS AND HECKE ALGEBRAS

ANTON EVSEEV AND SHUNSUKE TSUCHIOKA

Abstract. We consider graded Cartan matrices of the symmetric groups and the Iwahori-Hecke algebras of type A at roots of unity. These matrices are $\mathbb{Z}[v,v^{-1}]$-valued and may also be interpreted as Gram matrices of the Shapovalov form on sums of weight spaces of a basic representation of an affine quantum group. We present a conjecture predicting the invariant factors of these matrices and give evidence for the conjecture by proving its implications under a localization and certain specializations of the ring $\mathbb{Z}[v,v^{-1}]$. This proves and generalizes a conjecture of Ando-Suzuki-Yamada on the invariants of these matrices over $\mathbb{Q}[v,v^{-1}]$ and also generalizes the first author's recent proof of the Külshammer-Olsson-Robinson conjecture over $\mathbb{Z}$.

1. Introduction

The main object of study in this paper is the graded Cartan matrix $C^v_{\mathcal{H}_n(k_\ell;\eta_\ell)}$ of the Iwahori-Hecke algebra of type A (see Definition 1.1) in quantum characteristic ℓ, whose entries belong to the Laurent polynomial ring $\mathscr{A} = \mathbb{Z}[v,v^{-1}]$. To provide background and motivation, we begin by describing the relevant constructions and results for the ungraded case, obtained by substituting $v = 1$ (see §1.1). In §1.3 we move on to the graded case and state conjectures and results on the "invariant factors" of $C^v_{\mathcal{H}_n(k_\ell;\eta_\ell)}$, which are studied in the rest of the paper. We freely use the notation and conventions of §1.7.

1.1. **Generalized modular character theory of the symmetric groups.** In [KOR], Külshammer, Olsson, and Robinson initiated a study of an ℓ-analogue of the modular character theory of the symmetric group $\mathfrak{S}_n$ for an arbitrary integer $\ell \geq 2$. They showed that many of the classical combinatorial aspects of representation theory of $\mathfrak{S}_n$ over a field of a prime characteristic p (such as cores, blocks and Nakayama conjecture) generalize to the case when p is not necessarily a prime and is replaced by ℓ. Our interest focuses on the generalized Cartan matrices defined in [KOR, §1] (ℓ-Cartan matrices, for short) and, in particular, on their Smith normal forms over $\mathbb{Z}$. It is convenient to define ℓ-Cartan matrices in terms of Hecke algebras rather than the symmetric groups. Throughout, we consider the Hecke algebra $\mathcal{H}_n(k_\ell;\eta_\ell)$ defined as usual.

Definition 1.1. *For a field $\mathbb{F}$ and $q \in \mathbb{F}^\times$, $\mathcal{H}_n(\mathbb{F};q)$ is defined to be the $\mathbb{F}$-algebra generated by $\{T_r \mid 1 \leq r < n\}$ subject to the relations*

$$(T_r+1)(T_r-q)=0, \quad T_sT_{s+1}T_s = T_{s+1}T_sT_{s+1}, \quad T_tT_u = T_uT_t$$

for $1 \leq r \leq n-1$, $1 \leq s \leq n-2$ and $1 \leq t,u < n$ such that $|t-u|>1$. For $\ell \geq 2$, we fix a field k_ℓ which has a primitive ℓ-th root of unity η_ℓ.

Definition 1.2. *Let A be a finite-dimensional algebra over a field $\mathbb{F}$.*

(a) *We denote by $\mathsf{Mod}(A)$ the abelian category of finite-dimensional left A-modules and A-homomorphisms between them.*

(b) *We define the Cartan matrix C_A of A to be the matrix $([\mathsf{PC}(D) : D'])_{D,D' \in \mathsf{Irr}(\mathsf{Mod}(A))} \in \mathsf{Mat}_{|\mathsf{Irr}(\mathsf{Mod}(A))|}(\mathbb{Z})$ where $\mathsf{PC}(D)$ is the projective cover of D.*

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1.2. **The Külshammer-Olsson-Robinson conjecture.**

Definition 1.3. *Let X and Y be $n \times m$-matrices with entries in a commutative ring R. The matrices X and Y are said to be unimodularly equivalent over R if $Y = UXV$ for some $U \in \mathrm{GL}_n(R)$ and $V \in \mathrm{GL}_m(R)$. In this case, we write $X \equiv_R Y$.*

Due to a result of Donkin [Don, §2.2], the matrix $C_{\mathcal{H}_n(k_\ell;\eta_\ell)}$ is unimodularly equivalent over $\mathbb{Z}$ to the aforementioned ℓ-Cartan matrix of $\mathfrak{S}_n$. Since k_ℓ is a splitting field for $\mathcal{H}_n(k_\ell;\eta_\ell)$ (see also [Don, §2.2]), the Smith normal form of $C_{\mathcal{H}_n(k_\ell;\eta_\ell)}$ does not depend on the choice of k_ℓ or η_ℓ.

It is a standard result in modular representation theory (due to Brauer-Nesbitt) that, for a prime p and a finite group G, the elementary divisors of $C_{\overline{\mathbb{F}}_p G}$ are described in terms of p-defects of p-regular conjugacy classes of G. When p is replaced with a possibly composite number ℓ, the Smith normal form of $C_{\mathcal{H}_n(k_\ell;\eta_\ell)}$ is more complicated:

Theorem 1.4. *Let $\ell \geq 2$. If $k \in \mathbb{Z}$, write $\ell_k = \ell/(\ell,k)$. For a partition λ, define*

$$r_\ell(\lambda) = \prod_{k \in \mathbb{N}\setminus \ell\mathbb{Z}} \ell_k^{\lfloor \frac{m_k(\lambda)}{\ell} \rfloor} \cdot \lfloor \frac{m_k(\lambda)}{\ell} \rfloor !_{\pi(\ell_k)}. \tag{1.1}$$

Then

$$C_{\mathcal{H}_n(k_\ell;\eta_\ell)} \equiv_{\mathbb{Z}} \mathsf{diag}(\{r_\ell(\lambda) \mid \lambda \in \mathsf{CRP}_\ell(n)\}),$$

where $\pi(\ell_k)$ is the set of prime divisor of ℓ_k and $\mathsf{CRP}_\ell(n)$ is the set of ℓ-class regular partitions of n (see §1.7 below).

This result was proposed as a conjecture by Külshammer, Olsson and Robinson ([KOR, Conjecture 6.4]) and is known as the *KOR conjecture*. The determinant of the Cartan matrix $C_{\mathcal{H}_n(k_\ell;\eta_\ell)}$ was first computed by Brundan and Kleshchev [BK1, Corollary 1] and was shown to agree with the conjecture in [KOR]. Hill [Hil, Conjecture 10.5] gave a conjectural description of the invariant factors of the Cartan matrix of each individual block of $\mathcal{H}_n(k_\ell;\eta_\ell)$ and proved this description in the case when each prime divisor p of ℓ appears with multiplicity at most p in the prime decomposition of ℓ. The description was shown to imply Theorem 1.4 by Bessenrodt and Hill [BH, Theorem 5.2]. Finally, Hill's conjecture and hence Theorem 1.4 were proved in full generality by the first author [Evs, Theorem 1.1].

The proofs in [Hil] and [Evs] both use a reduction of the KOR conjecture to the problem of finding the Smith normal form of a certain $\mathsf{Par}(d) \times \mathsf{Par}(d)$-matrix which is smaller than $C_{\mathcal{H}_n(k_\ell;\eta_\ell)}$; here, d is not greater than the ℓ-weight of a fixed block of $\mathcal{H}_n(k_\ell;\eta_\ell)$. The reduction (for an individual block of $\mathcal{H}_n(k_\ell;\eta_\ell)$) is due to Hill: see [Hil, Theorem 1.1]; for an alternative approach, see [Evs, §3]. Among the main conjectures and results of the present paper are Conjecture 1.9, which is a graded version of the reduced problem, and Corollary 3.17, which is a graded version of the reduction. The ungraded versions are recovered by substituting $v = 1$.

1.3. **Graded Cartan matrices and Shapovalov forms.** While the KOR conjecture is now a theorem, the proof in [Evs] relies on technical combinatorial arguments and does not give a satisfactory conceptual understanding of the result. In particular, unlike in the special case when ℓ is a prime and the Brauer-Nesbitt result applies, it is hard to discern a link between the statement or the proof of the KOR conjecture and the group-theoretic structure of $\mathfrak{S}_n$. In a search for better understanding, we consider a remarkable grading on the Hecke algebras discovered independently by Brundan-Kleshchev [BK2, Theorem 1.1] and Rouquier [Ro1, Corollary 3.20]. It is a consequence of an isomorphism between $\mathcal{H}_n(k_\ell;\eta_\ell)$ and a cyclotomic KLR algebra $R_n^{\Lambda_0}(A_{\ell-1}^{(1)})$ defined by Khovanov-Lauda [KL, §3.4] and Rouquier [Ro1, §3.2.6]. A similar isomorphism and grading exist for the degenerate case, i.e., for the symmetric group algebra $\mathbb{F}_p\mathfrak{S}_n$ (see [BK2, Theorem 1.1] and [Ro1, Corollary 3.17]). Using the grading, one defines the graded Cartan matrix $C^v_{\mathcal{H}_n(k_\ell;\eta_\ell)}$ with entries in the ring $\mathscr{A} = \mathbb{Z}[v, v^{-1}]$ (see Definition 3.12). It is a refinement of $C_{\mathcal{H}_n(k_\ell;\eta_\ell)}$ in the sense that we have $C_{\mathcal{H}_n(k_\ell;\eta_\ell)} = C^v_{\mathcal{H}_n(k_\ell;\eta_\ell)}|_{v=1}$.

Remark 1.5. Rouquier [Ro2] has shown that interesting gradings are likely to exist for a large class of blocks of arbitrary finite groups. More precisely, he has constructed a grading on local blocks (i.e., blocks with normal defect group) whenever the defect group is abelian and has shown that, subject to the Broué abelian defect group conjecture, these gradings can be transferred to arbitrary blocks with abelian defect

groups. A study of the corresponding graded Cartan matrices up to unimodular equivalence may be of considerable interest, though is beyond the scope of this paper.

An alternative approach to defining $C_{\mathcal{H}_n(k_\ell;\eta_\ell)}$ is via the Shapovalov form on the basic representation $V(\Lambda_0)$ of the affine Kac-Moody Lie algebra of type $A^{(1)}_{\ell-1}$ (see [BK1, Hil]). Generalizing to the graded case is natural from this point of view as well, as one can replace the universal enveloping algebra of the Kac-Moody algebra with its quantized version $U_v(A^{(1)}_{\ell-1})$. The corresponding quantum Shapovalov forms were studied by the second author [Tsu] and are reviewed in §3.1 below. The matrix $C^v_{\mathcal{H}_n(k_\ell;\eta_\ell)}$ can be described in terms of Gram matrices of quantum Shapovalov forms on weight spaces of $V(\Lambda_0)$ (see Proposition 3.14). Since Shapovalov forms play an important role in representation theory of Lie algebras and quantum groups, this description provides further motivation for studying $C^v_{\mathcal{H}_n(k_\ell;\eta_\ell)}$.

1.4. **A graded analog of the Külshammer-Olsson-Robinson conjecture.** We propose the following graded version of the KOR conjecture.

Conjecture 1.6. For $\ell \geq 2$, we have (see also Definition 3.12)

$$C^v_{\mathcal{H}_n(k_\ell;\eta_\ell)} \equiv_{\mathscr{A}} \mathsf{diag}(\{r^v_\ell(\lambda) \mid \lambda \in \mathsf{CRP}_\ell(n)\}). \tag{1.2}$$

Here we put $\ell_k = \ell/(\ell,k)$ and for $\lambda \in \mathsf{Par}$ define

$$r^v_\ell(\lambda) = \prod_{k\geq 1} \prod_{t=1}^{\lfloor m_k(\lambda)/\ell \rfloor} \left[\ell_k t_{\pi(\ell_k)}\right]_{(\ell,k)t_{\pi(\ell_k)'}}, \tag{1.3}$$

where the right-hand side is interpreted according to §1.7.4 and §1.7.5.

The second author stated this conjecture in the special case when ℓ is a prime power (see [Tsu, Conjecture 6.18]) and computed the determinant of $C^v_{\mathcal{H}_n(k_\ell;\eta_\ell)}$, which agrees with the conjecture (see [Tsu, Theorem 6.11]).

Remark 1.7. Conjecture 1.6 implies Theorem 1.4: comparing (1.1) and (1.3), we have

$$\left(\prod_{t=1}^{\lfloor m_k(\lambda)/\ell \rfloor} \left[\ell_k t_{\pi(\ell_k)}\right]_{(\ell,k)t_{\pi(\ell_k)'}}\right)|_{v=1} = \ell_k^{\lfloor \frac{m_k(\lambda)}{\ell} \rfloor} \cdot \lfloor \frac{m_k(\lambda)}{\ell} \rfloor !_{\pi(\ell_k)}.$$

While $C^v_{\mathcal{H}_n(k_\ell;\eta_\ell)}$ has a description in terms of affine Kazhdan-Lusztig polynomials by virtue of the graded version of Lascoux-Leclerc-Thibon-Ariki theory [BK3, Corollary 5.15] (see also [Tsu, Remark 5.7]), there is no easy combinatorial description for the entries of $C^v_{\mathcal{H}_n(k_\ell;\eta_\ell)}$ in general. Nonetheless, we are able to reduce Conjecture 1.6 to a conjecture concerning matrices that do admit such a description up to unimodular equivalence over $\mathscr{A}$.

Definition 1.8. *For $\ell \geq 2$ and $\lambda \in \mathsf{Par}$, we define $I^v_\ell(\lambda), J^v_\ell(\lambda) \in \mathscr{A}$ by*

$$I^v_\ell(\lambda) = \prod_{k\geq 1} \prod_{t=1}^{m_k(\lambda)} \left[\ell_k t_{\pi(\ell_k)}\right]_{(\ell,k)t_{\pi(\ell_k)'}}, \quad J^v_\ell(\lambda) = \prod_{k\geq 1} [\ell]_k^{m_k(\lambda)}, \tag{1.4}$$

where again we put $\ell_k = \ell/(\ell,k)$.

The following conjecture involves a matrix M_n, which for the purposes of the statement may be assumed to be the character table of the symmetric group $\mathfrak{S}_n$ (see Definition 2.1 and Remark 2.2 for details).

Conjecture 1.9. For $\ell \geq 2$ and $n \geq 0$, we have the following unimodular equivalence over $\mathscr{A}$:

$$M_n\, \mathsf{diag}(\{J^v_\ell(\lambda) \mid \lambda \in \mathsf{Par}(n)\}) M_n^{-1} \equiv_{\mathscr{A}} \mathsf{diag}(\{I^v_\ell(\lambda) \mid \lambda \in \mathsf{Par}(n)\}). \tag{1.5}$$

In §3, we will show that Conjecture 1.9 implies Conjecture 1.6 (see Corollary 3.17). As is mentioned above, this generalizes a reduction for the ungraded case proved in [Hil, BH].

1.5. **Evidence for Conjecture 1.9.** Although there is no *a priori* reason to assert that $C^v_{\mathcal{H}_n(k_\ell;\eta_\ell)}$ is unimodularly equivalent to a diagonal matrix since $\mathscr{A}$ is not a principal ideal domain (PID, for short), we can give evidence that such an equivalence is likely to exist, which suggests that a hidden structure lies behind it and that one is unlikely to see this structure just by considering the ungraded case.

Theorem 1.10. *For $\ell \geq 2$ and $n \geq 0$, let X and D denote the matrices on the left-hand and right-hand sides of* (1.5)*. Then, we have*

(a) $X \equiv_{\mathbb{Q}[v,v^{-1}]} D$;
(b) *for any $0 \neq \theta \in \mathbb{Q}$, we have $X|_{v=\theta} \equiv_{\mathbb{Z}[\theta,\theta^{-1}]} D|_{v=\theta}$.*

Hence, the unimodular equivalence of Conjecture 1.6 holds over $\mathbb{Q}[v,v^{-1}]$ and holds over $\mathbb{Z}[\theta,\theta^{-1}]$ when one substitutes any $\theta \in \mathbb{Q}^\times$ for v.

The last statement follows from parts (a) and (b) due to Corollary 4.3.

Remark 1.11. We note the following consequence and special case:

(a) Combined with Proposition 3.15, Theorem 1.10 (a) settles affirmatively a conjecture of Ando-Suzuki-Yamada ([ASY, Conjecture 8.2]) and further generalizes it to the case of an arbitrary $\ell \geq 2$, not necessarily a prime.
(b) The case $\theta = 1$ of Theorem 1.10 (b) corresponds to the KOR conjecture (Theorem 1.4).

Our proof of Theorem 1.10 relies on the fact that the equivalences in the theorem are over PIDs (see Remark 6.1). In part, the proof is a generalization of the one in [Evs].

Since $\mathscr{A}$ is 2-dimensional, it appears that completely new ideas will be needed to prove a unimodular equivalence over $\mathscr{A}$. In particular, while the ungraded version of Conjecture 1.9 is easily reduced to the case when ℓ is a prime power (see [Hil]), there is no such apparent reduction in the graded case. The authors hope that this paper will help advertise Conjecture 1.9 (and its meaning) to a wide audience not restricted to representation theorists, as the conjecture is stated purely in the language of combinatorics and linear algebra.

1.6. **Organization of the paper.** In §2 we introduce the matrix M_n, which is the table of values of Young permutation characters of the symmetric group $\mathfrak{S}_n$. We also introduce a "p-local" and a multicolored version of M_n, and we prove a number of integrality results about these matrices that are needed later. In §3, we show how Conjecture 1.6 may be interpreted in terms of certain representations of quantum groups. We prove Theorem 3.10, which shows that the graded Cartan matrix $C^v_{\mathcal{H}_n(k_\ell;\eta_\ell)}$ (or $C^v_{\mathbb{F}_p\mathfrak{S}_n}$) is unimodularly equivalent to a block-diagonal matrix with blocks of the form given by the left-hand side of (1.5). Using this, we show that Conjecture 1.9 implies Conjecture 1.6. Theorem 1.10 is proved in §4 and §5. In §4, we prove Theorem 1.10 (a) and reduce Theorem 1.10 (b) to Theorem 4.14, which asserts a certain unimodular equivalence over the local ring $\mathbb{Z}_{(p)}$ and is proved in §5. In §6 (and §4.1), we discuss unimodular equivalences over arbitrary commutative rings and possible results that would be stronger than Theorem 1.10 but weaker than Conjecture 1.9, including possible further evidence in terms of equivalences over PIDs.

1.7. **Notation and conventions.**

1.7.1. *Commutative rings.* All commutative rings are assumed to contain a multiplicative identity, and homomorphisms between commutative rings are assumed to respect those identities. We denote by $\mathsf{max\text{-}Spec}(R)$ the set of maximal ideals of a commutative ring R.

1.7.2. *Matrices.* Let R be a commutative ring. For any integer $\ell \geq 0$, we denote by $\mathsf{Mat}_\ell(R)$ the algebra of all R-valued $\ell \times \ell$-matrices. More generally, $\mathsf{Mat}_S(R)$ is the algebra of $S \times S$-matrices for any finite set S. For a finite set S, 1_S denotes the identity $S \times S$-matrix. For an assignment $S \to R$, $s \mapsto r_s$, we denote by $\mathsf{diag}(\{r_s \mid s \in S\})$ the diagonal matrix with the (s,t)-entry equal to $\delta_{st} r_s$ for all $s,t \in S$. We often denote by M_{rs} the (r,s)-entry of a matrix M. If $S = \bigsqcup_i S_i$ is a disjoint union and $M_i \in \mathsf{Mat}_{S_i}(R)$ for each i, then $M = \bigoplus_i M_i$ is the block-diagonal matrix given by $M_{rs} = (M_i)_{rs}$ if r and s belong to the same subset S_i and $M_{rs} = 0$ otherwise. We say that matrices $X, Y \in \mathsf{Mat}_m(R)$ are row (resp. column) equivalent over R if there exists $U \in \mathrm{GL}_m(R)$ such that $X = UY$ (resp. $X = YU$).

1.7.3. *Discrete valuation rings.* When considering a discrete valuation ring R with valuation $\nu\colon K^\times \twoheadrightarrow \mathbb{Z}$, where K is the field of fractions of R, we set $\nu(0)=\infty$ where ∞ is a symbol satisfying $\infty > c$ for all $c\in\mathbb{Q}$. For a prime p, the valuation $\nu_p\colon \mathbb{Q}^\times \twoheadrightarrow \mathbb{Z}$ is defined by $\nu_p(p^m a/b)=m$ for $m\in\mathbb{Z}$ and $a,b\in\mathbb{Z}\setminus p\mathbb{Z}$. It corresponds to the discrete valuation ring $\mathbb{Z}_{(p)}=\{a/b\in\mathbb{Q}\mid b\notin p\mathbb{Z}\}$.

1.7.4. *Integers.* We write $\mathbb{N}=\{0,1,2,\dots\}$ and Prm for the set of all prime numbers. For $n\geq 1$, we denote by $\pi(n)$ the set of all prime divisors of n. For $n\geq 1$ and a subset $\Pi\subseteq\mathsf{Prm}$, we define the Π-part of n by $n_\Pi=\prod_{p\in\Pi} p^{\nu_p(n)}$. We write $\Pi'=\mathsf{Prm}\setminus\Pi$ and $p'=\mathsf{Prm}\setminus\{p\}$ for all $p\in\mathsf{Prm}$. For $a,b\geq 1$, (a,b) is the greatest common divisor of a and b.

1.7.5. *Quantum rings.* Let v be an indeterminate. In much of the paper, we work over the field $\mathbf{k}=\mathbb{Q}(v)$ and its subring $\mathscr{A}=\mathbb{Z}[v,v^{-1}]$. The $\mathbb{Q}$-algebra involution $\mathsf{bar}\colon \mathbf{k}\to\mathbf{k}$ is defined by $\mathsf{bar}(v)=v^{-1}$. For $t\in\mathbb{Z}$, we write $\mathsf{Infl}_t\colon \mathscr{A}\to\mathscr{A}$ for the ring homomorphism given by $v\mapsto v^t$. For $m\geq 1$ and $n\in\mathbb{Z}$, the quantum integer $[n]_m$ is defined by $[n]_m=(v^{mn}-v^{-mn})/(v^m-v^{-m})\in\mathscr{A}$. Note that $[n]_m|_{v=1}=n$. We set $[n]_m!=[n]_m[n-1]_m\cdots[1]_m$. For a field $\mathbb{F}$ and $q\in\mathbb{F}^\times$, the quantum characteristic of q is defined by $\mathsf{qchar}_q\,\mathbb{F}=\min\{k\geq 1\mid [k]|_{v=q}=0\}$ if the set on the right-hand side is non-empty and is set to be 0 otherwise.

1.7.6. *Groups and generalized characters.* Let G be a finite group. If R is a subring of $\mathbb{C}$, we say that a function $\chi\colon G\to\mathbb{C}$ is an *R-generalized character* of G if χ belongs to the R-span of the irreducible characters of G. By a *generalized character* we mean a $\mathbb{Z}$-generalized character. If $g,h\in G$, we write $g\equiv_G h$ if g and h are G-conjugate. If p is a prime, then, as usual, $g_p, g_{p'}\in\langle g\rangle\subseteq G$ are the p-part and the p'-part of g respectively, so that $g=g_pg_{p'}=g_{p'}g_p$, the order of g_p is a p-power and the order of $g_{p'}$ is prime to p.

1.7.7. *Partitions.* We write $\varnothing$ for the empty partition. For a partition $\lambda=(\lambda_1,\lambda_2,\dots)$, we define $m_k(\lambda)=|\{i\geq 1\mid \lambda_i=k\}|$ for $k\geq 1$. Also, $\ell(\lambda)=\sum_{i\geq 1} m_i(\lambda)$ and $|\lambda|=\sum_{i\geq 1}\lambda_i$. We denote by $\mathsf{Par}(n)$ (resp. $\mathsf{CRP}_s(n), \mathsf{RP}_s(n)$) the set of all (resp. s-class regular, s-regular) partitions of $n\geq 0$. Recall that, for $s\geq 1$, a partition λ is called

(i) *s-class regular* if we have $m_{ks}(\lambda)=0$ for all $k\geq 1$,
(ii) *s-regular* if we have $m_k(\lambda)<s$ for all $k\geq 1$.

We put $\mathsf{Par}=\bigsqcup_{n\geq 0}\mathsf{Par}(n)$ and $\mathsf{Par}_m(n)=\{(\lambda^{(i)})_{i=1}^m\in\mathsf{Par}^m\mid \sum_{i=1}^m|\lambda^{(i)}|=n\}$ for $m,n\geq 0$.

For $n\geq 0$, $p\in\mathsf{Prm}$ and $\nu\in\mathsf{CRP}_p(n)$, we define $\mathsf{Par}_p(n,\nu)=\{\lambda\in\mathsf{Par}(n)\mid \sum_{s\geq 0} m_{jp^s}(\lambda)p^s=m_j(\nu)\,\forall j\in\mathbb{N}\setminus p\mathbb{Z}\}$. Further, $\mathsf{Pow}_p(n)=\mathsf{Par}_p(n,(1^n))$ and $\mathsf{Pow}_p=\bigsqcup_{n\geq 0}\mathsf{Pow}_p(n)$ is the set of the partitions with all parts being powers of p.

For $\lambda,\mu\in\mathsf{Par}$, the partition $\lambda+\mu$ is defined by $m_i(\lambda+\mu)=m_i(\lambda)+m_i(\mu)$ for $i\geq 1$.

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2. The matrix M_n

2.1. **Definition of M_n.** As usual, let $\Lambda=\bigoplus_{n\geq 0}\varprojlim_{m\geq 0}\mathbb{Z}[u_1,\dots,u_m]_n^{\mathfrak{S}_m}$ be the ring of symmetric functions (see [Ful, §6] or [Mac, §I.2]) where $\mathbb{Z}[u_1,\dots,u_m]_n$ is the set of homogeneous polynomials of degree n.

The ring Λ is categorified by the module categories $\{\mathsf{Mod}(\mathbb{Q}\mathfrak{S}_n)\}_{n\geq 0}$. More precisely, let χ_V denote the character afforded by a module $V\in\mathsf{Mod}(\mathbb{Q}\mathfrak{S}_n)$. For $\mu\in\mathsf{Par}$, consider the power sum symmetric function $p_\mu=\prod_{i=1}^{\ell(\mu)}p_{\mu_i}$, where $p_k=\sum_{j\geq 1}u_j^k$ for $k\geq 1$. Let C_μ be the conjugacy class of elements of cycle type μ in $\mathfrak{S}_n$. For $\mu\in\mathsf{Par}$, let

$$z_\mu=\prod_{i\geq 1} m_i(\mu)!\cdot i^{m_i(\mu)}, \tag{2.1}$$

so that $\#C_\mu=|\mu|!/z_\mu$. Then the following character map is an isometry (see [Ful, §7.3]):

$$\mathsf{ch}\colon \bigoplus_{n\geq 0}\mathsf{K}_0(\mathsf{Mod}(\mathbb{Q}\mathfrak{S}_n))\xrightarrow{\sim}\Lambda,\quad [V]\longmapsto\sum_{\mu\in\mathsf{Par}(n)}\frac{1}{z_\mu}\chi_V(C_\mu)p_\mu, \tag{2.2}$$

where we write $\chi_V(C_\mu)$ for the value of χ_V on an arbitrary element of C_μ.

Definition 2.1. *Let $\lambda, \mu \in \mathsf{Par}(n)$. Consider the parabolic subgroup*

$$\mathfrak{S}_\lambda = \mathsf{Aut}(\{1, \dots, \lambda_1\}) \times \mathsf{Aut}(\{\lambda_1 + 1, \dots, \lambda_1 + \lambda_2\}) \times \dots \cong \otimes_{i \geq 1} \mathfrak{S}_{\lambda_i}$$

of $\mathfrak{S}_n$, and let $\mathsf{triv}_{\mathfrak{S}_\lambda}$ be its trivial representation. We set $M_{\lambda,\mu} = \chi_{\mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \mathsf{triv}_{\mathfrak{S}_\lambda}}(C_\mu)$ and put $M_n = (M_{\lambda,\mu}) \in \mathsf{Mat}_{\mathsf{Par}(n)}(\mathbb{Z})$.

Remark 2.2. Recall the complete symmetric function $h_\mu = \prod_{i \geq 1} h_{\mu_i}$ for $\mu \in \mathsf{Par}$ where

$$\sum_{n \geq 0} h_n t^n = \prod_{i \geq 1} (1 - u_i t)^{-1} = \prod_{r=1}^{\infty} \exp\left(\frac{p_r t^r}{r}\right). \tag{2.3}$$

There is a well-known identity $\mathsf{ch}([\mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \mathsf{triv}_{\mathfrak{S}_\lambda}]) = h_\lambda$ for $\lambda \in \mathsf{Par}(n)$ (see [Ful, §7.2, Lemma 4]). Further, we have

$$h_\lambda = \sum_{\mu \in \mathsf{Par}(n)} \frac{1}{z_\mu} M_{\lambda,\mu} p_\mu, \quad p_\lambda = \sum_{\mu \in \mathsf{Par}(n)} M_{\mu,\lambda} m_\mu \tag{2.4}$$

for $\lambda \in \mathsf{Par}(n)$, where m_μ is the monomial symmetric function (i.e., the function whose image in $\mathbb{Z}[u_1, \dots, u_m]_n$ for $m \geq \ell(\lambda)$ is the sum of the elements of the orbit of the monomial $\prod_{j=1}^{\ell(\mu)} u_j^{\mu_j}$ under the action of $\mathfrak{S}_m$ on the variables); see [Ful, §6, (11), (12)]. Using the second identity (2.4), we see that $M_{\lambda,\mu}$ has the following explicit combinatorial descriptions:

(a) $M_{\lambda,\mu}$ is the coefficient of $\prod_{j=1}^{\ell(\lambda)} u_j^{\lambda_j}$ in $\prod_{i \geq 1} (u_1^i + \dots + u_{\ell(\lambda)}^i)^{m_i(\mu)}$,
(b) $M_{\lambda,\mu} = \#\mathcal{M}_{\lambda,\mu}$ where

$$\mathcal{M}_{\lambda,\mu} = \{f \colon \{1, \dots, \ell(\mu)\} \to \{1, \dots, \ell(\lambda)\} \mid \textstyle\sum_{j \in f^{-1}(i)} \mu_j = \lambda_i \text{ whenever } 1 \leq i \leq l(\lambda)\}.$$

Remark 2.3. It is well known that the $\mathbb{Z}$-span of $\{\chi_{\mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \mathsf{triv}_{\mathfrak{S}_\lambda}} \mid \lambda \in \mathsf{Par}(n)\}$ is the whole set of generalized characters of $\mathfrak{S}_n$ (see [Ful, §7.2, Corollary]); equivalently, the matrix M_n is row equivalent over $\mathbb{Z}$ to the character table of $\mathfrak{S}_n$ (in which, as usual, rows correspond to irreducible characters and columns to conjugacy classes, labeled by their cycle types). Therefore, as we claimed in §1.4, the matrix on the left-hand side of (1.5) stays in the same unimodular equivalence class if one replaces M_n by the character table of $\mathfrak{S}_n$.

In the remainder of this section, we prove a number of results on the matrix M_n and some of its analogues, mainly of a combinatorial nature. Proposition 2.4 will not be used until §5.4. The results in §2.2 are used in §4 and §5, whereas the results of §2.3 are needed in §3.

Proposition 2.4. *Let $n \geq 0$ and let $\lambda, \mu \in \mathsf{Par}(n)$.*

(a) *$M_{\lambda,\lambda} = \prod_{j \geq 1} m_j(\lambda)!$ and $M_{\lambda,\lambda}$ divides $M_{\lambda,\mu}$;*
(b) *$\ell(\lambda) \leq \ell(\mu)$ if $M_{\lambda,\mu} > 0$;*
(c) *Let $p \geq 3$ be a prime, and assume that $M_{\lambda,\mu} > 0$ and $\lambda \neq \mu$. Then $\nu_p(M_{\lambda,\mu}) > \ell(\lambda) - \ell(\mu) + \sum_{j \geq 1} \nu_p(m_j(\mu)!)$.*

Proof. (a) and (b) follow immediately from the combinatorial descriptions in Remark 2.2. To prove (c), let C be the set of maps $c \colon \{1, \dots, \ell(\lambda)\} \to \mathsf{Par} \setminus \{\varnothing\}$ such that $\sum_{k=1}^{\ell(\lambda)} c(k) = \mu$ and $|c(k)| = \lambda_k$ for $1 \leq k \leq \ell(\lambda)$. For $c \in C$, we define $\mathcal{M}_{\lambda\mu}^c$ to be the set of maps $f \in \mathcal{M}_{\lambda\mu}$ such that, whenever $1 \leq k \leq \ell(\lambda)$, there is a multiset equality

$$\{\mu_j \mid j \in f^{-1}(k)\} = \{c(k)_j \mid 1 \leq j \leq \ell(c(k))\}.$$

It is clear that $\mathcal{M}_{\lambda,\mu} = \bigsqcup_{c \in C} \mathcal{M}_{\lambda,\mu}^c$ (thus, we have $C \neq \emptyset$) and

$$\#\mathcal{M}_{\lambda,\mu}^c = \prod_{j \geq 1} \binom{m_j(\mu)}{m_j(c(1)), m_j(c(2)), \dots, m_j(c(\ell(\lambda)))}.$$

It suffices to prove that $\nu_p(\#\mathcal{M}^c_{\lambda,\mu}) > \ell(\lambda) - \ell(\mu) + \sum_{j\geq 0}\nu_p(m_j(\mu)!)$ for $c \in C$. By Lemma 2.5,

$$\nu_p(\#\mathcal{M}^c_{\lambda,\mu}) - \nu_p(m_j(\mu)!) - \ell(\lambda) + \ell(\mu) = \sum_{k=1}^{\ell(\lambda)} \left(\ell(c(k)) - \sum_{j\geq 1}\nu_p(m_j(c(k))!) - 1 \right) \geq 0$$

and the equality holds exactly when $\ell(c(k)) = 1$ for $1 \leq k \leq \ell(\lambda)$, i.e., when $\lambda = \mu$. □

Lemma 2.5. *Let $p \geq 3$ be a prime and $\lambda \in \mathsf{Par} \setminus \{\varnothing\}$. We have $\ell(\lambda) - \sum_{j\geq 1}\nu_p(m_j(\lambda)!) \geq 1$, and the equality holds exactly when $\ell(\lambda) = 1$.*

Proof. Note that

$$\nu_p(a!) = \sum_{i=1}^{\infty} \lfloor a/p^i \rfloor \leq \sum_{i=1}^{\infty} a/p^i = a/(p-1) \tag{2.5}$$

for $a \geq 0$. Thus,

$$\ell(\lambda) - \sum_{j\geq 1}\nu_p(m_j(\lambda)!) \geq (1 - 1/(p-1))\,\ell(\lambda) > 1$$

when $\ell(\lambda) \geq 3$. When $\ell(\lambda) = 1, 2$, we have $\nu_p(m_j(\lambda)!) = 0$ for all $j \geq 1$. □

2.2. **p-local version $N_n^{(p)}$ of M_n.** As in [Evs, §4], we consider a submatrix $N_n^{(p)}$ of M_n and use it to construct a certain block-diagonal matrix $L_n^{(p)}$, which is row equivalent over $\mathbb{Z}_{(p)}$ to M_n, for any fixed prime p.

Definition 2.6. *For $p \in \mathsf{Prm}$ and $n \geq 0$, we define $N_n^{(p)} = M_n|_{\mathsf{Pow}_p(n)\times\mathsf{Pow}_p(n)}$ and*

$$L_n^{(p)} = \bigoplus_{\nu\in\mathsf{CRP}_p(n)} \bigotimes_{j\in\mathbb{N}\setminus p\mathbb{Z}} N^{(p)}_{m_j(\nu)}.$$

We regard $L_n^{(p)}$ as an element of $\mathsf{Mat}_{\mathsf{Par}(n)}(\mathbb{Z})$ by using the following identification:

(a) $\mathsf{Par}(n) = \bigsqcup_{\nu\in\mathsf{CRP}_p(n)} \mathsf{Par}_p(n,\nu)$,

(b) $\mathsf{Par}_p(n,\nu) \xrightarrow{\sim} \prod_{j\in\mathbb{N}\setminus p\mathbb{Z}} \mathsf{Pow}_p(m_j(\nu))$, $\lambda \longmapsto (\lambda^{(j)})_{j\in\mathbb{N}\setminus p\mathbb{Z}}$ where $m_{p^s}(\lambda^{(j)}) = m_{jp^s}(\lambda)$.

Proposition 2.7. *Let $p \in \mathsf{Prm}$. For a $\mathbb{Z}_{(p)}$-algebra R and a family of homomorphisms $(r_j\colon R \to R)_{j\in\mathbb{N}\setminus p\mathbb{Z}}$, assume that*

(i) *there are maps $f, g\colon \mathsf{Par} \to R$ such that*

$$f(\lambda) = \prod_{j\in\mathbb{N}\setminus p\mathbb{Z}} r_j(f(\lambda^{(j)})), \quad g(\lambda) = \prod_{j\in\mathbb{N}\setminus p\mathbb{Z}} r_j(g(\lambda^{(j)}))$$

for all $k \geq 0$, $\nu \in \mathsf{CRP}_p(k)$ and $\lambda \in \mathsf{Par}_p(k,\nu)$, where the assignment $\lambda \mapsto (\lambda^{(j)})_{j\in\mathbb{N}\setminus p\mathbb{Z}}$ is defined as above.

(ii) *for all $n \geq 0$, $M_n\,\mathsf{diag}(\{f(\lambda) \mid \lambda \in \mathsf{Par}(n)\})M_n^{-1}$ is R-valued.*

Then, we have

(a) *$N_k^{(p)}\,\mathsf{diag}(\{f(\lambda) \mid \lambda \in \mathsf{Pow}_p(k)\})(N_k^{(p)})^{-1}$ is R-valued for all $k \geq 0$,*

(b) *For a $\mathbb{Z}_{(p)}$-algebra R' with a homomorphism $\phi\colon R \to R'$, the following implication holds:*

$$\begin{aligned}&\forall k \geq 0, \phi(N_k^{(p)}\,\mathsf{diag}(\{f(\lambda) \mid \lambda \in \mathsf{Pow}_p(k)\})(N_k^{(p)})^{-1}) \equiv_{R'} \phi(\mathsf{diag}(\{g(\lambda) \mid \lambda \in \mathsf{Pow}_p(k)\}))\\ &\Longrightarrow \forall n \geq 0, \phi(M_n\,\mathsf{diag}(\{f(\lambda) \mid \lambda \in \mathsf{Par}(n)\})M_n^{-1} \equiv_{R'} \phi(\mathsf{diag}(\{g(\lambda) \mid \lambda \in \mathsf{Par}(n)\})).\end{aligned}$$

Proof. By [Evs, Lemma 4.8], the matrices M_n and $L_n^{(p)}$ are row equivalent over $\mathbb{Z}_{(p)}$ and hence over R. Thus, by (ii) we have

$$M_n\,\mathsf{diag}(\{f(\lambda) \mid \lambda \in \mathsf{Par}(n)\})M_n^{-1} \equiv_R L_n^{(p)}\,\mathsf{diag}(\{f(\lambda) \mid \lambda \in \mathsf{Par}(n)\})(L_n^{(p)})^{-1}.$$

By (i), the right-hand side is just

$$\bigoplus_{\nu\in\mathsf{CRP}_p(n)} \bigotimes_{j\in\mathbb{N}\setminus p\mathbb{Z}} N^{(p)}_{m_j(\nu)}\,\mathsf{diag}(\{r_j(f(\lambda^{(j)})) \mid \lambda^{(j)} \in \mathsf{Pow}_p(m_j(\nu))\})(N^{(p)}_{m_j(\nu)})^{-1}.$$

We have shown that $N_n^{(p)}\,\mathsf{diag}(\{f(\lambda) \mid \lambda \in \mathsf{Pow}_p(n)\})(N_n^{(p)})^{-1}$ is a block submatrix of an R-valued matrix which is unimodularly equivalent to $M_n\,\mathsf{diag}(\{f(\lambda) \mid \lambda \in \mathsf{Par}(n)\})M_n^{-1}$ over R (note that, by (i), $r_1(f(\lambda)) = f(\lambda)$ for $\lambda \in \mathsf{Pow}_p(n)$). Thus, (a) is proved. Part (b) follows from the above equivalences and hypothesis (i). □

Our next aim is to prove an integrality result (Proposition 2.12), which will be used in §5.3.

Definition 2.8. *Let $p \in \mathsf{Prm}$. For a sequence $\theta = (\theta_j)_{j\geq 0} \in \mathbb{Z}_{(p)}^{\mathbb{N}}$ and $n \geq 0$, we define*

$$a_\theta^{(p)}(n) = \sum_{\nu \in \mathsf{Pow}_p(n)} \frac{1}{z_\nu} \prod_{j\geq 0} \theta_j^{m_{p^j}(\nu)}.$$

Lemma 2.9. *Let $p \in \mathsf{Prm}$. For any $\theta \in \mathbb{Z}_{(p)}^{\mathbb{N}}$ and $n \geq 0$, we have*

(a) $a_{\theta+\theta'}^{(p)}(n) = \sum_{k=0}^n a_\theta^{(p)}(k) a_{\theta'}^{(p)}(n-k)$, *where* $(\theta+\theta')_j := \theta_j + \theta'_j$ *for* $j \geq 0$,
(b) $a_\theta^{(p)}(n) \in \mathbb{Z}_{(p)}$ *if* $\nu_p(\theta_j) \geq j+1$ *for all* $j \geq 0$,
(c) $a_{(\theta)}^{(p)}(n) \in \mathbb{Z}_{(p)}$ *if there exist* $s \in \mathbb{Z}_{\geq 1}$ *and* $c \in \mathbb{Z}_{(p)}$ *such that* $\theta_j = sc^{p^j}$ *for all* $j \geq 0$.

Proof. Consider the generating function $A_\theta = \sum_{n\geq 0} a_\theta^{(p)}(n) t^n$. By a straightforward calculation similar to the one in the proof of [Mac, Equation (I.2.14)], we obtain the identity $A_\theta = \exp(\sum_{j\geq 0} p^{-j}\theta_j t^{p^j})$. Hence, $A_{\theta+\theta'} = A_\theta A_{\theta'}$, and part (a) follows by equating coefficients in t^n. Part (b) follows from the identity

$$a_\theta^{(p)}(n) = \sum_{\nu \in \mathsf{Pow}_p(n)} \prod_{j\geq 0} \frac{1}{m_{p^j}(\nu)!} \left(\frac{\theta_j}{p^j}\right)^{m_{p^j}(\nu)}.$$

and the inequality $\nu_p(d!) \leq d$ (see (2.5)).

To prove (c), we recall a corollary of Brauer's characterization of characters. Let G be a finite group. Then the characteristic function of a p'-section $\mathsf{Sec}_{p'}(x) := \{y \in G \mid y_{p'} \equiv_G x\}$ of any p'-element $x \in G$ is an $\mathscr{O}$-generalized character of G (see [Isa, Lemma 8.19]) for a certain DVR $\mathscr{O}$ with $\mathbb{Z}_{(p)} \subseteq \mathscr{O} \subseteq \mathbb{C}$. In particular, the characteristic function of $\mathsf{Sec}_{p'}(1_{\mathfrak{S}_n}) = \bigsqcup_{\nu \in \mathsf{Pow}_p(n)} C_\nu$ is an $\mathscr{O}$-generalized character of $\mathfrak{S}_n$.

We denote by $\langle \cdot, \cdot \rangle_G$ the usual inner product on the complex-valued class functions on G, so that $\{\chi_V \mid V \in \mathsf{Irr}(\mathsf{Mod}(\mathbb{C}G))\}$ is an orthonormal basis. Due to (a), we may assume that $s = 1$, so that $\theta_j = c^{p^j}$ for all j. We have

$$a_\theta^{(p)}(n) = \sum_{\nu \in \mathsf{Pow}_p(n)} z_\mu^{-1} c^n = c^n \langle \chi_{\mathsf{triv}_{\mathfrak{S}_n}}|_{\bigsqcup_{\nu \in \mathsf{Pow}_p(n)} C_\nu}, \chi_{\mathsf{triv}_{\mathfrak{S}_n}} \rangle_{\mathfrak{S}_n} \in \mathbb{Q} \cap \mathscr{O} = \mathbb{Z}_{(p)}.$$ □

Proposition 2.10. *Let $R \subseteq \mathbb{C}$ be a ring, and consider a map $\xi\colon \mathsf{Par}(n) \to \mathbb{C}$ be a map for some $n \geq 0$. If the class function ξ^{cl} defined by $\xi^{\mathsf{cl}}(C_\lambda) = \xi(\lambda)$ for $\lambda \in \mathsf{Par}(n)$ is an R-generalized character of $\mathfrak{S}_n$, then $M_n\,\mathsf{diag}(\{\xi(\lambda) \mid \lambda \in \mathsf{Par}(n)\})M_n^{-1}$ is R-valued.*

Proof. Let $T_n = (\chi_V(C_\lambda))_{V \in \mathsf{Irr}(\mathsf{Mod}(\mathbb{Q}\mathfrak{S}_n)),\, \lambda \in \mathsf{Par}(n)}$ be the character table of $\mathfrak{S}_n$. Then, for $V, W \in \mathsf{Irr}(\mathsf{Mod}(\mathbb{Q}\mathfrak{S}_n))$, the (V,W)-entry of $T_n\,\mathsf{diag}(\{\xi(\lambda) \mid \lambda \in \mathsf{Par}(n)\})T_n^{-1}$ is equal to $\langle \xi^{\mathsf{cl}}\chi_V, \chi_W \rangle_{\mathfrak{S}_n}$. Indeed, we have

$$\langle \xi^{\mathsf{cl}}\chi_V, \chi_W \rangle_{\mathfrak{S}_n} = \sum_{\lambda \in \mathsf{Par}(n)} \frac{1}{z_\lambda} \chi_V(C_\lambda)\xi(\lambda)\chi_W(C_\lambda),$$

and $z_\lambda^{-1}\chi_W(C_\lambda)$ is the (λ, W)-entry of T_n^{-1} due to the orthogonality relations. The result follows since M_n and T_n are row equivalent over $\mathbb{Z}$ (see Remark 2.3). □

Corollary 2.11. *Let $p \in \mathsf{Prm}$ and $n \geq 0$. For a map $\xi\colon \mathsf{Pow}_p(n) \to \mathbb{C}$, if the class function ξ^{cl} defined by*

$$\xi^{\mathsf{cl}}(C_\lambda) = \begin{cases} \xi(\lambda) & \text{if } \lambda \in \mathsf{Pow}_p(n), \\ 0 & \text{if } \lambda \in \mathsf{Par}(n) \setminus \mathsf{Pow}_p(n) \end{cases}$$

is a $\mathbb{Z}_{(p)}$-generalized character of $\mathfrak{S}_n$, then $N_n^{(p)}\,\mathsf{diag}(\{\xi(\lambda) \mid \lambda \in \mathsf{Pow}_p(n)\})(N_n^{(p)})^{-1}$ is $\mathbb{Z}_{(p)}$-valued.

Proof. Put $\widehat{M}_n = \bigoplus_{\nu\in\mathsf{CRP}_p(n)} M_n|_{\mathsf{Par}_p(n,\nu)\times\mathsf{Par}_p(n,\nu)} \in \mathsf{Mat}_{\mathsf{Par}(n)}(\mathbb{Z})$. Then M_n and $\widehat{M}_n$ are row equivalent over $\mathbb{Z}_{(p)}$ by [Evs, Lemma 4.6]. Thus, by Proposition 2.10, $\widehat{M}_n\,\mathsf{diag}(\{\xi^{\mathsf{cl}}(C_\lambda) \mid \lambda \in \mathsf{Par}(n)\})\widehat{M}_n^{-1} \in \mathsf{Mat}_{\mathsf{Par}(n)}(\mathbb{Z}_{(p)})$. Now $N_n^{(p)}\,\mathsf{diag}(\{\xi(\lambda) \mid \lambda \in \mathsf{Pow}_p(n)\})(N_n^{(p)})^{-1}$ is simply the $\mathsf{Pow}_p(n)\times\mathsf{Pow}_p(n)$-submatrix of this matrix, so the result follows. □

Proposition 2.12. *Let $p \in \mathsf{Prm}$ and $n \geq 0$, $\ell \geq 2$ be integers. Put $r = \nu_p(\ell)$. Then, for any $a/b \in \mathbb{Z}_{(p)}$ with $a, b \in \mathbb{Z}\setminus p\mathbb{Z}$ and $a^2 - b^2 \in p\mathbb{Z}$, we have*

$$N_n^{(p)}\,\mathsf{diag}(\{p^{-r\ell(\lambda)}\prod_{j\geq 0}[\ell]_{p^{r+j}}^{m_{p^j}(\lambda)}|_{v=a/b} \mid \lambda\in\mathsf{Pow}_p(n)\})(N_n^{(p)})^{-1} \in \mathsf{Mat}_{\mathsf{Pow}_p(n)}(\mathbb{Z}_{(p)}).$$

Proof. Put $\theta = (\theta_j)_{j\geq 0} \in \mathbb{Z}_{(p)}^{\mathbb{N}}$ where $\theta_j = p^{-r}[\ell]_{p^{r+j}}|_{v=a/b}$. Consider the map $\xi\colon \mathsf{Pow}_p(n) \to \mathbb{Q}$ given by $\nu \mapsto \prod_{j\geq 0}\theta_j^{m_{p^j}(\nu)}$. By Corollary 2.11, it is enough to show that ξ^{cl} is a $\mathbb{Z}_{(p)}$-generalized character of $\mathfrak{S}_n$. By Frobenius reciprocity, for all $\lambda \in \mathsf{Par}(n)$ we have

$$\langle \xi^{\mathsf{cl}}, \chi_{\mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}\mathsf{triv}_{\mathfrak{S}_\lambda}}\rangle_{\mathfrak{S}_n} = \langle \mathrm{Res}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}\xi^{\mathsf{cl}}, \chi_{\mathsf{triv}_{\mathfrak{S}_\lambda}}\rangle_{\mathfrak{S}_\lambda} = \textstyle\prod_{i=1}^{\ell(\lambda)} a_\theta^{(p)}(\lambda_i).$$

Therefore, since $\{\chi_{\mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}\mathsf{triv}_{\mathfrak{S}_\lambda}} \mid \lambda\in\mathsf{Par}(n)\}$ is a $\mathbb{Z}$-basis of the abelian group of generalized characters of $\mathfrak{S}_n$, it suffices to show that $a_\theta^{(p)}(k) \in \mathbb{Z}_{(p)}$ for all $k \geq 0$.

Let $\theta_j'' = \ell_{p'}(a/b)^{-(\ell-1)p^{r+j}}$ and $\theta_j' = \theta_j - \theta_j''$ for $j \geq 0$, so that $\theta = \theta' + \theta''$. We know that $a_{\theta''}^{(p)}(k) \in \mathbb{Z}_{(p)}$ by Lemma 2.9 (c). Thus, by Lemma 2.9 (a), it is enough to show that $a_{\theta'}^{(p)}(k) \in \mathbb{Z}_{(p)}$. By Lemma 2.9 (b), it will suffice to prove that $\nu_p(\theta_j') \geq j+1$. Note that

$$\theta_j' = \frac{\sum_{i=0}^{\ell-1}(a/b)^{2ip^{r+j}} - \ell}{p^r}\cdot\Big(\frac{a}{b}\Big)^{-(\ell-1)p^{r+j}} = \frac{\sum_{i=0}^{\ell-1}\Big((a/b)^{2ip^{r+j}} - 1\Big)}{p^r}\cdot\Big(\frac{a}{b}\Big)^{-(\ell-1)p^{r+j}}.$$

Since the assumption that $a^2 - b^2 \in p\mathbb{Z}$ implies that $a^{2ip^{r+j}} - b^{2ip^{r+j}} \in p^{1+r+j}\mathbb{Z}$ for all $i \geq 0$ (see e.g. Proposition 5.1 and its proof), we are done. □

2.3. **ℓ-colored version $M_{\ell,d}$ of M_n.** Let R be a commutative ring and $A \in \mathsf{Mat}_\ell(R)$ for some $\ell \geq 1$. Let $\{v_1, \ldots, v_\ell\}$ be the standard basis of the free R-module R^ℓ. Then the symmetric power $\mathsf{Sym}^m(R^\ell)$ has a basis $\{v_{i_1}v_{i_2}\cdots v_{i_m} \mid (i_1,\ldots,i_m) \in \mathsf{Mult}_m(\ell)\}$ where

$$\mathsf{Mult}_m(\ell) = \{(i_1,\ldots,i_m) \in \mathbb{Z}^m \mid 1 \leq i_1 \leq \cdots \leq i_m \leq \ell\}.$$

Since Sym^m is a functor from the category of finitely generated R-modules to itself, the endomorphism of R^ℓ given by A induces an endomorphism of $\mathsf{Sym}^m(R^\ell)$, and the m-th symmetric power $\mathsf{Sym}^m(A)$ is defined to be the matrix of this endomorphism with respect to the given basis (see e.g. [Evs, Equation (3.15)] for a more explicit description). Thus, $\mathsf{Sym}^m(A) \in \mathsf{Mat}_{\mathsf{Mult}_m(\ell)}(R)$.

For $\ell, d \geq 0$, we define

$$\Omega_{\ell,d} = \bigsqcup_{\lambda\in\mathsf{Par}(d)} \{(\lambda,(i_1,\ldots,i_{\ell(\lambda)})) \mid 1 \leq i_j \leq \ell\ \forall j \text{ and } \lambda_j = \lambda_{j+1} \Rightarrow i_j \leq i_{j+1}\}.$$

There is a bijection $\Omega_{\ell,d} \xrightarrow{\sim} \mathsf{Par}_\ell(d)$ given by $(\lambda,\underline{i}) \mapsto (\lambda^{(1)},\ldots,\lambda^{(\ell)})$ where $\lambda^{(j)}$ consists of the parts λ_k such that $i_k = j$ (see [Hil, Notation 3.1]).

Definition 2.13. *For positive integers ℓ, d and $A \in \mathsf{Mat}_\ell(\mathscr{A})$, we define (see §1.7.5)*

$$S^d(A) = \bigoplus_{\lambda\in\mathsf{Par}(d)}\ \bigotimes_{t\geq 1}\mathsf{Sym}^{m_t(\lambda)}(\mathsf{Infl}_t(A)).$$

We may view $S^d(A)$ as an $\Omega_{\ell,d}\times\Omega_{\ell,d}$-matrix via the identification

$$\bigsqcup_{\lambda\in\mathsf{Par}(d)}\ \prod_{t\geq 1}\mathsf{Mult}_{m_t(\lambda)}(\ell) \xrightarrow{\sim} \Omega_{\ell,d},$$
$$\big((i_{t,1},\ldots,i_{t,m_t(\lambda)})\big)_{t\geq 1} \mapsto (\lambda,(i_{1,1},i_{1,2},\ldots,i_{1,m_1(\lambda)},i_{2,1},i_{2,2},\ldots,i_{2,m_2(\lambda)},\ldots)).$$

Further, combining this with the above identification, we may (and do) view $S^d(A)$ as an element of $\mathsf{Mat}_{\mathsf{Par}_\ell(d)}(\mathscr{A})$.

Definition 2.14. *The ℓ-colored ring of symmetric functions is defined by $\Lambda_\ell = \bigotimes_{t=1}^{\ell} \Lambda^{(t)}$, where each $\Lambda^{(t)}$ is a copy of Λ. We write $m_\mu^{(t)}$ for the image of m_μ in $\Lambda^{(t)}$ and adopt a similar convention for the functions h_μ and p_μ. For $\ell \geq 1$ and $d \geq 0$, we define the matrices $M_{\ell,d}, K_{\ell,d} \in \mathsf{Mat}_{\mathsf{Par}_\ell(d)}(\mathbb{Z})$ by the following equations:*

$$\begin{aligned} p^{(1)}_{\lambda^{(1)}} \cdots p^{(\ell)}_{\lambda^{(\ell)}} &= \sum_{(\mu^{(1)},\dots,\mu^{(\ell)}) \in \mathsf{Par}_\ell(d)} (M_{\ell,d})_{(\lambda^{(1)},\dots,\lambda^{(\ell)}),(\mu^{(1)},\dots,\mu^{(\ell)})} m^{(1)}_{\mu^{(1)}} \cdots m^{(\ell)}_{\mu^{(\ell)}} \\ &= \sum_{(\mu^{(1)},\dots,\mu^{(\ell)}) \in \mathsf{Par}_\ell(d)} (K_{\ell,d})_{(\lambda^{(1)},\dots,\lambda^{(\ell)}),(\mu^{(1)},\dots,\mu^{(\ell)})} h^{(1)}_{\mu^{(1)}} \cdots h^{(\ell)}_{\mu^{(\ell)}}. \end{aligned}$$

Remark 2.15. $M_{1,n} = {}^{\mathrm{tr}}M_n$ by (2.4) and $M_{\ell,n} = \bigoplus_{\sum_{i=1}^{\ell} n_i = n, n_i \geq 0} \bigotimes_{i=1}^{\ell} M_{1,n_i}$.

Remark 2.16. $M_{\ell,d}$ and $K_{\ell,d}$ are column equivalent over $\mathbb{Z}$ since both of

$$\{\textstyle\prod_{i\in I} m^{(i)}_{\mu^{(i)}} \mid (\mu^{(i)})_{i\in I} \in \mathsf{Par}_\ell(d)\}, \quad \{\textstyle\prod_{i\in I} h^{(i)}_{\mu^{(i)}} \mid (\mu^{(i)})_{i\in I} \in \mathsf{Par}_\ell(d)\}$$

are bases of the same $\mathscr{A}$-lattice of the degree d part of Λ_ℓ (see [Ful, §6, Proposition 1]).

The following result is similar to [Tsu, Proposition 2.3] and is proved by essentially the same argument as that given in [BK1, §5]. We include a proof for clarity.

Proposition 2.17. *Let $\mathbb{F}$ be a field of characteristic 0 and $I = \{1, \dots, \ell\}$ for a fixed integer $\ell \geq 0$. We regard the polynomial ring $V = \mathbb{F}[y^{(i)}_n \mid i \in I, n \geq 1]$ as a graded $\mathbb{F}$-algebra via $\deg y^{(i)}_n = n$ and denote by V_d the $\mathbb{F}$-vector subspace of V consisting of homogeneous elements of degree d for $d \geq 0$. Assume that we are given the following data:*

(a) *a ring involution $\sigma\colon \mathbb{F} \xrightarrow{\sim} \mathbb{F}$,*
(b) *a family of invertible matrices $A = (A^{(m)})_{m\geq 1}$ where $A^{(m)} = (a^{(m)}_{ij})_{i,j\in I} \in \mathrm{GL}_I(\mathbb{F})$,*
(c) *two bi-additive forms $\langle\cdot,\cdot\rangle_S$ and $\langle\cdot,\cdot\rangle_K\colon V \times V \to \mathbb{F}$ such that*
- *$\langle cf, g\rangle_X = \sigma(c)\langle f, g\rangle_X$ and $\langle f, cg\rangle_X = c\langle f, g\rangle_X$,*
- *$\langle 1, 1\rangle_X = 1$, and $\langle 1, f\rangle_X = 0$ if $f \in V_d$ for some $d > 0$,*
- *$\langle m y^{(i)}_m f, g\rangle_S = \langle f, \sum_{j\in I} a^{(m)}_{ij} \frac{\partial g}{\partial y^{(j)}_m}\rangle_S$ and $\langle m y^{(i)}_m f, g\rangle_K = \langle f, \frac{\partial g}{\partial y^{(i)}_m}\rangle_K$*

for $X \in \{S, K\}$ and $f, g \in V$, $c \in \mathbb{F}$, $m \geq 1$,
(d) *a family of new variables $(x^{(i)}_n)_{i\in I, n\geq 1}$ such that $x^{(i)}_n - y^{(i)}_n \in \mathbb{F}[y^{(i)}_m \mid 1 \leq m < n] \cap V_n$ for all $n \geq 1$.*

Set $x^{(\underline{i})}_\lambda = \prod_{k=1}^{\ell(\lambda)} x^{(i_k)}_{\lambda_k}$ and $y^{(\underline{i})}_\lambda = \prod_{k=1}^{\ell(\lambda)} y^{(i_k)}_{\lambda_k}$ for $(\lambda, \underline{i}) \in \Omega_{\ell,d}$, and define the transition matrix $P = (p^{(\underline{i},\underline{j})}_{\lambda,\mu}) \in \mathrm{GL}_{\Omega_{\ell,d}}(\mathbb{F})$ by $x^{(\underline{i})}_\lambda = \sum_{(\mu,\underline{j})\in\Omega_{\ell,d}} p^{(\underline{i},\underline{j})}_{\lambda,\mu} y^{(\underline{j})}_\mu$. Then the Gram matrices $M_S = (\langle x^{(\underline{i})}_\lambda, x^{(\underline{j})}_\mu\rangle_S)_{(\lambda,\underline{i}),(\mu,\underline{j})\in\Omega_{\ell,d}}$ and $M_K = (\langle x^{(\underline{i})}_\lambda, x^{(\underline{j})}_\mu\rangle_K)_{(\lambda,\underline{i}),(\mu,\underline{j})\in\Omega_{\ell,d}}$ are related by the identity

$$M_S = \sigma(P) \left(\bigoplus_{\lambda\in\mathsf{Par}(d)} \bigotimes_{t\geq 1} \mathsf{Sym}^{m_t(\lambda)}(A^{(t)}) \right) \sigma(P)^{-1} M_K. \tag{2.6}$$

Proof. Let $z^{(i)}_n = \sum_{j\in I} \sigma(a^{(n)}_{ij}) y^{(j)}_n$ for $n > 0$ and $i \in I$, and define $z^{(\underline{i})}_\lambda = \prod_{k=1}^{\ell(\lambda)} z^{(i_k)}_{\lambda_k}$ for all $(\lambda, \underline{i}) \in \Omega_{\ell,d}$. First, we will prove by induction on d that, for all for all $f \in V$ and $(\lambda, \underline{i}) \in \Omega_{\ell,d}$, we have

$$\langle y^{(\underline{i})}_\lambda, f\rangle_S = \langle z^{(\underline{i})}_\lambda, f\rangle_K \tag{2.7}$$

(cf. [BK1, Lemma 5.2]). We have $\langle 1, f\rangle_S = \langle 1, f\rangle_K$ for all $f \in V$, as both sides are equal to the constant term of f, so (2.7) holds when $d = 0$. If (2.7) holds for some $(\lambda, \underline{i}) \in \bigcup_{d\geq 0} \Omega_{\ell,d}$ and all $f \in V$, then for all $n > 0$, $i \in I$ and $f \in V$ we have

$$\begin{aligned} \left\langle y^{(i)}_n y^{(\underline{i})}_\lambda, f \right\rangle_S &= \left\langle n^{-1} y^{(\underline{i})}_\lambda, \textstyle\sum_{j\in I} a^{(n)}_{ij} \frac{\partial f}{\partial y^{(j)}_n} \right\rangle_S &&= \left\langle n^{-1} z^{(\underline{i})}_\lambda, \textstyle\sum_{j\in I} a^{(n)}_{ij} \frac{\partial f}{\partial y^{(j)}_n} \right\rangle_K \\ &= \textstyle\sum_{j\in I} a^{(n)}_{ij} \left\langle n^{-1} z^{(\underline{i})}_\lambda, \frac{\partial f}{\partial y^{(j)}_n} \right\rangle_K &&= \textstyle\sum_{j\in I} a^{(n)}_{ij} \left\langle y^{(i)}_n z^{(\underline{i})}_\lambda, f \right\rangle_K \\ &= \left\langle \textstyle\sum_{j\in I} \sigma(a^{(n)}_{ij}) y^{(i)}_n z^{(\underline{i})}_\lambda, f \right\rangle_K &&= \left\langle z^{(i)}_n z^{(\underline{i})}_\lambda, f \right\rangle_K, \end{aligned}$$

and therefore (2.7) holds in all cases.

Let $Q = (q_{\lambda,\mu}^{(\underline{i},\underline{j})})_{(\lambda,\underline{i}),(\mu,\underline{j})\in\Omega_{\ell,d}} \in \mathsf{Mat}_{\Omega_{\ell,d}}(\mathbb{F})$ be the transition matrix defined by

$$z_\lambda^{(\underline{i})} = \sum_{(\mu,\underline{j})\in\Omega_{\ell,d}} q_{\lambda,\mu}^{(\underline{i},\underline{j})} y_\mu^{(\underline{j})}.$$

For any $(\lambda,\underline{i}) \in \Omega_{\ell,d}$, we have

$$z_\lambda^{(\underline{i})} = \sum_{\underline{j}\in I^{\ell(\lambda)}} \sigma(a_{i_1,j_1}^{(\lambda_1)})\cdots\sigma(a_{i_{\ell(\lambda)},j_{\ell(\lambda)}}^{(\lambda_{\ell(\lambda)})}) y_{\lambda_1}^{(j_1)}\cdots y_{\lambda_{\ell(\lambda)}}^{(j_{\ell(\lambda)})}, \qquad \text{whence}$$

$$Q = \bigoplus_{\lambda\in\mathsf{Par}(d)} \bigotimes_{t\geq 1} \mathsf{Sym}^{m_t(\lambda)}(\sigma(A^{(t)}))$$

(cf. [BK1, Lemma 5.3]). Writing $P^{-1} = (\tilde{p}_{\lambda,\mu}^{(\underline{i},\underline{j})})_{(\lambda,\underline{i}),(\mu,\underline{j})\in\Omega_{\ell,d}}$, we have

$$\begin{aligned}
\left\langle x_\lambda^{(\underline{i})}, x_\mu^{(\underline{j})} \right\rangle_S &= \sum_{(\nu,\underline{k})\in\Omega_{\ell,d}} \sigma(p_{\lambda,\nu}^{(\underline{i},\underline{k})}) \left\langle y_\nu^{(\underline{k})}, x_\mu^{(\underline{j})} \right\rangle_S \\
&= \sum_{(\nu,\underline{k})\in\Omega_{\ell,d}} \sigma(p_{\lambda,\nu}^{(\underline{i},\underline{k})}) \left\langle z_\nu^{(\underline{k})}, x_\mu^{(\underline{j})} \right\rangle_K \\
&= \sum_{(\nu,\underline{k}),(\eta,\underline{r})\in\Omega_{\ell,d}} \sigma(p_{\lambda,\nu}^{(\underline{i},\underline{k})})\sigma(q_{\nu,\eta}^{(\underline{k},\underline{r})}) \left\langle y_\eta^{(\underline{r})}, x_\mu^{(\underline{j})} \right\rangle_K \\
&= \sum_{(\nu,\underline{k}),(\eta,\underline{r}),(\theta,\underline{s})\in\Omega_{\ell,d}} \sigma(p_{\lambda,\nu}^{(\underline{i},\underline{k})})\sigma(q_{\nu,\eta}^{(\underline{k},\underline{r})})\sigma(\tilde{p}_{\eta,\theta}^{(\underline{r},\underline{s})}) \left\langle x_\theta^{(\underline{s})}, x_\mu^{(\underline{j})} \right\rangle_K
\end{aligned}$$

for any $(\lambda,\underline{i}),(\mu,\underline{j}) \in \Omega_{\ell,d}$, where the second equality holds by (2.7). Therefore,

$$M_S = \sigma(P)\sigma(Q)\sigma(P)^{-1}M_K = \sigma(P)\left(\bigoplus_{\lambda\in\mathsf{Par}(d)} \bigotimes_{t\geq 1} \mathsf{Sym}^{m_t(\lambda)}(A^{(t)})\right)\sigma(P)^{-1}M_K. \qquad \square$$

The following is a corollary of the boson-fermion correspondence over $\mathbb{Z}$ (see [DcKK, Corollary 2.1] and [Tsu, Proposition 2.4]).

Proposition 2.18. *Let $\mathbb{F}$, ℓ, I, σ, V and V_d be as in Proposition 2.17.*

(a) *There exists a unique bi-additive non-degenerate map $\langle\cdot,\cdot\rangle_K : V\times V\to\mathbb{F}$ such that*
   (i) $\langle af, g\rangle_K = \sigma(a)\langle f,g\rangle_K, \langle f, ag\rangle_K = a\langle f,g\rangle_K$ *and* $\langle f,g\rangle_K = \sigma(\langle g,f\rangle_K)$,
   (ii) $\langle 1,1\rangle_K = 1$ *and* $\langle m y_m^{(i)} f, g\rangle_K = \langle f, \frac{\partial g}{\partial y_m^{(i)}}\rangle_K$,
   for all $f,g\in V$, $a\in\mathbb{F}$ and $i\in I$.

(b) *Suppose further that for each $1\leq i\leq\ell$ the variables $\{x_n^{(i)} \mid n\geq 1\}$ and $\{y_n^{(i)} \mid n\geq 1\}$ are related by the formal identity*

$$1 + \sum_{n\geq 1} x_n^{(i)} t^n = \exp\left(\sum_{r\geq 1} y_r^{(i)} t^r\right). \tag{2.8}$$

Then, for any $d\geq 0$, the set of Schur functions

$$\left\{\prod_{i\in I} s_{\lambda^{(i)}}(x^{(i)}) \;\middle|\; \sum_{i\in I}|\lambda^{(i)}| = d\right\}$$

forms an orthonormal basis of the $\mathbb{Z}$-lattice $\mathbb{Z}[x_n^{(i)} \mid i\in I, n\geq 1]\cap V_d$ of V_d with respect to $\langle\cdot,\cdot\rangle_K$. Here, $s_\lambda(x^{(i)}) := \det(x_{\lambda_k+j-k}^{(i)})_{1\leq j,k\leq|\lambda|}$ for $\lambda\in\mathsf{Par}$ and $x_m^{(i)} = \delta_{m,0}$ for $m\leq 0$.

Note that the form $\langle\cdot,\cdot\rangle_K : V\times V\to\mathbb{F}$ satisfying the conditions of Proposition 2.17 is clearly unique. Also, those conditions are implied by the properties satisfied by the form $\langle\cdot,\cdot\rangle_K$ of Proposition 2.18 (a).

Corollary 2.19. *Assume all the hypotheses of Proposition 2.17. Suppose further that the variables* $\{x_n^{(i)} \mid i \in I, n \geq 1\}$ *and* $\{y_n^{(i)} \mid i \in I, n \geq 1\}$ *are related as in Proposition 2.18* (b)*. Then*

$$M_S = K_{\ell,d}^{-1} \left(\bigoplus_{\lambda \in \mathsf{Par}(d)} \bigotimes_{t \geq 1} \mathsf{Sym}^{m_t(\lambda)}(A^{(t)}) \right) K_{\ell,d} M_K. \tag{2.9}$$

and $M_K \in \mathrm{GL}_{\mathsf{Par}_\ell(d)}(\mathbb{Z})$.

Proof. Let $Y = \bigoplus_{\lambda \in \mathsf{Par}(d)} \bigotimes_{t \geq 1} \mathsf{Sym}^{m_t(\lambda)}(A^{(t)})$. We identify the ring V with $\mathbb{F} \otimes \Lambda_\ell$ by setting $y_n^{(i)} = p_n^{(i)}/n$. Then, comparing the hypothesis with (2.3), we see that $x_n^{(i)} = h_n^{(i)}$. Define $w_\mu = \mu_1 \cdots \mu_{\ell(\mu)}$ for $\mu \in \mathsf{Par}$ and $i \in I$, and let $W = \mathsf{diag}\{w_{\mu^{(1)}} \cdots w_{\mu^{(\ell)}} \mid (\mu^{(1)}, \dots, \mu^{(\ell)}) \in \mathsf{Par}_\ell(d)\}$. It follows from Definition 2.14 that the change-of-basis matrix P of Proposition 2.17 is given by $P = K_{\ell,d}^{-1} W$. Hence, Proposition 2.17 yields

$$M_S = \sigma(K_{\ell,d})^{-1} \sigma(W) Y \sigma(W)^{-1} \sigma(K_{\ell,d}) M_K.$$

Observing that, when we view W as an $\Omega_{\ell,d} \times \Omega_{\ell,d}$-matrix, each block of W corresponding to a fixed $\lambda \in \mathsf{Par}(d)$ is a scalar matrix and also that $\sigma(K_{\ell,d}) = K_{\ell,d}$ because $K_{\ell,d}$ is $\mathbb{Q}$-valued, we obtain (2.9).

Thanks to Proposition 2.18, there exists $Q \in \mathrm{GL}_{\mathsf{Par}_\ell(d)}(\mathbb{Z})$ such that $M_K = {}^{\mathrm{tr}}Q \cdot Q$. □

The following result is a quantized version of [Hil, Proposition 3.3], though our proof is different.

Theorem 2.20. *For* $\ell \geq 1$ *and* $A \in \mathsf{Mat}_\ell(\mathscr{A})$*, we have* $M_{\ell,d}^{-1} S^d(A) M_{\ell,d} \in \mathsf{Mat}_{\mathsf{Par}_\ell(d)}(\mathscr{A})$ *for any* $d \geq 0$.

Proof. Let $I = \{1, \dots, \ell\}$. By Remark 2.16, it will suffice to prove that $K_{\ell,d}^{-1} S^d(P) K_{\ell,d} \in \mathsf{Mat}_{\mathsf{Par}_\ell(d)}(\mathscr{A})$.

In the rest of the proof, we identify $\mathbf{k} \otimes \Lambda_\ell$ with $V = \mathbf{k}[y_n^{(i)} \mid i \in I, n \geq 1]$ by identifying $p_n^{(i)}/n$ with $y_n^{(i)}$. Write $A = (a_{ij})_{i,j \in I}$. Define new variables $x_n^{(i)} \in V$ by the identity (2.8). Clearly, there exists a unique bi-additive map $\langle \cdot, \cdot \rangle_S \colon V \times V \to \mathbf{k}$ such that

(a) $\langle cf, g \rangle_S = \mathsf{bar}(c) \langle f, g \rangle_S$, $\langle f, cg \rangle_S = c \langle f, g \rangle_S$,

(b) $\langle 1, 1 \rangle_S = 1$, and $\langle 1, f \rangle_S = 0$ if f has zero constant term as a polynomial in the variables $y_n^{(j)}$,

(c) $\langle m y_m^{(i)} f, g \rangle_S = \langle f, \sum_{j \in I} \mathsf{Infl}_m(a_{ij}) \frac{\partial g}{\partial y_m^{(j)}} \rangle_S$

for $f, g \in V$, $c \in \mathbf{k}$, $m \geq 1$, $i \in I$. Applying Corollary 2.19 with $\mathbb{F} = \mathbf{k}$, $\sigma = \mathsf{bar}$ and the form $\langle \cdot, \cdot \rangle_K$ supplied by Proposition 2.18 (a), we obtain $M_S = K_{\ell,d}^{-1} S^d(A) K_{\ell,d} M_K$ (in the notation of Proposition 2.17) and $M_K \in \mathrm{GL}_{\mathsf{Par}_\ell(d)}(\mathbb{Z})$.

Thus, it is enough to show that $\langle x_\lambda^{(\underline{i})}, x_\mu^{(\underline{j})} \rangle_S \in \mathscr{A}$ for $(\lambda, \underline{i}), (\mu, \underline{j}) \in \Omega_{\ell,d}$, where $x_\lambda^{(\underline{i})}$ is defined as in Proposition 2.17. We argue by induction on $|\lambda|$. Expanding (2.8), we obtain

$$x_n^{(i)} = \textstyle\sum_{\lambda \in \mathsf{Par}(n)} \prod_{k \geq 1} \frac{(y_k^{(i)})^{m_k(\lambda)}}{m_k(\lambda)!}, \tag{2.10}$$

and therefore $\partial x_m^{(i)} / \partial y_n^{(j)} = \delta_{ij} x_{m-n}^{(i)}$ for $i, j \in I$ and $m, n \geq 1$, where we put $x_\gamma^{(i)} = \delta_{\gamma,0}$ for $\gamma \leq 0$ (see also [DcKK, page 129]). Combining (2.10) with the defining property (c) of $\langle \cdot, \cdot \rangle_S$, we obtain the identity $\langle x_n^{(i)} f, g \rangle_S = \langle f, D_n^{(i)} g \rangle_S$ for all $f, g \in V$, $n \geq 1$, $i \in I$, where the differential operator $D_n^{(i)} \colon V \to V$ is defined by

$$D_n^{(i)} = \sum_{\lambda \in \mathsf{Par}(n)} \prod_{k \geq 1} \frac{1}{k^{m_k(\lambda)} m_k(\lambda)!} \left(\sum_{j \in I} \mathsf{Infl}_k(a_{ij}) \frac{\partial}{\partial y_k^{(j)}} \right)^{m_k(\lambda)}$$

Let $V^{\mathscr{A}} = \mathscr{A}[x_\lambda^{(\underline{i})} \mid (\lambda, \underline{i}) \in \Omega_{\ell,d}]$. By the inductive hypothesis, it is enough to show that $D_n^{(i)}(V^{\mathscr{A}}) \subset V^{\mathscr{A}}$ for all $i \in I$, $n \geq 1$. By a straightforward calculation, one obtains the product rule $D_n^{(i)}(fg) = \sum_{s=0}^n D_s^{(i)}(f) D_{n-s}^{(i)}(g)$ for $f, g \in V$. Hence, it suffices to prove that $D_n^{(i)}(x_m^{(j)}) \in V^{\mathscr{A}}$ for all $i, j \in I$ and $n, m \geq 1$. We have

$$D_n^{(i)}(x_m^{(j)}) = \left(\sum_{\lambda \in \mathsf{Par}(n)} \prod_{k \geq 1} \frac{\mathsf{Infl}_k(a_{ij})^{m_k(\lambda)}}{k^{m_k(\lambda)} m_k(\lambda)!} \right) x_{m-n}^{(j)},$$

and the result now follows from Lemma 2.21. □

Lemma 2.21. *For any $f \in \mathscr{A}$, we have $\sum_{\lambda\in\mathsf{Par}(n)} \frac{1}{z_\lambda} \prod_{k\geq 1} \mathsf{Infl}_k(f)^{m_k(\lambda)} \in \mathscr{A}$.*

Proof. For $\theta = (\theta_k)_{k\geq 1} \in \mathscr{A}^{\mathbb{Z}_{\geq 1}}$ and $n \geq 0$, we define $b_\theta(n) = \sum_{\lambda\in\mathsf{Par}(n)} \frac{1}{z_\lambda} \prod_{k\geq 1} \theta_k^{m_k(\lambda)}$ (cf. Definition 2.8). Similarly to Lemma 2.9 (a), we have $b_{\theta+\theta'}(n) = \sum_{k=0}^{n} b_\theta(k) b_{\theta'}(n-k)$. Thus, it is enough to show that $b_{\theta_m^\pm}(n) \in \mathscr{A}$ for $m \in \mathbb{Z}$ where $\theta_m^\pm = (\pm v^m, \pm v^{2m}, \pm v^{3m}, \ldots)$. By the orthogonality relations, we have $\sum_{\lambda\in\mathsf{Par}(n)} \frac{(\pm 1)^{\ell(\lambda)}}{z_\lambda} = (1 \pm 1)/2$, which implies that $b_{\theta_m^\pm}(n) = (v^{mn} \pm v^{mn})/2$. □

3. Graded Cartan matrices of symmetric groups and Hecke algebras

In this section we recall the definition of graded Cartan matrices $C^v_{\mathcal{H}_n(k_\ell;\eta_\ell)}$ and reduce the problem of finding their unimodular equivalence classes to the same problem for the matrix $M_n \,\mathsf{diag}(\{J^v_\ell(\lambda) \mid \lambda \in \mathsf{Par}(n)\})M_n^{-1}$ (cf. Conjecture 1.9).

3.1. **Gram matrices of quantized Shapovalov forms.** We now recall some of the definitions and results from [Tsu] and, in particular, define the Gram matrix $\mathsf{QSh}^{\mathsf{M}}_{\lambda,\mu}(X)$ of a quantized Shapovalov form (cf. [Tsu, Definition 3.13]). For the theory of quantum groups, the book [Lus] is a standard reference.

Let $X = (a_{ij})_{i,j\in I}$ be a symmetrizable generalized Cartan matrix and take the symmetrization $d = (d_i)_{i\in I}$ of X, i.e., the unique $d \in \mathbb{Z}^I_{\geq 1}$ such that $d_i a_{ij} = d_j a_{ji}$ for all $i, j \in I$ and $\gcd(d_i)_{i\in I} = 1$. We consider a root datum $(\mathcal{P}, \mathcal{P}^\vee, \Pi, \Pi^\vee)$ in the following sense:

(a) $\mathcal{P}^\vee$ is a free $\mathbb{Z}$-module of rank $(2|I| - \operatorname{rank} X)$ and $\mathcal{P} = \operatorname{Hom}_{\mathbb{Z}}(\mathcal{P}^\vee, \mathbb{Z})$,
(b) $\Pi^\vee = \{h_i \mid i \in I\}$ is a $\mathbb{Z}$-linearly independent subset of $\mathcal{P}^\vee$,
(c) $\Pi = \{\alpha_i \mid i \in I\}$ is a $\mathbb{Z}$-linearly independent subset of $\mathcal{P}$,
(d) $\alpha_j(h_i) = a_{ij}$ for all $i, j \in I$.

We denote by $Q^+ = \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$ the positive part of the root lattice and denote by $\mathcal{P}^+$ the set of dominant integral weights $\{\lambda \in \mathcal{P} \mid \forall i \in I, \lambda(h_i) \in \mathbb{Z}_{\geq 0}\}$. For each $i \in I$, $\Lambda_i \in \mathcal{P}^+$ is a dominant integral weight determined modulo the subgroup $\{\lambda \in \mathcal{P} \mid \forall i \in I, \lambda(h_i) = 0\}$ of $\mathcal{P}$ by the condition that $\Lambda_i(h_j) = \delta_{ij}$ for all $j \in I$.

Recall that the Weyl group $W = W(X)$ is the subgroup of $\mathsf{Aut}(\mathcal{P})$ generated by $\{s_i : \mathcal{P} \xrightarrow{\sim} \mathcal{P}, \lambda \longmapsto \lambda - \lambda(h_i)\alpha_i \mid i \in I\}$.

Definition 3.1. *The quantum group $U_v = U_v(X)$ is the unital associative* $\mathbf{k}$*-algebra generated by $\{e_i, f_i \mid i \in I\} \cup \{v^h \mid h \in \mathcal{P}^\vee\}$ with the following defining relations:*

(a) $v^0 = 1$ *and* $v^h v^{h'} = v^{h+h'}$ *for any* $h, h' \in \mathcal{P}^\vee$,
(b) $v^h e_i v^{-h} = v^{\alpha_i(h)} e_i, v^h f_i v^{-h} = v^{-\alpha_i(h)} f_i$ *for any* $i \in I$ *and* $h \in \mathcal{P}^\vee$,
(c) $e_i f_j - f_j e_i = \delta_{ij}(K_i - K_i^{-1})/(v_i - v_i^{-1})$ *for any* $i, j \in I$,
(d) $\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(k)} e_j e_i^{(1-a_{ij}-k)} = 0 = \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(k)} f_j f_i^{(1-a_{ij}-k)}$ *for any* $i \neq j \in I$,

where $K_i = v^{d_i h_i}, v_i = v^{d_i}$ and $e_i^{(n)} = e_i^n/[n]_{d_i}!, f_i^{(n)} = f_i^n/[n]_{d_i}!$.

Let U_v^+, U_v^0, U_v^- be the $\mathbf{k}$-subalgebras of U_v defined by

$$U_v^+ = \langle e_i \mid i \in I\rangle, \quad U_v^- = \langle f_i \mid i \in I\rangle, \quad U_v^0 = \langle v^h \mid h \in \mathcal{P}^\vee\rangle.$$

Then, the following is a triangular decomposition theorem for quantum groups [Lus, §3.2]:

(i) the canonical map $U_v^- \otimes_{\mathbf{k}} U_v^0 \otimes_{\mathbf{k}} U_v^+ \to U_v$ is a $\mathbf{k}$-vector space isomorphism,
(ii) U_v^0 is canonically isomorphic to the group $\mathbf{k}$-algebra $\mathbf{k}[\mathcal{P}^\vee]$.

For each $\lambda \in \mathcal{P}^+$, we denote by $V(\lambda)$ the integrable highest weight U_v-module with highest weight λ and a fixed highest weight vector $1_\lambda \in V(\lambda)$.

Proposition 3.2 ([Tsu, Proposition 3.8])**.** *For $\lambda \in \mathcal{P}^+$, there exist unique bi-additive non-degenerate maps $\langle\cdot,\cdot\rangle_{\mathsf{QSh}} : V(\lambda) \times V(\lambda) \to \mathbf{k}$ and $\langle\cdot,\cdot\rangle_{\mathsf{RSh}} : V(\lambda) \times V(\lambda) \to \mathbf{k}$ with*

(i) $\langle aw_1, w_2\rangle_Y = \mathsf{bar}(a)\langle w_1, w_2\rangle_Y$, $\langle w_1, aw_2\rangle_Y = a\langle w_1, w_2\rangle_Y$ *and* $\langle w_1, w_2\rangle_Y = \mathsf{bar}(\langle w_2, w_1\rangle_Y)$,
(ii) $\langle 1_\lambda, 1_\lambda\rangle_Y = 1$ *and* $\langle uw_1, w_2\rangle_{\mathsf{QSh}} = \langle w_1, \Omega(u)w_2\rangle_{\mathsf{QSh}}$, $\langle uw_1, w_2\rangle_{\mathsf{QSh}} = \langle w_1, \Upsilon(u)w_2\rangle_{\mathsf{RSh}}$.

for all $Y \in \{\mathsf{QSh}, \mathsf{RSh}\}$ *and for all* $w_1, w_2 \in V(\lambda)$, $u \in U_v$ *and* $a \in \mathbf{k}$. *Here,* Ω *and* Υ *are the* $\mathbb{Q}$*-antiinvolution and* $\mathbb{Q}$*-antiautomorphism of* U_v *defined by*

$$\Omega(e_i) = f_i, \quad \Omega(f_i) = e_i, \quad \Omega(v^h) = v^{-h}, \quad \Omega(v) = v^{-1},$$
$$\Upsilon(e_i) = v_i f_i K_i^{-1}, \quad \Upsilon(f_i) = v_i^{-1} K_i e_i, \quad \Upsilon(v^h) = v^{-h}, \quad \Upsilon(v) = v^{-1}.$$

We denote by $P(\lambda) := \{\mu \in \mathcal{P} \mid V(\lambda)_\mu \neq 0\}$ the set of weights of $V(\lambda)$, which is W-invariant [Lus, Proposition 5.2.7]. Let $(U_v^-)^{\mathscr{A}}$ be the $\mathscr{A}$-subalgebra of U_v^- generated by $\{f_i^{(n)} \mid i \in I, n \geq 0\}$. The constructions below use the following deep results:

(a) $(U_v^-)^{\mathscr{A}}$ is an $\mathscr{A}$-lattice of U_v^- (see [Lus, Theorem 14.4.3]),
(b) $V(\lambda)_\nu^{\mathscr{A}} := V(\lambda)_\nu \cap V(\lambda)^{\mathscr{A}}$ is an $\mathscr{A}$-lattice of $V(\lambda)_\nu$ for $\nu \in P(\lambda)$ where $V(\lambda)^{\mathscr{A}} := (U_v^-)^{\mathscr{A}} 1_\lambda \subseteq V(\lambda)$ (see [Lus, Theorem 14.4.11]).

Definition 3.3 ([Tsu, Proposition 3.13])**.** *For* $\lambda \in \mathcal{P}^+$ *and* $\mu \in P(\lambda)$, *we define*

$$\mathsf{QSh}^{\mathsf{M}}_{\lambda,\mu}(X) = (\langle w_i, w_j \rangle_{\mathsf{QSh}})_{1 \leq i,j \leq \dim V(\lambda)_\mu}, \quad \mathsf{RSh}^{\mathsf{M}}_{\lambda,\mu}(X) = (\langle w_i, w_j \rangle_{\mathsf{RSh}})_{1 \leq i,j \leq \dim V(\lambda)_\mu}$$

where $\{w_i \mid 1 \leq i \leq \dim V(\lambda)_\mu\}$ *is an* $\mathscr{A}$*-basis of* $V(\lambda)_\mu^{\mathscr{A}}$.

For any $n \geq 0$, define the equivalence relation $\coloneqq$ on $\mathsf{Mat}_n(\mathscr{A})$ as follows:

$$Y \coloneqq Z \overset{\text{def}}{\iff} \exists P \in \mathrm{GL}_n(\mathscr{A}),\ \mathsf{bar}({}^{\mathrm{tr}}P) Y P = Z.$$

For $Z \in \{\mathsf{QSh}^{\mathsf{M}}_{\lambda,\mu}(X), \mathsf{RSh}^{\mathsf{M}}_{\lambda,\mu}(X)\}$, the equivalence class of Z under $\coloneqq$ does not depend on the choice of the basis in Definition 3.3. Thus, the $\mathscr{A}$-unimodular equivalence classes of Z are uniquely determined. Note that by construction ${}^{\mathrm{tr}}Z = \mathsf{bar}(Z)$. The following is implicit in [Tsu, Proposition 3.16].

Proposition 3.4. *For* $\lambda \in \mathcal{P}^+$ *and* $\mu \in P(\lambda)$, *there exists an* $\mathscr{A}$*-basis of* $V(\lambda)_\mu^{\mathscr{A}}$ *whose associated* $\mathsf{QSh}^{\mathsf{M}}_{\lambda,\mu}(X)$ *is an* $\mathscr{A}^{\mathsf{bar}}$*-valued symmetric matrix.*

Proof. Take an $\mathscr{A}$-basis $(v_b)_b$ of $V(\lambda)_\mu^{\mathscr{A}}$ of the form $v_b = G_b 1_\lambda$ with $G_b \in (U_v^-)^{\mathscr{A}}$ and $G_b = \overline{G_b}$, where the bar involution $\overline{\ } : U_v \to U_v$ is defined by

$$\overline{e_i} = e_i, \quad \overline{f_i} = f_i, \quad \overline{v^h} = v^{-h}, \quad \overline{v} = v^{-1}.$$

This is possible using the lower canonical basis of U_v^- (see the last paragraph of [Ka2]) or using [Lak, Theorem 6.5].

Let $\mathsf{HC} \colon U_v \to U_v^0$ and $\mathsf{ev}_\lambda \colon U_v^0 \to \mathbf{k}$ be the following maps:

(i) the Harish-Chandra projection $\mathsf{HC} \colon U_v \twoheadrightarrow U_v^0$, which is the $\mathbf{k}$-linear projection from $U_v = U_v^0 \oplus ((\sum_{i \in I} f_i U_v) + (\sum_{i \in I} U_v e_i))$ onto U_v^0,
(ii) the evaluation map $\mathsf{ev}_\lambda \colon U_v^0 \to \mathbf{k}$, which is the $\mathbf{k}$-algebra homomorphism determined by the assignment $\mathsf{ev}_\lambda(v^h) = v^{\lambda(h)}$ for each $h \in \mathcal{P}^\vee$.

These maps exist by parts (i), (ii) in the triangular decomposition theorem respectively.

By the construction of $\langle \cdot, \cdot \rangle_{\mathsf{QSh}}$ (see the proof of [Tsu, Proposition 3.8]), we have

$$\langle v_b, v_b' \rangle_{\mathsf{QSh}} = \mathsf{ev}_\lambda(\mathsf{HC}(\Omega(G_b) G_{b'})). \tag{3.1}$$

Since $\mathsf{HC}(\Omega(G_b)G_{b'}) \in U_v^0 \cap U_v^{\mathscr{A}}$, where $U_v^{\mathscr{A}}$ is an $\mathscr{A}$-subalgebra of U_v generated by $\{v^h, e_i^{(n)}, f_i^{(n)} \mid i \in I, n \geq 0, h \in \mathcal{P}^\vee\}$, and it is known (see [Lus2, Theorem 4.5] or [DDPW, Theorem 6.49]) that $U_v^0 \cap U_v^{\mathscr{A}}$ is the $\mathscr{A}$-subalgebra of U_v^0 generated by

$$\left\{ v^h, \begin{bmatrix} K_i; 0 \\ n \end{bmatrix} := \prod_{j=1}^n \frac{K_i v_i^{-j+1} - K_i^{-1} v_i^{j-1}}{v_i^j - v_i^{-j}} \;\middle|\; i \in I, n \geq 1, h \in \mathcal{P}^\vee \right\},$$

(3.1) is $\mathscr{A}$-valued. Since $\Omega(G_b)G_{b'}$ is bar-invariant, (3.1) is $\mathscr{A}^{\mathsf{bar}}$-valued due to the isomorphism $U_v^0 \cong \mathbf{k}[\mathcal{P}^\vee]$. (For an estimate of (3.1) when G_b is the lower canonical basis, see [Ka1, Problem 2].) □

Corollary 3.5. *For* $\lambda \in \mathcal{P}^+$ *and* $\mu \in P(\lambda)$, *we have* $\mathsf{QSh}^{\mathsf{M}}_{\lambda,\mu}(X) \equiv_{\mathscr{A}} {}^{\mathrm{tr}}\mathsf{QSh}^{\mathsf{M}}_{\lambda,\mu}(X)$.

The proof of Proposition 3.4 also shows that $\mathsf{RSh}^{\mathsf{M}}_{\lambda,\mu}(X)$ is $\mathscr{A}$-valued, which is again implicit in [Tsu, Proposition 3.16].

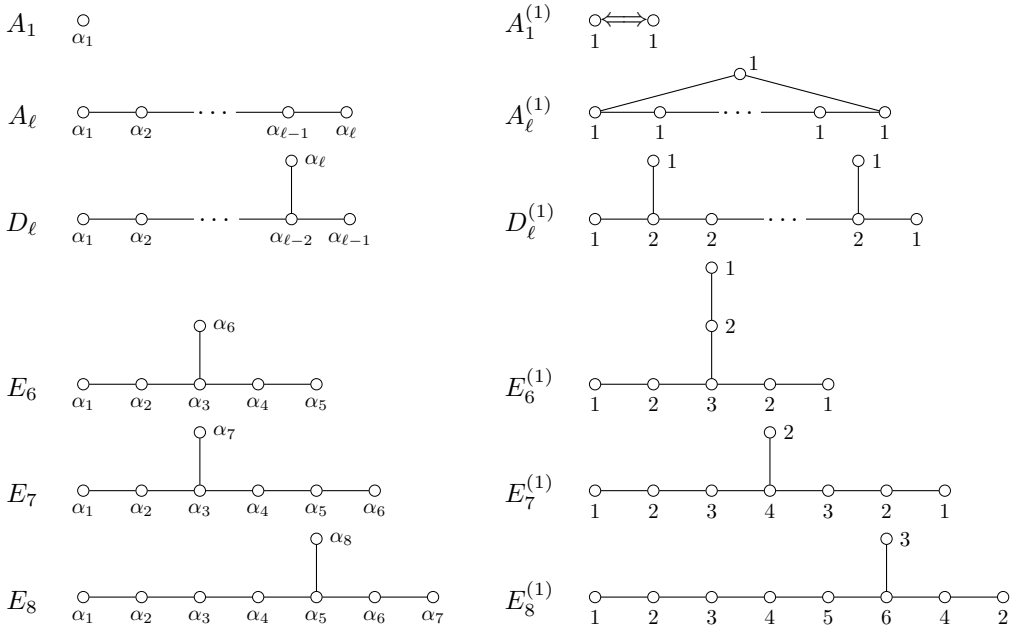

FIGURE 1. Finite and untwisted affine Dynkin diagrams of types A,D,E.

Proposition 3.6 ([Tsu, Proposition 3.16])**.** *For $\lambda \in \mathcal{P}^+$ and $\mu \in P(\lambda)$, there exists an $\mathscr{A}$-basis of $V(\lambda)^{\mathscr{A}}_\mu$ whose associated $\mathsf{QSh}^{\mathsf{M}}_{\lambda,\mu}(X)$ and $\mathsf{RSh}^{\mathsf{M}}_{\lambda,\mu}(X)$ satisfy $D\mathsf{QSh}^{M}_{\lambda,\mu}(X) = \mathsf{RSh}^{M}_{\lambda,\mu}(X)$ for a diagonal matrix D all of whose diagonal entries belong to $v^{\mathbb{Z}}$.*

3.2. **Specialization to the basic representations.** Let $X = (a_{ij})_{i,j\in I}$ be a Cartan matrix of type A,D,E and let $\widehat{X} = X^{(1)}$ be the extended (generalized) Cartan matrix of X indexed by $\widehat{I} = \{0\} \sqcup I$ as in Figure 1. Let $(a_i)_{i\in\widehat{I}}$ be the numerical labels of $\widehat{X}$ in Figure 1 and let $\delta = \sum_{i\in\widehat{I}} a_i\alpha_i$. We set $U_v = U_v(\widehat{X})$ and apply the notation of §3.1 to this algebra. By [Kac, Lemma 12.6], we have $P(\Lambda_0) = \{w\Lambda_0 - d\delta \mid w \in W, d \geq 0\}$.

Definition 3.7. *For $d \geq 0$ and $w \in W$, we define $C^v_d(X)$ to be $\mathsf{QSh}^{\mathsf{M}}_{\Lambda_0, w\Lambda_0 - d\delta}(\widehat{X})$. For $\ell \geq 2$, we put $C^v_{\ell,d} = C^v_d(A_{\ell-1})$.*

The equivalence class of $C^v_d(X)$ under $\coloneqq$ does not depend on the choice of $w \in W$ [Tsu, Proposition 3.18]. The following is implicit in the proof of [Tsu, Theorem 4.4]. For convenience, we give a proof.

Theorem 3.8. *Let $X = (a_{ij})_{i,j\in I}$ be a Cartan matrix of type A, D or E, where $I = \{1, \dots, \ell\}$. For any $d \geq 0$, we have $C^v_d(X) \equiv_{\mathscr{A}} M^{-1}_{\ell,d} S^d([X]) M_{\ell,d}$ where $[X] = ([a_{ij}]) \in \mathsf{Mat}_I(\mathscr{A})$.*

Proof. Let $I = \{1, \dots, \ell\}$. As in the proof of [Tsu, Theorem 4.4], $V(\Lambda_0)^{\mathscr{A}}_{\Lambda_0 - d\delta}$ can be regarded as an $\mathscr{A}$-lattice of the polynomial ring $\mathbf{k}[h_{i,-r} \mid i \in I, r \geq 1]$. More precisely, defining new variables $y^{(i)}_r$ and $x^{(i)}_r$ (for $i \in I$, $r \geq 1$) by $y^{(i)}_r = h_{i,-r}/[r]$ and (2.8), we have

(i) $V(\Lambda_0)_{\Lambda_0 - d\delta}$ has a $\mathbf{k}$-basis $\{y^{(\underline{i})}_\lambda \mid (\lambda, \underline{i}) \in \Omega_{\ell,d}\}$,
(ii) $V(\Lambda_0)^{\mathscr{A}}_{\Lambda_0 - d\delta}$ has an $\mathscr{A}$-basis $\{x^{(\underline{i})}_\lambda \mid (\lambda, \underline{i}) \in \Omega_{\ell,d}\}$,

where $x^{(\underline{i})}_\lambda$ and $y^{(\underline{i})}_\lambda$ are defined as in Proposition 2.17. Moreover, by an identity in the proof of [Tsu, Theorem 4.4][1] (together with the definition of $\langle\cdot,\cdot\rangle_{\mathsf{QSh}}$), we have

$$\left\langle s y^{(i)}_s H, y^{(i_1)}_{r_1} \cdots y^{(i_m)}_{r_m} \right\rangle_{\mathsf{QSh}} = \left\langle H, \textstyle\sum_{k=1}^m \delta_{s,r_k} [a_{i,i_k}]_s y^{(i_1)}_{r_1} \cdots y^{(i_{k-1})}_{r_{k-1}} y^{(i_{k+1})}_{r_{k+1}} \cdots y^{(i_m)}_{r_m} \right\rangle_{\mathsf{QSh}}$$

[1] Our $x^{(i)}_r$ and $y^{(i)}_r$ correspond respectively to $\tilde{P}^-_{i,r}$ and $h'_{i,-r}$ in *loc. cit.*

for $H \in \mathbf{k}[h_{i,-r} \mid i \in I, r \geq 1]$ and $i, i_k \in I, s, r_k \geq 1$. We can rewrite this identity as

$$\Big\langle s y_s^{(i)} H, H' \Big\rangle_{\mathsf{QSh}} = \Big\langle H, \textstyle\sum_{j=1}^{\ell} [a_{i,j}]_s \dfrac{\partial}{\partial y_s^{(j)}} H' \Big\rangle_{\mathsf{QSh}}.$$

Therefore, by Corollary 2.19, we have $(\langle x_\lambda^{(\underline{i})}, x_\mu^{(\underline{j})} \rangle_{\mathsf{QSh}})_{(\lambda,\underline{i}),(\mu,\underline{j}) \in \Omega_{\ell,d}} = K_{\ell,d}^{-1} S^d([X]) K_{\ell,d} M_K$ where $M_K \in \mathrm{GL}_{\mathsf{Par}_{\ell-1}(d)}(\mathbb{Z})$. By Remark 2.16, we are done. □

Lemma 3.9. *For $\ell \geq 1$, there exist $Q_\ell, T_\ell \in \mathrm{GL}_\ell(\mathscr{A})$ such that $Q_\ell[A_\ell]T_\ell = [A'_\ell]$ where $A'_\ell = \mathsf{diag}(\{1, \dots, 1, \ell+1\}) \in \mathsf{Mat}_\ell(\mathbb{Z})$.*

Proof. Define $Q_\ell \in \mathsf{Mat}_\ell(\mathscr{A})$ by

$$(Q_\ell)_{ij} = \begin{cases} v^j[i] & \text{if } j = i \text{ or } j = i+1, \\ v^i[j] & \text{if } j < i, \\ 0 & \text{otherwise.} \end{cases}$$

A straightforward calculation shows that the matrix $Q_\ell[A_\ell]$ is upper-triangular with diagonal entries $1, \dots, 1, v^\ell[\ell+1]$ and hence is column equivalent to $[A'_\ell]$ over $\mathscr{A}$. Also, $\det(Q_\ell[A_\ell]) = v^\ell[\ell+1]$. We have $\det([A_\ell]) = [\ell+1]$ by an easy inductive argument (cf. [Tsu], proof of Corollary 4.5). Hence, $\det(Q_\ell) = v^\ell$, so $Q_\ell \in \mathrm{GL}_\ell(\mathscr{A})$. □

Theorem 3.10. *For $\ell \geq 2$ and $d \geq 0$, we have*

$$C^v_{\ell,d} \equiv_{\mathscr{A}} \bigoplus_{s=0}^{d} \Big(M_s \, \mathsf{diag}(\{\textstyle\prod_{i\geq 1} [\ell]_i^{m_i(\lambda)} \mid \lambda \in \mathsf{Par}(s)\}) M_s^{-1} \Big)^{\oplus |\mathsf{Par}_{\ell-2}(d-s)|}. \tag{3.2}$$

Proof. By Theorem 3.8, we have $C^v_{\ell,d} \equiv_{\mathscr{A}} M_{\ell-1,d}^{-1} S^d([A_{\ell-1}]) M_{\ell-1,d}$. Let $Q_{\ell-1}$ and $T_{\ell-1}$ be the matrices supplied by Lemma 3.9. By the functoriality of symmetric powers, $S^d(Q_{\ell-1}) S^d([A_{\ell-1}]) S^d(T_{\ell-1}) = S^d([A'_{\ell-1}])$. Further, the matrices $M_{\ell-1,d}^{-1} S^d(Q_{\ell-1}) M_{\ell-1,d}$ and $M_{\ell-1,d}^{-1} S^d(T_{\ell-1}) M_{\ell-1,d}$ belong to $\mathrm{GL}_{\mathsf{Par}_{\ell-1}(d)}(\mathscr{A})$. Indeed, these matrices are $\mathscr{A}$-valued by Theorem 2.20, and their determinants are invertible elements of $\mathscr{A}$ since that is the case for the determinants of $Q_{\ell-1}$, $T_{\ell-1}$. Therefore,

$$C^v_{\ell,d} \equiv_{\mathscr{A}} M_{\ell-1,d}^{-1} S^d([A_{\ell-1}]) M_{\ell-1,d} = M_{\ell-1,d}^{-1} S^d([A'_{\ell-1}]) M_{\ell-1,d}. \tag{3.3}$$

It follows from Definition 2.13 that (see §1.7.2)

$$S^d([A'_{\ell-1}]) = \bigoplus_{\sum_{i=1}^{\ell-1} d_i = d} \Big(\Big(\bigotimes_{j=1}^{\ell-2} 1_{\mathsf{Par}(d_j)} \Big) \otimes \mathsf{diag}(\{\textstyle\prod_{i\geq 1} [\ell]_i^{m_i(\lambda)} \mid \lambda \in \mathsf{Par}(d_{\ell-1})\}) \Big).$$

Substituting this identity and the formula of Remark 2.15 into (3.3), we obtain

$$C^v_{\ell,d} \equiv_{\mathscr{A}} \bigoplus_{s=0}^{d} \Big(M_{1,s}^{-1} \, \mathsf{diag}(\{\textstyle\prod_{i\geq 1} [\ell]_i^{m_i(\lambda)} \mid \lambda \in \mathsf{Par}(s)\}) M_{1,s} \Big)^{\oplus |\mathsf{Par}_{\ell-2}(d-s)|}. \tag{3.4}$$

By Corollary 3.5, we have $C^v_{\ell,d} \equiv_{\mathscr{A}} {}^{\mathrm{tr}}C^v_{\ell,d}$. Hence, transposing both sides of (3.4) and using the fact that ${}^{\mathrm{tr}}M_{1,s} = M_s$ (see Remark 2.15), we obtain (3.2). □

Remark 3.11. In the rest of the paper, we will see an implication of Conjecture 1.9 for "invariant factors" of $C^v_{\ell,d}$ (Proposition 3.15) and give evidence for Conjecture 1.9 (Theorem 1.10). For Cartan matrices X of the other simply-laced finite types (D and E), we can prove the existence of $Q_X, T_X \in \mathrm{GL}_I(\mathscr{A})$ such that

(a) $Q_X[X]T_X = \mathsf{diag}(\{1, \dots, 1, \det[X]\})$ for $X \neq D_{2m}$,
(b) $Q_X[X]T_X = \mathsf{diag}(\{1, \dots, 1, [2], [2]_{2m-1}\})$ for $X = D_{2m}$,

where $m \geq 2$ (for the ungraded case $v = 1$, see [Hil, Table 1]). For the value of $\det[X]$, see [Tsu, proof of Corollary 4.5]. These results allow us to analyze $C^v_d(X)$ further: a conjectural formula for invariant factors of $\mathsf{QSh}^{\mathsf{M}}_{\Lambda_0,\mu}(Z)$ for $\mu \in P(\Lambda_0)$ and evidence for it in the spirit of this paper when $Z = X^{(1)}$ and X is of type D or E as well as for the *twisted* affine A,D,E cases will be given elsewhere. Results on these invariant factors would provide information on modular reductions of $V(\Lambda_0)^{\mathscr{A}}$, namely, on the structure of the $F \otimes_{\mathscr{A}} U_v^{\mathscr{A}}$-module $F \otimes_{\mathscr{A}} V(\Lambda_0)^{\mathscr{A}}$ and its unique simple quotient, where F is any field, viewed as an $\mathscr{A}$-module via a fixed ring homomorphism $\mathscr{A} \to F$.

3.3. **Graded Cartan matrices and implications of Conjecture 1.9.**

Definition 3.12. *Let A be a finite-dimensional graded algebra over a field $\mathbb{F}$, i.e., A has a decomposition $A = \bigoplus_{i\in\mathbb{Z}} A_i$ into $\mathbb{F}$-vector spaces such that $A_iA_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}$.*

(a) *We denote by $\mathsf{Mod}_{\mathsf{gr}}(A)$ the abelian category of finite-dimensional left graded A-modules and degree preserving A-homomorphisms between them. The n-component of $M \in \mathsf{Mod}_{\mathsf{gr}}(A)$ is denoted by M_n. For $M \in \mathsf{Mod}_{\mathsf{gr}}(A)$ and $k \in \mathbb{Z}$, the shifted graded module $M\langle k\rangle$ of M is defined to be the same module as M with the grading given by $(M\langle k\rangle)_n = M_{k+n}$ for all $n \in \mathbb{Z}$.*

(b) *Fix a grading on each simple A-module, and let $\mathcal{S}(A)$ be the resulting set of graded simple modules. We define the graded Cartan matrix C_A^v of A by*

$$C_A^v = \big(\textstyle\sum_{k\in\mathbb{Z}}[\mathsf{PC}(D) : D'\langle -k\rangle]v^k\big)_{D,D'\in\mathcal{S}(A)} \in \mathsf{Mat}_{\mathcal{S}(A)}(\mathscr{A}),$$

where $\mathsf{PC}(D)$ is the projective cover of $D \in \mathsf{Mod}_{\mathsf{gr}}(A)$.

(c) *Let $\mathsf{Proj}_{\mathsf{gr}}(A)$ be the full subcategory of $\mathsf{Mod}_{\mathsf{gr}}(A)$ consisting of graded projective A-modules. The Cartan pairing is defined as follows:*

$$\langle\cdot,\cdot\rangle\colon [\mathsf{Proj}_{\mathsf{gr}}(A)] \times [\mathsf{Mod}_{\mathsf{gr}}(A)] \longrightarrow \mathscr{A}, \quad \langle [P], [M]\rangle = \sum_{k\in\mathbb{Z}} \dim_{\mathbb{F}} \mathrm{Hom}_A(P, M\langle k\rangle)v^k,$$

where $[M]$ denotes the image of M in the graded Grothendieck group $[\mathsf{Mod}_{\mathsf{gr}}(A)]$ of $\mathsf{Mod}_{\mathsf{gr}}(A)$, which has an $\mathscr{A}$-module structure given by $v[N] = [N\langle -1\rangle]$ for $N \in \mathsf{Mod}_{\mathsf{gr}}(A)$.

Remark 3.13. (a) Each simple A-module has a unique grading up to grading shift (see [NVO, Theorem 9.6.8]). Moreover, each simple graded A-module has a unique graded projective cover. Consequently, changing $\mathcal{S}(A)$ results in C_A^v being conjugated by a diagonal matrix with integer powers of v on the diagonal. Certainly, the $\mathscr{A}$-unimodular equivalence class of C_A^v does not depend on the choice of $\mathcal{S}(A)$,

(b) $C_A^v = (\langle [\mathsf{PC}(D)], [\mathsf{PC}(D')]\rangle)_{D',D\in\mathcal{S}(A)}$ when $\mathbb{F}$ is a splitting field for A,

(c) C_A^v is a refinement of C_A in the sense that $C_A^v|_{v=1} = C_A$.

Let $\ell \geq 2$ and $n \geq 0$. As usual, a partition ρ is an *ℓ-core* if ρ contains no rim ℓ-hooks. We denote by $\mathsf{Bl}_\ell(n)$ the set of tuples (ρ, d) where ρ is an ℓ-core and $d \geq 0$ is an integer such that $|\rho| + \ell d = n$. It is well known that the set $\mathsf{Bl}_\ell(n)$ parameterizes the blocks of $\mathcal{H}_n(k_\ell; \eta_\ell)$ (see [DJ]). When $\ell = p$ is a prime, $\mathsf{Bl}_\ell(n)$ parameterizes the blocks of $\mathbb{F}_p\mathfrak{S}_n$. We denote by $B_{\rho,d}^{(\ell)}$ the corresponding block algebra of $A := \mathcal{H}_n(k_\ell; \eta_\ell)$ or of $A := \mathbb{F}_p\mathfrak{S}_n$ for $(\rho, d) \in \mathsf{Bl}_\ell(n)$ (for the latter case, $\ell = p$ is a prime).

From now on, we view $B_{\rho,d}^{(\ell)}$ as a graded algebra, with the grading defined by [BK1, Corollary 1] (cf. §1.3). Consequently, A becomes graded. Clearly, we have

$$C_A^v \equiv_{\mathscr{A}} \bigoplus\nolimits_{(\rho,d)\in\mathsf{Bl}_\ell(n)} C^v_{B_{\rho,d}^{(\ell)}}. \tag{3.5}$$

In fact, the two sides are equal if appropriate choices are made.

By [BK3, Theorem 4.18], there is an isomorphism $\iota\colon [\mathsf{Proj}_{\mathsf{gr}}(A)] \xrightarrow{\sim} V(\Lambda_0)^{\mathscr{A}}$ as $U_v(A_{\ell-1}^{(1)})$-modules, which identifies the Cartan pairing $\langle\cdot,\cdot\rangle$ with the form $\langle\cdot,\cdot\rangle_{\mathsf{RSh}}$ on $V(\Lambda_0)^{\mathscr{A}}$. For $(\rho, d) \in \mathsf{Bl}_\ell(n)$, we have $\iota([\mathsf{Proj}_{\mathsf{gr}}(B_{\rho,d}^{(\ell)})]) = V(\Lambda_0)^{\mathscr{A}}_{\Lambda_0-\beta_{\rho,d}}$ where $\beta_{\rho,d} \in \sum_{i\in\widehat{I}}\mathbb{Z}_{\geq 0}\alpha_i$ is defined as in [Tsu, Definition 5.5(c)] under the identification $\widehat{I} \cong \mathbb{Z}/\ell\mathbb{Z}$. Noting Remark 3.13 (b), we have $C^v_{B_{\rho,d}^{(\ell)}} \coloneqq \mathsf{RSh}^{\mathsf{M}}_{\Lambda_0,\Lambda_0-\beta_{\rho,d}}(A_{\ell-1}^{(1)})$ (see Definition 3.3).

By Proposition 3.6, Definition 3.7 and the fact that $\Lambda_0 - \beta_{\rho,d} = w\Lambda_0 - d\delta$ for some $w \in W(A_{\ell-1}^{(1)})$, we obtain the following result, which is implicit in the proof of [Tsu, Theorem 5.6].

Proposition 3.14. *Let $\ell \geq 2$ and $n \geq 0$. For any $(\rho, d) \in \mathsf{Bl}_\ell(n)$, we have $C^v_{B_{\rho,d}^{(\ell)}} \equiv_{\mathscr{A}} C^v_{\ell,d}$.*

The following is an immediate consequence of Theorem 3.10.

Proposition 3.15. *Let $\ell \geq 2$ and let $d \geq 0$. If Conjecture 1.9 is true, then*

$$C^v_{\ell,d} \equiv_{\mathscr{A}} \mathsf{diag}\left(\bigsqcup_{s=0}^{d}\{I_\ell^v(\lambda) \mid \lambda \in \mathsf{Par}(s)\}^{|\mathsf{Par}_{\ell-2}(d-s)|}\right). \tag{3.6}$$

Lemma 3.16 ([BH, Lemma 5.5]). *For any $\ell \geq 2$ and $n \geq 0$, we have the multiset identity*

$$\bigsqcup_{(\rho,d)\in \mathsf{Bl}_\ell(n)} \bigsqcup_{s=0}^{d} \bigsqcup_{\lambda\in \mathsf{Par}(s)} \{\mathsf{cut}_\ell(\lambda)\}^{|\mathsf{Par}_{\ell-2}(d-s)|} = \{\mathsf{red}_\ell(\lambda) \mid \lambda \in \mathsf{CRP}_\ell(n)\}$$

where the maps $\mathsf{cut}_\ell, \mathsf{red}_\ell \colon \mathsf{Par} \to \mathsf{Par}$ are defined as follows for $k \geq 1$:

$$m_k(\mathsf{red}_\ell(\lambda)) = \lfloor m_k(\lambda)/\ell \rfloor, \quad m_k(\mathsf{cut}_\ell(\lambda)) = \begin{cases} m_k(\lambda) & \text{if } k \notin \ell\mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $r^v_\ell(\lambda) = I^v_\ell(\mathsf{red}(\lambda))$ and $I^v_\ell(\lambda) = I^v_\ell(\mathsf{cut}(\lambda))$ for all $\lambda \in \mathsf{Par}$. Combining these identities and Lemma 3.16 with (3.5) and Proposition 3.14, we see the following implication.

Corollary 3.17. *Conjecture 1.9 implies Conjecture 1.6.*

Remark 3.18. When $\ell = p^r$ is a prime power, the equivalence (3.6) is nothing but [Tsu, Conjecture 6.8]. Similarly, Conjecture 1.6 reduces to [Tsu, Conjecture 6.18] in this case. Indeed, the Laurent polynomials $I^v_{p,r}(\lambda)$ and $r^v_{p,r}(\lambda)$ defined in *loc. cit.* satisfy $I^v_{p,r}(\lambda) = I^v_{p^r}(\lambda)$ and $r^v_{p,r}(\lambda) = r^v_{p^r}(\lambda)$.

4. Combinatorial reductions

4.1. **Variants of unimodular equivalences.**

Definition 4.1. *Let R be a commutative ring, and let Y and Z be $n \times m$-matrices with entries in R. We say that Y and Z are*

(a) *unimodularly pseudo-equivalent over R (abbreviated as $Y \equiv'_R Z$) if we have $\mathsf{Cok}_Y \cong \mathsf{Cok}_Z$ as R-modules where $\mathsf{Cok}_T = \mathsf{Coker}(R^m \to R^n, \boldsymbol{v} \mapsto T\boldsymbol{v})$ for $T \in \{Y, Z\}$,*

(b) *Fitting equivalent (abbreviated as $Y \equiv^F_R Z$) if Cok_Y and Cok_Z have the same Fitting invariants (see* [Nor, §3.1]*), i.e., we say that $Y \equiv^F_R Z$ if $\mathsf{Fitt}_d(Y) = \mathsf{Fitt}_d(Z)$ whenever $0 \leq d < r := \min\{m, n\}$ where the d-th Fitting ideal $\mathsf{Fitt}_d(T)$ of $T \in \{Y, Z\}$ over R is the ideal of R generated by all $(r-d) \times (r-d)$-minors of T.*

Proposition 4.2. *The following general statements hold:*

(a) $Y \equiv_R Z \Longrightarrow Y \equiv'_R Z \Longrightarrow Y \equiv^F_R Z$.

(b) *for a ring homomorphism $\phi \colon R \to R'$ (see §1.7.1), we have the implications $Y \equiv_R Z \Longrightarrow \phi(Y) \equiv_{R'} \phi(Z)$, $Y \equiv'_R Z \Longrightarrow \phi(Y) \equiv'_{R'} \phi(Z)$ and $Y \equiv^F_R Z \Longrightarrow \phi(Y) \equiv^F_{R'} \phi(Z)$.*

(c) *Let $(X_\lambda)_{\lambda\in\Lambda}$ and $(Y_\lambda)_{\lambda\in\Lambda}$ be families of R-valued matrices where Λ is a finite set and for each $\lambda \in \Lambda$ the matrix X_λ has the same dimensions as Y_λ. Then, for any $\sim \in \{\equiv_R, \equiv'_R, \equiv^F_R\}$, we have the implication $\forall \lambda \in \Lambda, X_\lambda \sim Y_\lambda \Longrightarrow \bigoplus_{\lambda\in\Lambda} X_\lambda \sim \bigoplus_{\lambda\in\Lambda} Y_\lambda$.*

(d) $Y \equiv^F_R Z \Longrightarrow Y \equiv_R Z$ *when R is a PID.*

(e) $Y \equiv^F_R Z \Longleftrightarrow \forall \mathfrak{m} \in \mathsf{max\text{-}Spec}(R), Y \equiv^F_{R_\mathfrak{m}} Z$,

(f) $Y \equiv'_R Z \Longrightarrow Y \equiv_R Z$ *when R is a semiperfect ring.*

Proof. (a) is obvious and (d) is "Elementary Divisor Theorem". The cases of $\equiv$ and $\equiv^F$ in (b) are obvious. The right exactness of the functor $R' \otimes_R$ - implies that $R' \otimes_R \mathsf{Cok}_Y \cong \mathsf{Cok}_{\phi(Y)}$. Thus, the case $\equiv'$ follows. When $\sim \in \{\equiv_R, \equiv'_R\}$, (c) is obvious. The case $\equiv^F_R$ follows from the equality $\mathsf{Fitt}_d(Y \oplus Z) = \sum_{d_1+d_2=d} \mathsf{Fitt}_{d_1}(Y)\,\mathsf{Fitt}_{d_2}(Z)$ (see [Nor, §3.1, Exercise 3]). (e) follows from the fact that for ideals I and J in R, we have $I = J \Longleftrightarrow \forall \mathfrak{m} \in \mathsf{max\text{-}Spec}(R), I_\mathfrak{m} = J_\mathfrak{m}$ (see e.g. [Kun, Chapter IV, Corollary 1.4]). For (f), when R is a local ring, for any given two R-module surjections $\alpha \colon R^k \twoheadrightarrow M$, $\beta \colon R^k \twoheadrightarrow N$, we can lift any R-module isomorphism $f \colon M \xrightarrow{\sim} N$ to the isomorphism $g \colon R^k \xrightarrow{\sim} R^k$ such that $f \circ \alpha = \beta \circ g$ by the Nakayama Lemma. Thus, (f) holds when R is local. Since a semiperfect ring is the same thing as a finite direct product of local rings [Lam, (23.11)], (f) follows by (b) (see also [LR, (4.3)]). □

By the reasoning used to prove Corollary 3.15 and Corollary 3.17, Proposition 4.2 (b) and (c) imply:

Corollary 4.3. *Let R be a commutative ring with a ring homomorphism $\phi \colon \mathscr{A} \to R$ and $\sim \in \{\equiv_R, \equiv'_R, \equiv^F_R\}$. Suppose that Conjecture 1.9 holds when we specialize $\mathscr{A}$ and $\equiv_\mathscr{A}$ to R and $\sim$ respectively via ϕ, i.e., that*

$$\phi\left(M_n\,\mathsf{diag}(\{J^v_\ell(\lambda) \mid \lambda \in \mathsf{Par}(n)\})M_n^{-1}\right) \sim \mathsf{diag}(\{\phi(I^v_\ell(\lambda)) \mid \lambda \in \mathsf{Par}(n)\})$$

for all $n \geq 0$. Then we have $\phi(Y) \sim \phi(Z)$ if either

(i) Y and Z are the matrices on the two sides of (3.6)*, or*
(ii) Y and Z are the matrices on the two sides of (1.2)*.*

Throughout, we omit ϕ(-) if ϕ is evident when we apply Proposition 4.2 (b).

4.2. **A pseudo-equivalence over $\mathbb{Z}_{(p)}[v, v^{-1}]$.**

Definition 4.4. *For $n \geq 3$, we denote by $\Phi_n \in \mathbb{Z}[v]$ the n-th cyclotomic polynomial and put $\Psi_n = v^{-\phi(n)/2}\Phi_n \in \mathscr{A}^{\mathsf{bar}}$ where ϕ is the Euler function: $\phi(n) = \#(\mathbb{Z}/n\mathbb{Z})^\times$.*

It is easy to see that, for $n, m \geq 1$,

$$[n]_m = \textstyle\prod_{b\leq 3,\, 2mn\in b\mathbb{Z},\, 2m\notin b\mathbb{Z}} \Psi_b. \tag{4.1}$$

Thus, each $I^v_\ell(\lambda)$ and $J^v_\ell(\lambda)$ is a product of certain scaled cyclotomic polynomials Ψ_b.

Definition 4.5. *Let $p \in \mathsf{Prm}$ and $z \in \mathbb{N} \setminus p\mathbb{Z}$. Let $P = \prod_{i\in I} \Psi_{b_i}$ be a finite product of scaled cyclotomic polynomials (with $b_i \geq 3$ for all $i \in I$, as in Definition 4.4). We define $\rho^{(p)}_z(P) = \prod_{(b_i)_{p'}=z} \Psi_{b_i}$.*

Recall the famous equality $\#\mathsf{CRP}_s(n) = \#\mathsf{RP}_s(n)$ for $s \geq 1$ and $n \geq 0$. We reserve the symbol $\varphi_{s,n}$ for an arbitrary bijection $\varphi_{s,n}\colon \mathsf{RP}_s(n) \xrightarrow{\sim} \mathsf{CRP}_s(n)$ and put $\varphi_s = \sqcup_{n\geq 0}\varphi_{s,n}$. As a standard choice, we can take the Glaisher bijection (see [ASY, §4], for example) for $s \geq 2$ or the Sylvester bijection for $s = 2$ (see [Bes], for example).

Definition 4.6. *Fix $M \geq 1$. For any $\lambda \in \mathsf{Par}$, consider the decomposition $\lambda = \lambda_{\mathsf{div}} + \lambda_{\mathsf{reg}}$ defined by $m_a(\lambda_{\mathsf{div}}) = M\lfloor m_a(\lambda)/M\rfloor$, $m_a(\lambda_{\mathsf{reg}}) = m_a(\lambda) - m_a(\lambda_{\mathsf{div}})$ for $a \geq 1$. We define a size-preserving auto-bijection $\beta_M\colon \mathsf{Par} \xrightarrow{\sim} \mathsf{Par}$ by $\beta_M(\lambda) = \mu + \varphi_M(\lambda_{\mathsf{reg}})$ where $m_{aM}(\mu) = m_a(\lambda_{\mathsf{div}})/M$ for $a \geq 1$ and $m_b(\mu) = 0$ for all $b \notin M\mathbb{Z}$.*

Definition 4.7. *For $\ell \geq 2$, $k, t \geq 1$ and $p \in \mathsf{Prm}$, define*

$$g^{(\ell,p)}_{k,t} = \begin{cases} [\ell_{p'}]_{k_{p'}(\ell t)_{\{p\}}} & \text{if } \nu_p(k) \geq \nu_p(\ell), \\ [\ell t_{\{p\}}/k_{\{p\}}]_k & \text{if } \nu_p(k) < \nu_p(\ell), \end{cases}$$

and set $I^v_{\ell,p}(\lambda) = \prod_{k\geq 1}\prod_{t=1}^{m_k(\lambda)} g^{(\ell,p)}_{k,t}$ for $\lambda \in \mathsf{Par}$.

Further, we define $f^{(\ell)}_{k,t} = [\ell_k t_{\pi(\ell_k)}]_{(\ell,k)t_{\pi(\ell_k)'}}$ and note that $I^v_\ell(\lambda) = \prod_{k\geq 1}\prod_{t=1}^{m_k(\lambda)} f^{(\ell)}_{k,t}$.

Proposition 4.8. *Let p be a prime, $\ell \geq 2$, and $z \in \mathbb{N} \setminus p\mathbb{Z}$. For any $\lambda \in \mathsf{Par}$, we have*

$$\rho^{(p)}_z(I^v_\ell(\lambda)) = \rho^{(p)}_z(I^v_{\ell,p}(\beta_{z/(z,2\ell)}(\lambda))).$$

First, we need two lemmas. Fix p and z to be as in the statement of the proposition. For any $k, t \geq 1$, define

$$\mathcal{F}^{(\ell,p)}_{k,t,z} = \{s \geq 0 \mid 2\ell t \in zp^s\mathbb{Z} \text{ and } 2(\ell,k)t_{\pi(\ell_k)'} \notin zp^s\mathbb{Z}\},$$

$$\mathcal{G}^{(\ell,p)}_{k,t,z} = \begin{cases} \{s \geq 0 \mid 2\ell t_{\{p\}}k_{p'} \in zp^s\mathbb{Z} \text{ and } 2(\ell t)_{\{p\}}k_{p'} \notin zp^s\mathbb{Z}\} & \text{if } \nu_p(k) \geq \nu_p(\ell), \\ \{s \geq 0 \mid 2\ell t_{\{p\}}k_{p'} \in zp^s\mathbb{Z} \text{ and } 2k \notin zp^s\mathbb{Z}\} & \text{if } \nu_p(k) < \nu_p(\ell). \end{cases}$$

The following is an immediate consequence of (4.1) and the definitions:

Lemma 4.9. *For $k, t \geq 1$, we have $\rho^{(p)}_z(f^{(\ell)}_{k,t}) = \prod_{s\in\mathcal{F}^{(\ell,p)}_{k,t,z}} \Psi_{zp^s}$ and $\rho^{(p)}_z(g^{(\ell,p)}_{k,t}) = \prod_{s\in\mathcal{G}^{(\ell,p)}_{k,t,z}} \Psi_{zp^s}$.*

Define $M = z/(z, 2\ell)$.

Lemma 4.10. *For any $k, t \geq 1$, we have $\rho^{(p)}_z(f^{(\ell)}_{k,tM}) = \rho^{(p)}_z(g^{(\ell,p)}_{kM,t})$.*

Proof. Due to Lemma 4.9, it is enough to show that $\mathcal{F}^{(\ell,p)}_{k,tM,z} = \mathcal{G}^{(\ell,p)}_{kM,t,z}$. Fix $s \geq 0$: we will show that $s \in \mathcal{F}^{(\ell,p)}_{k,tM,z}$ if and only if $s \in \mathcal{G}^{(\ell,p)}_{kM,t,z}$. Note that $M \notin p\mathbb{Z}$. If $2\ell t \notin p^s\mathbb{Z}$, then s belongs to neither of the sets in question, for the first conditions in the definitions of those sets fail. Thus, we may assume that $2\ell t \in p^s\mathbb{Z}$. Since we always have $2\ell M \in z\mathbb{Z}$ (due to the definition of M), now the first conditions in the

definitions of $\mathcal{F}^{(\ell,p)}_{k,tM,z}$ and $\mathcal{G}^{(\ell,p)}_{kM,t,z}$ are guaranteed to hold. So we may focus on the second conditions: it remains to show that

$$2(\ell,k)(tM)_{\pi(\ell_k)'} \in zp^s\mathbb{Z} \iff \begin{cases} 2(kM)_{p'}(\ell t)_{\{p\}} \in zp^s\mathbb{Z} & \text{if } \nu_p(k) \geq \nu_p(\ell), \\ 2kM \in zp^s\mathbb{Z} & \text{if } \nu_p(k) < \nu_p(\ell). \end{cases}$$

This follows from the conjunction of the following two equivalences:

$$2(\ell,k)(tM)_{\pi(\ell_k)'} \in p^s\mathbb{Z} \iff \begin{cases} 2\ell t \in p^s\mathbb{Z} & \text{if } \nu_p(k) \geq \nu_p(\ell), \\ 2k \in p^s\mathbb{Z} & \text{if } \nu_p(k) < \nu_p(\ell) \end{cases} \quad \text{and} \tag{4.2}$$

$$2(\ell,k)(tM)_{\pi(\ell_k)'} \in z\mathbb{Z} \iff 2kM \in z\mathbb{Z}. \tag{4.3}$$

The equivalence (4.2) is immediate in each of the cases on its right-hand side, so it remains only to prove (4.3).

We always have

$$(2(\ell,k)tM)_{\pi(\ell_k)'} = (2\ell tM)_{\pi(\ell_k)'} \in z_{\pi(\ell_k)'}\mathbb{Z}$$

since $2\ell M \in z\mathbb{Z}$. Further, $(2k)_{\pi(\ell_k)'} \in (2\ell)_{\pi(\ell_k)'}\mathbb{Z}$, so $(2kM)_{\pi(\ell_k)'} \in (2\ell M)_{\pi(\ell_k)'}\mathbb{Z} \subseteq z_{\pi(\ell_k)'}\mathbb{Z}$. This means that the truth values of the statements on both sides of (4.3) do not change if we replace z by $z_{\pi(\ell_k)}$. In other words, it is enough to show that for all $q \in \pi(\ell_k)$,

$$\nu_q(2(\ell,k)) \geq \nu_q(z) \iff \nu_q(2kM) \geq \nu_q(z). \tag{4.4}$$

Now $\nu_q(k) < \nu_q(\ell)$, so $\nu_q(2(\ell,k)) = \nu_q(2k)$. Using the definition of M, we obtain $\nu_q(2kM) = \nu_q(2k) + \nu_q(z) - \nu_q((2\ell,z))$. If $\nu_q(2\ell) \geq \nu_q(z)$, then $\nu_q(2kM) = \nu_q(2k)$ and the equivalence (4.4) is clear. Otherwise, we have $\nu_q(2k) < \nu_q(2\ell) < \nu_q(z)$ and neither side of (4.4) holds. □

Proof of Proposition 4.8. Fix $\lambda \in \mathsf{Par}$, and let $\lambda_{\mathsf{div}}, \lambda_{\mathsf{reg}}, \mu$ be as in Definition 4.6. It is clear that $I^v_{\ell,p}(\beta_M(\lambda)) = I^v_{\ell,p}(\mu) I^v_{\ell,p}(\varphi_M(\lambda_{\mathsf{reg}}))$. In the expansion $I^v_\ell(\lambda) = \prod_{k\geq 1}\prod_{t=1}^{m_k(\lambda)} f^{(\ell)}_{k,t}$, only $t \in M\mathbb{Z}$ contribute to $\rho^{(p)}_z(I^v_\ell(\lambda))$ by Lemma 4.9, as $2\ell t \in z\mathbb{Z}$ implies $t \in M\mathbb{Z}$. Further, $\rho^{(p)}_z(I^v_{\ell,p}(\varphi_M(\lambda_{\mathsf{reg}}))) = 1$ by the same lemma, as $2\ell t_{\{p\}} k_{p'} \notin z\mathbb{Z}$ for any $k \notin M\mathbb{Z}$ and $t \geq 1$. It follows that

$$\rho^{(p)}_z(I^v_\ell(\lambda)) = \prod_{k\geq 1} \prod_{t=1}^{\lfloor m_k(\lambda)/M \rfloor} \rho^{(p)}_z(f^{(\ell)}_{k,tM}),$$

$$\rho^{(p)}_z(I^v_{\ell,p}(\beta_M(\lambda))) = \prod_{k\geq 1} \prod_{t=1}^{\lfloor m_k(\lambda)/M \rfloor} \rho^{(p)}_z(g^{(\ell,p)}_{kM,t}).$$

The two right-hand sides are equal by Lemma 4.10. □

Proposition 4.11. *Let R be a commutative ring and let $a \in R$. Suppose that $a = \prod_{\lambda\in\Lambda}\prod_{x\in T_\lambda} x$ for a finite set Λ and a family of finite multisets $(T_\lambda \subseteq R)_{\lambda\in\Lambda}$ such that any $x \in T_\lambda$ and $x' \in T_{\lambda'}$ are coprime (i.e., $xy + x'y' = 1$ for some $y, y' \in R$) whenever $\lambda \neq \lambda'$. Then, as R-modules, we have*

$$R/(a) \cong \bigoplus_{\lambda\in\Lambda} R/(\textstyle\prod_{x\in T_\lambda} x).$$

Proof. Observe that $(\prod_{x\in T_\lambda} x)_{\lambda\in\Lambda}$ are pairwise coprime (in the above sense): this follows from the elementary fact that if $x, y, z \in R$ and x, y are both coprime to z, then xy is coprime to z. Now the proposition follows from the Chinese remainder theorem for ideals. □

Corollary 4.12. *For $p \in \mathsf{Prm}$ and $n \geq 0, \ell \geq 2$, we have*

$$\mathsf{diag}(\{I^v_\ell(\lambda) \mid \lambda \in \mathsf{Par}(n)\}) \equiv'_{\mathbb{Z}_{(p)}[v,v^{-1}]} \mathsf{diag}(\{I^v_{\ell,p}(\lambda) \mid \lambda \in \mathsf{Par}(n)\}).$$

Proof. Whenever $3 \leq b < c$ and c/b is not a p-power, there exist $u, w \in \mathbb{Z}_{(p)}[v, v^{-1}]$ such that $\Psi_b u + \Psi_c w = 1$ (see [Fil, Lemma 2]). By Proposition 4.11, we have

$$\mathsf{Cok}_{\mathsf{diag}(\{f(\lambda)\mid\lambda\in\mathsf{Par}(n)\})} \cong \bigoplus_{\lambda\in\mathsf{Par}(n)} \bigoplus_{z\in\mathbb{N}\setminus p\mathbb{Z}} \mathbb{Z}_{(p)}[v,v^{-1}]/(\rho^{(p)}_z(f(\lambda))) \tag{4.5}$$

as $\mathbb{Z}_{(p)}[v, v^{-1}]$-modules for any $f \in \{I^v_\ell, I^v_{\ell,p}\}$. By Proposition 4.8, the isomorphism class of (4.5) does not depend on the choice of f. □

4.3. **A conditional proof of Theorem 1.10.**

Proof of Theorem 1.10 (a). For $\lambda \in \mathsf{Par}$ and $\ell \geq 2$ we have $I^v_{\ell,p}(\lambda) = J^v_\ell(\lambda)$ for every sufficiently large $p \in \mathsf{Prm}$ (as $g^{(\ell,p)}_{k,t} = [\ell]_k$ for $p > \max(k,t,\ell)$). Thus, Theorem 1.10(a) is a consequence of Corollary 4.12 (note that $M_n \in \mathrm{GL}_{\mathsf{Par}(n)}(\mathbb{Q})$). □

Recall the matrix $N^{(p)}_n$ defined in §2.2. Applying Proposition 2.7 (a) for

$$R = \mathbb{Z}_{(p)}[v, v^{-1}], \quad f = J^v_\ell, \quad g = I^v_{\ell,p}, \quad (r_j = \mathsf{Infl}_j \colon R \to R, v \mapsto v^j)_{j \in \mathbb{N} \setminus p\mathbb{Z}}, \tag{4.6}$$

we get the following.

Proposition 4.13. *For any $p \in \mathsf{Prm}$, $\ell \geq 2$ and $n \geq 0$, the matrix $N^{(p)}_n \,\mathsf{diag}(\{J^v_\ell(\lambda) \mid \lambda \in \mathsf{Pow}_p(n)\})(N^{(p)}_n)^{-1}$ is $\mathbb{Z}_{(p)}[v, v^{-1}]$-valued.*

Further, in §5, we will prove the following result.

Theorem 4.14. *Suppose that $0 \neq \theta = a/b \in \mathbb{Q}$, where $a, b \in \mathbb{Z}$ and $(a,b) = 1$. Let p be a prime such that $a, b \notin p\mathbb{Z}$. Then, for any $\ell \geq 2$ and $n \geq 0$, we have*

$$N^{(p)}_n \,\mathsf{diag}(\{J^v_\ell(\lambda)|_{v=\theta} \mid \lambda \in \mathsf{Pow}_p(n)\})(N^{(p)}_n)^{-1} \equiv_{\mathbb{Z}_{(p)}} \mathsf{diag}(\{I^v_{\ell,p}(\lambda)|_{v=\theta} \mid \lambda \in \mathsf{Pow}_p(n)\}). \tag{4.7}$$

Proof of Theorem 1.10 (b) assuming Theorem 4.14. Fix $0 \neq \theta = a/b \in \mathbb{Q}$ with $a, b \in \mathbb{Z}$, $(a,b) = 1$. By Proposition 4.2 (d) and (e), it is enough to show that,

$$M_n \,\mathsf{diag}(\{J^v_\ell(\lambda)|_{v=\theta} \mid \lambda \in \mathsf{Par}(n)\})M_n^{-1} \equiv_{\mathbb{Z}_{(p)}} \mathsf{diag}(\{I^v_\ell(\lambda)|_{v=\theta} \mid \lambda \in \mathsf{Par}(n)\}) \tag{4.8}$$

for any $p \in \{p \in \mathsf{Prm} \mid p\mathbb{Z} \not\ni ab\} \cong \mathsf{Spec}(\mathbb{Z}[a/b, b/a])$. Applying Proposition 2.7 (b) for $R' = \mathbb{Z}_{(p)}, \phi = (R \to R', v \mapsto \theta)$ in addition to (4.6), we obtain

$$M_n \,\mathsf{diag}(\{J^v_\ell(\lambda)|_{v=\theta} \mid \lambda \in \mathsf{Par}(n)\})M_n^{-1} \equiv_{\mathbb{Z}_{(p)}} \mathsf{diag}(\{I^v_{\ell,p}(\lambda)|_{v=\theta} \mid \lambda \in \mathsf{Par}(n)\}).$$

The unimodular equivalence (4.8) now follows by substituting $v = \theta$ to Corollary 4.12 and Proposition 4.2 (a). □

5. Proof of Theorem 4.14

5.1. **Elementary prime power estimates.** The following fact is classical.

Proposition 5.1. *Let p be a prime. Suppose that $x, y \in \mathbb{Z} \setminus p\mathbb{Z}$ satisfy $d := \nu_p(x-y) \geq 1$. If either $p \geq 3$ or $d \geq 2$, then $\nu_p(x^n - y^n) = d + \nu_p(n)$ for all $n \geq 1$.*

Proof. We have $x = y + p^d z$ for some $z \in \mathbb{Z} \setminus p\mathbb{Z}$. The binomial expansion yields

$$x^n - y^n = np^d z y^{n-1} + \sum_{k=2}^{n} \binom{n}{k} p^{kd} z^k y^{n-k},$$

so it suffices to show that $\nu_p(\binom{n}{k}) + kd > d + \nu_p(n)$ for $2 \leq k \leq n$. Since $\nu_p(\binom{n}{k}) \geq \nu_p(n) - \nu_p(k!)$, it is enough to prove the inequality $kd - \nu_p(k!) - d > 0$. Using (2.5), we easily see that $\nu_p(k!) \leq k-1$ and that this inequality is strict unless $k = p = 2$. It follows that the desired inequality holds unless we have $d = 1$ and $k = p = 2$, which is ruled out by the hypothesis. □

Corollary 5.2. *Let $p \in \mathsf{Prm}$ and let $a, b \in \mathbb{Z} \setminus p\mathbb{Z}$ with $a^2 - b^2 \in p\mathbb{Z}$. Then, we have $\nu_p([n]_m|_{v=a/b}) = \nu_p(n)$ for all $n, m \geq 1$.*

Proof. We may assume that $a^2 \neq b^2$: otherwise, $[n]_m|_{v=a/b} = \pm n$. Consider $d \geq 1$ and $z \in \mathbb{Z} \setminus p\mathbb{Z}$ such that $a^2 - b^2 = p^d z$. Note that $d \geq 2$ if $p = 2$. By Proposition 5.1, we have

$$\nu_p([n]_m|_{v=a/b}) = \nu_p\left(\frac{a^{2nm} - b^{2nm}}{a^{2m} - b^{2m}}\right) = (\nu_p(nm) + d) - (\nu_p(m) + d) = \nu_p(n). \qquad \square$$

Corollary 5.3. *Let $p \geq 3$ be a prime and $a, b \in \mathbb{Z} \setminus p\mathbb{Z}$. Suppose that $a^2 - b^2 \notin p\mathbb{Z}$ and $a^{2n} - b^{2n} \in p\mathbb{Z}$ for some $n \geq 2$. Put $\gamma = \nu_p(a^{t_0} - b^{t_0})$ where $t_0 = \min\{t \geq 1 \mid a^{2t} - b^{2t} \in p\mathbb{Z}\}$ (t_0 exists and divides n). Then $\nu_p([n]_{p^s}|_{v=a/b}) = \nu_p(n) + s + \gamma$ for any $s \geq 0$.*

Proof. Note that $t_0 \notin p\mathbb{Z}$. We have

$$\nu_p([n]_{p^s}|_{v=a/b}) = \nu_p\left(\frac{a^{2np^s} - b^{2np^s}}{a^{2p^s} - b^{2p^s}}\right) = \nu_p(a^{2np^s} - b^{2np^s}) \\ = \nu_p(2np^s/t_0) + \gamma = \nu_p(n) + s + \gamma,$$

where the third equality follows from Proposition 5.1. □

Proposition 5.4. *Let $p \in \mathsf{Prm}$ and $n \geq 1$. Suppose that $a, b \in \mathbb{Z} \setminus p\mathbb{Z}$ satisfy $a^{2n} - b^{2n} \notin p\mathbb{Z}$. Then, $\nu_p([n]_{p^s}|_{v=a/b}) = 0$ for any $s \geq 0$.*

Proof. The hypothesis implies that $a^{2np^s} - b^{2np^s} \notin p\mathbb{Z}$, whence we also have $a^{2p^s} - b^{2p^s} \notin p\mathbb{Z}$. Since $\nu_p([n]_{p^s}|_{v=a/b}) = \nu_p((a^{2np^s} - b^{2np^s})/(a^{2p^s} - b^{2p^s}))$, the result follows. □

5.2. **Some definitions and results from** [Evs, §5]**.** For the remainder of §5, we fix a prime p and an integer $n \geq 0$. The matrices considered in the sequel implicitly depend on these parameters. Let $\ell \geq 2$ and $\theta = a/b \in \mathbb{Q} \setminus \{0\}$ be as in the statement of Theorem 4.14. We set $r = \nu_p(\ell)$. In what follows, diagonal matrices are generally denoted by lower-case letters.

Define the matrices $b^{(\ell,\theta)} = \mathsf{diag}(\{J^v_\ell(\lambda)|_{v=\theta} \mid \lambda \in \mathsf{Pow}_p(n)\})$ and $z = \mathsf{diag}(\{z_\lambda \mid \lambda \in \mathsf{Pow}_p(n)\})$, where z_λ is given by (2.1).

Lemma 5.5 ([Evs, Lemma 5.1])**.** *The matrices $(N^{(p)}_n)^{-1}$ and $z^{-1}({}^{\mathrm{tr}}N^{(p)}_n)$ are column equivalent over $\mathbb{Z}_{(p)}$.*

We write $N = N^{(p)}_n$. It follows from the lemma that the left-hand side of Theorem 4.14 is unimodularly equivalent over $\mathbb{Z}_{(p)}$ to the matrix $Y := Nb^{(\ell,\theta)}z^{-1}({}^{\mathrm{tr}}N)$, so Theorem 4.14 is equivalent to the identity

$$Y \equiv_{\mathbb{Z}_{(p)}} \mathsf{diag}(\{I^v_{\ell,p}(\lambda)|_{v=\theta} \mid \lambda \in \mathsf{Pow}_p(n)\}). \tag{5.1}$$

Definition 5.6. *(a) For $\lambda \in \mathsf{Pow}_p$, we define partitions $\lambda^{<r}, \lambda^{\geq r}, \overline{\lambda}^{\,r} \in \mathsf{Pow}_p$ by setting $m_{p^i}(\lambda^{\geq r}) = m_{p^{r+i}}(\lambda)$,*

$$m_{p^i}(\lambda^{<r}) = \begin{cases} m_{p^i}(\lambda) & \text{if } i < r, \\ 0 & \text{if } i \geq r, \end{cases} \quad m_{p^i}(\overline{\lambda}^{\,r}) = \begin{cases} m_{p^i}(\lambda) & \text{if } i < r, \\ \sum_{j \geq r} p^{j-r} m_{p^j}(\lambda) & \text{if } i = r, \\ 0 & \text{if } i > r, \end{cases}$$

for all $i \geq 0$.

(b) For $\lambda \in \mathsf{Pow}_p$, we set $x_\lambda = \prod_{s \geq 0} m_{p^s}(\lambda)!$ and $y_\lambda = \prod_{s \geq 0} p^{s m_{p^s}(\lambda)}$, so that $z_\lambda = x_\lambda y_\lambda$.

(c) We define the following seven elements of $\mathsf{Mat}_{\mathsf{Pow}_p(n)}(\mathbb{Z})$: $x = \mathsf{diag}(\{x_\lambda\}_\lambda)$, $x_{<r} = \mathsf{diag}(\{x_{\lambda^{<r}}\}_\lambda)$, $x^{\geq r} = \mathsf{diag}(\{x_{\lambda^{\geq r}}\}_\lambda)$, $y^{<r} = \mathsf{diag}(\{y_{\lambda^{<r}}\}_\lambda)$, $y^{\geq r} = \mathsf{diag}(\{y_{\lambda^{\geq r}}\}_\lambda)$, $\widetilde{y}^{(r)} = \mathsf{diag}(\{\prod_{i \geq r} p^{r m_{p^i}(\lambda)}\}_\lambda)$ and $C^{(r)}$, where the latter is given by

$$(C^{(r)})_{\lambda,\mu} = \begin{cases} (N^{(p)}_{|\lambda^{\geq r}|})_{\lambda^{\geq r}, \mu^{\geq r}} & \text{if } \overline{\lambda}^{\,r} = \overline{\mu}^{\,r}, \\ 0 & \text{if } \overline{\lambda}^{\,r} \neq \overline{\mu}^{\,r}. \end{cases}$$

Here, λ, μ run over all elements of $\mathsf{Pow}_p(n)$.

Put $\mathcal{K}^{(p,r)} = \{\lambda \in \mathsf{Pow}_p \mid \overline{\lambda}^{\,r} = \lambda\} \subseteq \mathsf{Pow}_p$. For $\kappa \in \mathcal{K}^{(p,r)}$, set $\mathsf{Pow}_{p,r}(n,\kappa) := \{\lambda \in \mathsf{Pow}_p(n) \mid \overline{\lambda}^{\,r} = \kappa\}$. Observe that there is a bijection

$$\mathsf{Pow}_{p,r}(n,\kappa) \xrightarrow{\sim} \mathsf{Pow}_p(m_{p^r}(\kappa)), \quad \lambda \longmapsto \lambda^{\geq r}. \tag{5.2}$$

We will call a matrix $Z \in \mathsf{Mat}_{\mathsf{Pow}_p(n)}(\mathbb{Q})$ *block-diagonal* if $Z_{\lambda,\mu} = 0$ for all $\lambda, \mu \in \mathsf{Pow}_p(n)$ with $\overline{\lambda}^{\,r} \neq \overline{\mu}^{\,r}$. In particular, $C^{(r)}$ is block-diagonal. Applying Lemma 5.5 to each $\kappa \in \mathcal{K}^{(p,r)}_n := \mathsf{Pow}_p(n) \cap \mathcal{K}^{(p,r)}$ and noting that $\mathsf{Pow}_p(n) = \bigsqcup_{\kappa \in \mathcal{K}^{(p,r)}_n} \mathsf{Pow}_{p,r}(n,\kappa)$, we see that there exists a block-diagonal matrix $W^{(r)} \in \mathrm{GL}_{\mathsf{Pow}_p(n)}(\mathbb{Z}_{(p)})$ such that $(C^{(r)})^{-1}W^{(r)} = (x^{\geq r}y^{\geq r})^{-1} \cdot {}^{\mathrm{tr}}C^{(r)}$. We define $A^{(r)} = N(C^{(r)})^{-1}$ and $U^{(r)} = (x^{<r})^{-1}A^{(r)}$, so that $N = x^{<r}U^{(r)}C^{(r)}$.

In §5.3, §5.4, and §5.5, we consider separate cases and use Corollaries 5.2 and 5.3 and Proposition 5.4 respectively. The cases of §5.3 and §5.4 will require the following specialization of [Evs, Lemma 5.6].

Lemma 5.7. *Let R be a DVR with valuation $\nu\colon K^\times \twoheadrightarrow \mathbb{Z}$, where K is the field of fractions of R. Let I be a finite set. Suppose that P, Q, $s = \mathsf{diag}(\{s_i \mid i \in I\})$ and $t = \mathsf{diag}(\{t_i \mid i \in I\})$ are elements of $\mathrm{GL}_I(K)$ such that*

$$\nu(P_{ij} - \delta_{ij}) > \frac{\nu(t_i) - \nu(t_j)}{2} \quad \textit{and} \quad \nu(Q_{ij} - \delta_{ij}) > \frac{\nu(t_j) - \nu(t_i)}{2}$$

for all $i, j \in I$. Then $sPtQs \equiv_R s^2 t$.

Proof. Apply [Evs, Lemma 5.6] with $\alpha_i = v(t_i)/2$ and $\beta_i = -v(t_i)/2$. Verifying the hypotheses is straightforward. □

5.3. **Case $a^2 - b^2 \in p\mathbb{Z}$.** This is a generalization of the case $v = 1$, and we generalize the proof in [Evs, §5], Proposition 2.12 being an extra needed ingredient.

Observe that $z = x^{<r}x^{\geq r}y^{<r}y^{\geq r}\widetilde{y}^{(r)}$. Put $b^{(<r,\ell,\theta)} = \mathsf{diag}(\{J^v_\ell(\lambda^{<r})|_{v=\theta} \mid \lambda \in \mathsf{Pow}_p(n)\})$, $b^{(\geq r,\ell,\theta)} = b^{(\ell,\theta)}(b^{(<r,\ell,\theta)})^{-1}$ and $d = b^{(<r,\ell,\theta)}(x^{<r}y^{<r})^{-1}$. Let

$$X = C^{(r)} b^{(\geq r,\ell,\theta)} (\widetilde{y}^{(r)})^{-1} (C^{(r)})^{-1} W^{(r)}.$$

Note that all the matrices in this product are block-diagonal, so X is block-diagonal. Setting also $V = X \cdot {}^{\mathrm{tr}}U^{(r)} \cdot X^{-1}$, we have

$$\begin{aligned}
Y &= N b_n^{(\ell,\theta)} z^{-1} \cdot {}^{\mathrm{tr}}N \\
&= x^{<r}U^{(r)}C^{(r)}b^{(<r,\ell,\theta)}b^{(\geq r,\ell,\theta)}(x^{<r}x^{\geq r}y^{<r}y^{\geq r}\widetilde{y}^{(r)})^{-1} \cdot {}^{\mathrm{tr}}C^{(r)} \cdot {}^{\mathrm{tr}}U^{(r)}x^{<r} \\
&= x^{<r}U^{(r)}C^{(r)}d \cdot b^{(\geq r,\ell,\theta)}(\widetilde{y}^{(r)})^{-1}(x^{\geq r}y^{\geq r})^{-1} \cdot {}^{\mathrm{tr}}C^{(r)} \cdot {}^{\mathrm{tr}}U^{(r)}x^{<r} && (5.3)\\
&= x^{<r}U^{(r)}C^{(r)}d \cdot b^{(\geq r,\ell,\theta)}(\widetilde{y}^{(r)})^{-1}(C^{(r)})^{-1}W^{(r)} \cdot {}^{\mathrm{tr}}U^{(r)}x^{<r} && (5.4)\\
&= x^{<r}U^{(r)}C^{(r)}d \cdot (C^{(r)})^{-1}C^{(r)}b^{(\geq r,\ell,\theta)}(\widetilde{y}^{(r)})^{-1}(C^{(r)})^{-1}W^{(r)} \cdot {}^{\mathrm{tr}}U^{(r)}x^{<r} \\
&= x^{<r}U^{(r)}C^{(r)}d(C^{(r)})^{-1}X \cdot {}^{\mathrm{tr}}U^{(r)}x^{<r} && (5.5)\\
&= x^{<r}U^{(r)}d \cdot X \cdot {}^{\mathrm{tr}}U^{(r)}x^{<r} && (5.6)\\
&= x^{<r}U^{(r)}dVXx^{<r} && (5.7)\\
&= x^{<r}U^{(r)}dVx^{<r}X && (5.8)\\
&\equiv_{\mathbb{Z}_{(p)}} x^{<r}U^{(r)}dVx^{<r}. && (5.9)
\end{aligned}$$

Here, Equations (5.3), (5.4), (5.5) and (5.7) follow from the defining equations of the matrices d, $W^{(r)}$, X and V respectively. Equations (5.6) and (5.8) follow from the facts that the matrices $C^{(r)}$ and X are block diagonal and that any block-diagonal matrix commutes with $b^{(<r,\ell,\theta)}$, $x^{<r}$, $y^{<r}$, and hence also with d.

The equivalence (5.9) is due to the fact that $X \in \mathrm{GL}_{\mathsf{Pow}_p(n)}(\mathbb{Z}_{(p)})$, which may be proved as follows. Note that $(\widetilde{y}^{(r)})_{\lambda,\lambda} = p^{r\ell(\lambda^{\geq r})}$ for all $\lambda \in \mathsf{Pow}_p(n)$. We have

$$\begin{aligned}
&C^{(r)}b^{(\geq r,\ell,\theta)}(\widetilde{y}^{(r)})^{-1}(C^{(r)})^{-1} \\
&= \bigoplus_{\kappa \in \mathcal{K}^{(p)}} N^{(p)}_{m_{p^r}(\kappa)} \,\mathsf{diag}(\{p^{-r\ell(\mu)} \prod_{j\geq 0} [\ell]_{p^{r+j}}^{m_{p^j}(\mu)} \mid \mu \in \mathsf{Pow}_p(m_{p^r}(\kappa))\})(N^{(p)}_{m_{p^r}(\kappa)})^{-1}, && (5.10)
\end{aligned}$$

where the right-hand side is interpreted via the identification (5.2): this identity is readily verified from the definitions of the matrices involved. By Proposition 2.12, the right-hand side of (5.10) is $\mathbb{Z}_{(p)}$-valued. By Corollary 5.2, we have

$$\nu_p((b^{(\geq r,\ell,\theta)})_{\lambda,\lambda}) = \nu_p\left(\prod_{i\geq r} [\ell]_{p^i}^{m_{p^i}(\lambda)}\right) = r\ell(\lambda^{\geq r}) = \nu_p((\widetilde{y}^{(r)})_{\lambda,\lambda}).$$

for $\lambda \in \mathsf{Pow}_p(n)$. So the p-adic valuation of the determinant of the left-hand side of (5.10) is 0. Therefore, the left-hand side of (5.10) belongs to $\mathrm{GL}_{\mathsf{Pow}_p(n)}(\mathbb{Z}_{(p)})$. Since $W^{(r)} \in \mathrm{GL}_{\mathsf{Pow}_p(n)}(\mathbb{Z}_{(p)})$, we see that $X \in \mathrm{GL}_{\mathsf{Pow}_p(n)}(\mathbb{Z}_{(p)})$, as claimed.

We will complete the proof by applying Lemma 5.7 to the product $x^{<r}U^{(r)}dVx^{<r}$. For $\lambda \in \mathsf{Pow}_p(n)$, define

$$f_\lambda^{(r)} = \sum_{0\le s<r} (r-s)m_{p^s}(\lambda) \quad \text{and} \quad e_\lambda^{(r)} = \sum_{0\le s<r} \nu_p(m_{p^s}(\lambda)!).$$

We have $\nu_p((x_n^{<r})_{\lambda,\lambda}) = e_\lambda^{(r)}$. Using Corollary 5.2, we obtain

$$\begin{aligned}\nu_p(d_{\lambda,\lambda}) &= \nu_p(b_{\lambda,\lambda}^{(<r,\ell,\theta)}) - \nu_p(y_{\lambda,\lambda}^{<r}) - \nu_p(x_{\lambda,\lambda}^{<r}) = \sum_{0\le s<r} \nu_p([\ell]_{p^s}^{m_{p^s}(\lambda)}) - \sum_{0\le s<r} s m_{p^s}(\lambda) - e_\lambda^{(r)} \\ &= f_\lambda^{(r)} - e_\lambda^{(r)} =: k_\lambda^{(r)} \qquad \text{and} \\ \nu_p(I_{\ell,p}^v(\lambda)|_{v=\theta}) &= \sum_{s\ge 0}\sum_{t=1}^{m_{p^s}(\lambda)} \nu_p(g_{p^s,t}^{(\ell,p)}|_{v=\theta}) = \sum_{0\le s<r}\sum_{t=1}^{m_{p^s}(\lambda)} (r+\nu_p(t)-s) = f_\lambda^{(r)} + e_\lambda^{(r)}\end{aligned}$$

(cf. Definition 4.7). The hypotheses of Lemma 5.7 are verified as follows. By [Evs, Lemma 5.4], we have $\nu_p(U_{\lambda,\mu}^{(r)} - \delta_{\lambda,\mu}) > \max\{k_\lambda^{(r)} - k_\mu^{(r)}, -1\}$ for all $\lambda, \mu \in \mathsf{Pow}_p(n)$, which implies the first desired inequality, namely

$$\nu_p(U_{\lambda,\mu}^{(r)} - \delta_{\lambda,\mu}) > \frac{k_\lambda^{(r)} - k_\mu^{(r)}}{2}. \tag{5.11}$$

The second desired inequality concerns $V = X \cdot {}^{\mathrm{tr}}(U^{(r)}) \cdot X^{-1}$ and follows from (5.11) because $X \in \mathrm{GL}_{\mathsf{Pow}_p(n)}(\mathbb{Z}_{(p)})$ is block-diagonal and the right-hand side of (5.11) depends only on $\overline{\lambda}^r$ and $\overline{\mu}^r$.

By Lemma 5.7, we have $Y \equiv_{\mathbb{Z}_{(p)}} (x^{<r})^2 d$, and (5.1) follows because $\nu_p((x_{\lambda,\lambda}^{<r})^2 d_{\lambda,\lambda}) = f_\lambda^{(r)} + e_\lambda^{(r)} = \nu_p(I_{\ell,p}^v(\lambda)|_{v=\theta})$ for all $\lambda \in \mathsf{Pow}_p(n)$.

5.4. **Case $a^2 - b^2 \notin p\mathbb{Z}$ and $a^{2\ell} - b^{2\ell} \in p\mathbb{Z}$.** Note that the assumption implies that $p \ge 3$. Let γ be as in Corollary 5.3. Applying that corollary, we obtain $\nu_p(g_{p^s,t}^{(\ell,p)}|_{v=\theta}) = \gamma + r + \nu_p(t)$ for all $t \ge 1$ and $s \ge 0$ (see Definition 4.7). Hence,

$$\nu_p(I_{\ell,p}^v(\lambda)|_{v=\theta}) = (\gamma+r)\ell(\lambda) + \sum_{s\ge 0} \nu_p(m_{p^s}(\lambda)!) = (\gamma+r)\ell(\lambda) + \nu_p(x_\lambda). \tag{5.12}$$

Consider the matrix $K \in \mathsf{Mat}_{\mathsf{Pow}_p(n)}(\mathbb{Q})$ such that $N = xK$. For each $\lambda \in \mathsf{Pow}_p(n)$, we have $M_{\lambda,\lambda} = x_\lambda$ by Proposition 2.4 (a), so $K_{\lambda,\lambda} = 1$ (in fact, K is $\mathbb{Z}$-valued by the same Proposition). We have $Y = xKb'({}^{\mathrm{tr}}K)x$ where $b' = b^{(\ell,\theta)}z^{-1}$. We will apply Lemma 5.7 to this product. Using Corollary 5.3, we obtain

$$\begin{aligned}\nu_p(b'_{\lambda,\lambda}) &= \nu_p(J_\ell^v(\lambda)|_{v=\theta}) - \nu_p(z_\lambda) = \sum_{s\ge 0} m_{p^s}(\lambda)(r+s+\gamma) - \sum_{s\ge 0}(s m_{p^s}(\lambda) + \nu_p(m_{p^s}(\lambda))) \\ &= (\gamma+r)\ell(\lambda) - \nu_p(x_\lambda). \qquad (5.13)\end{aligned}$$

In order to verify the hypotheses of Lemma 5.7, we only need to show that $\nu_p(K_{\lambda,\mu} - \delta_{\lambda,\mu}) > (\nu_p(b'_{\lambda,\lambda}) - \nu_p(b'_{\mu,\mu}))/2$ for all $\lambda, \mu \in \mathsf{Pow}_p(n)$. This inequality is immediate if $M_{\lambda,\mu} = 0$ or if $\lambda = \mu$ (as $K_{\lambda,\lambda} = 1$). In the remaining case, we have

$$\begin{aligned}&\nu_p(K_{\lambda,\mu}) - \frac{\nu_p(b'_{\lambda,\lambda}) - \nu_p(b'_{\mu,\mu})}{2} \\ &= (\nu_p(M_{\lambda,\mu}) - \nu_p(x_\lambda)) + \frac{\nu_p(x_\lambda) - \nu_p(x_\mu) + (\gamma+r)(\ell(\mu) - \ell(\lambda))}{2} \\ &= \frac{\nu_p(M_{\lambda,\mu}) - \nu_p(x_\lambda)}{2} + \frac{\nu_p(M_{\lambda,\mu}) - \nu_p(x_\mu) + \ell(\mu) - \ell(\lambda)}{2} + \frac{\ell(\mu) - \ell(\lambda)}{2}(\gamma + r - 1) > 0,\end{aligned}$$

as the first, second, and third summands are nonnegative by parts (a), (c), and (b) of Proposition 2.4 respectively; moreover, the second summand is positive.

By Lemma 5.7, $Y \equiv_{\mathbb{Z}_{(p)}} b'x^2$. It follows from (5.12) and (5.13) that $\nu_p(b'_{\lambda,\lambda}x_\lambda^2) = \nu_p(I_{\ell,p}^v(\lambda)|_{v=\theta})$, so (5.1) holds.

5.5. **Case $a^{2\ell} - b^{2\ell} \notin p\mathbb{Z}$.** By Proposition 5.4, the determinants of the matrices on both sides of (4.7) are invertible in $\mathbb{Z}_{(p)}$. Since both of these matrices are $\mathbb{Z}_{(p)}$-valued (see Proposition 4.13), they are both unimodularly equivalent over $\mathbb{Z}_{(p)}$ to the identity matrix.

This completes the proof of Theorem 4.14 and hence of Theorem 1.10.

6. Remarks on possible generalizations of Theorem 1.10

Our aim here is to demonstrate how far we still are from proving Conjecture 1.9 and to discuss natural statements that are stronger than Theorem 1.10 but are weaker than Conjecture 1.9, as well as implications between those statements. Proving some of them – if indeed they are true – would provide further evidence for Conjecture 1.9 and would be of interest in its own right.

Remark 6.1. In the proof of Theorem 1.10 (cf. §4.3), we have used the fact that the local-global correspondence holds when R is a PID by Proposition 4.2 (d) and (e), i.e.,

$$Y \equiv_R Z \Longleftrightarrow \forall \mathfrak{m} \in \mathsf{max\text{-}Spec}(R), Y \equiv_{R_{\mathfrak{m}}} Z.$$

An advantage of considering unimodular pseudo-equivalences $\equiv'_R$ is that:

Proposition 6.2. *Let R be a 1-dimensional Noetherian domain. For $n \times m$-matrices Y, Z with entries in R, we have*

$$Y \equiv'_R Z \Longleftrightarrow \forall \mathfrak{m} \in \mathsf{max\text{-}Spec}(R), Y \equiv'_{R_{\mathfrak{m}}} Z \tag{6.1}$$

if $\mathsf{Cok}_T = \mathsf{Tor}_R(\mathsf{Cok}_T)(:= \{x \in \mathsf{Cok}_T \mid \exists a \in R \setminus \{0\}, ax = 0\})$ for $T \in \{Y, Z\}$ (for example, when $n = m$ and $\det T \neq 0$ for $T \in \{Y, Z\}$).

Proof. The $\Rightarrow$ direction follows from Proposition 4.2 (b), so we need only prove the $\Leftarrow$ direction. Since R is an integral domain, the intersection of any two non-zero ideals of R is non-zero, and in particular $I := \mathsf{Ann}_R(\mathsf{Cok}_Y) \cap \mathsf{Ann}_R(\mathsf{Cok}_Z) \neq 0$. Clearly, $Y \equiv'_R Z \Leftrightarrow Y \equiv'_{R'} Z$ where $R' := R/I$. Since R' is Artinian, $\mathsf{max\text{-}Spec}(R')$ is a finite set and the natural ring homomorphism $R' \to \prod_{\mathfrak{m} \in \mathsf{max\text{-}Spec}(R')} R'_{\mathfrak{m}}$ is an isomorphism (see [Mat, (24.C)]). Thus, $Y \equiv'_{R'} Z \Leftrightarrow \forall \mathfrak{m} \in \mathsf{max\text{-}Spec}(R'), Y \equiv'_{R'_{\mathfrak{m}}} Z$ by Proposition 4.2 (b). Let $\phi\colon R \twoheadrightarrow R'$ be the natural surjection. Since $R'_{\mathfrak{m}} \cong R_{\phi^{-1}(\mathfrak{m})}/I_{\phi^{-1}(\mathfrak{m})}$ (see [Kun, Example 4.18 (a)], if $Y \equiv'_{R_{\phi^{-1}(\mathfrak{m})}} Z$ for all $\mathfrak{m} \in \mathsf{max\text{-}Spec}(R')$, then $Y \equiv'_R Z$. Noting that $I \subseteq \phi^{-1}(\mathfrak{m}) \in \mathsf{max\text{-}Spec}(R)$ for all $\mathfrak{m} \in \mathsf{max\text{-}Spec}(R')$, we deduce the result. □

An advantage of considering Fitting equivalences $\equiv^F_R$ is that for a large class of rings R we have an algorithm to decide whether two explicitly given matrices Y and Z are Fitting equivalent or not (see [RFW, Chapter VIII] and references therein). If we have $Y \equiv^F_R Z$, then by Proposition 4.2 (d) it is not possible to demonstrate that $Y \not\equiv_R Z$ by localization or specialization to a PID R'. Thus, as far as unimodular equivalences over PIDs are concerned, the ultimate piece of evidence for Conjecture 1.9 would be to prove that $X \equiv^F_{\mathscr{A}} D$, where X and D are the matrices on the two sides of (1.5).

Remark 6.3. If $X \equiv^F_{\mathscr{A}} D$, then, in particular, $X \equiv_{\mathbb{F}_p[v,v^{-1}]} D$ for any prime p. Whether or not the latter equivalence holds is an interesting intermediate open problem.

Proposition 6.4. *Let X and Y be $n \times m$-matrices with entries in $\mathscr{A}$. If $X|_{v=\theta} \equiv^F_{\mathbb{Z}[\theta,\theta^{-1}]} Y|_{v=\theta}$ for all $\theta \in \overline{\mathbb{Q}} \setminus \{0\}$, then $X \equiv^F_{\mathscr{A}} Y$.*

We conclude the paper by proving Proposition 6.4, which implies that, in order to show that $X \equiv^F_{\mathscr{A}} D$, it would suffice to generalize Theorem 1.10 (b) by proving that $X|_{v=\theta} \equiv^F_{\mathbb{Z}[\theta,\theta^{-1}]} D|_{v=\theta}$ for all non-zero algebraic numbers θ. Despite Proposition 4.2 (a) and Proposition 6.2, proving the equivalence $X \equiv^F_{\mathbb{Z}[\theta,\theta^{-1}]} D$ for an arbitrary $\theta \in \overline{\mathbb{Q}} \setminus \{0\}$ (if it is true) is likely to be considerably more difficult than proving Theorem 1.10 because $\mathbb{Z}[\theta, \theta^{-1}]$ is not integrally closed (equivalently, it is not a Dedekind domain) in general. However, it may be possible to use the methods of the present paper to prove that $X \equiv_{\mathscr{O}_\theta} D$, where $\mathscr{O}_\theta$ is the integral closure of the ring $\mathbb{Z}[\theta, \theta^{-1}]$ in its field of fractions, at least for some classes of algebraic numbers θ. Establishing whether X and D are Fitting equivalent – or, indeed, settling Conjecture 1.9 – is likely to require new ideas.

Proof of Proposition 6.4. The proposition is an immediate corollary of Theorem 6.5. □

In the following, let S be the set of non-constant irreducible polynomials in $\mathbb{Z}[v]$. Let $\theta \in \overline{\mathbb{Q}} \setminus \{0\}$ be a root of $f \in S$. For an ideal I of $\mathscr{A}$, we denote by $I|_{v=\theta}$ the image of I under the ring surjection $\pi_\theta \colon \mathscr{A} \twoheadrightarrow \mathbb{Z}[\theta, \theta^{-1}]$ given by $v \mapsto \theta$. Then, by Gauss's Lemma we have $\mathsf{Ker}\, \pi_\theta = \mathscr{A} f$.

Theorem 6.5. *Let I and J be ideals of $\mathscr{A}$. If $I|_{v=\theta} = J|_{v=\theta}$ in $\mathbb{Z}[\theta, \theta^{-1}]$ for all $\theta \in \overline{\mathbb{Q}} \setminus \{0\}$, then $I = J$.*

Lemma 6.6. *Let R be a Noetherian commutative ring. For any ideal I of R, we have*

$$I = \bigcap_{I \subseteq \mathfrak{m} \in \mathsf{max\text{-}Spec}(R)} \bigcap_{n \geq 1} (I + \mathfrak{m}^n).$$

Proof. Replacing R with R/I, we may assume that $I = 0$. Let $J = \cap_{\mathfrak{m} \in \mathsf{max\text{-}Spec}(R)} \cap_{n \geq 1} \mathfrak{m}^n$. For $\mathfrak{m} \in \mathsf{max\text{-}Spec}(R)$, we have $J_\mathfrak{m} \subseteq \cap_{n \geq 1} \mathfrak{m}_\mathfrak{m}^n$, and $\cap_{n \geq 1} \mathfrak{m}_\mathfrak{m}^n = 0$ in $R_\mathfrak{m}$ by Krull intersection theorem. So $J_\mathfrak{m} = 0$ for all $\mathfrak{m}$, whence $J = 0$ (see the proof of Proposition 4.2 (e)). □

Lemma 6.7. *Let $\mathfrak{m} \in \mathsf{max\text{-}Spec}(\mathbb{Z}[v])$ and let $n \geq 1$. Then, $\mathfrak{m}^n \cap S$ is an infinite set.*

Proof. It is well known that $\mathfrak{m} = (p, h)$ for some $p \in \mathsf{Prm}$ and non-constant monic irreducible polynomial h which remains irreducible in $\mathbb{F}_p[v]$ (see [GP, Exercise 7.9]). For any $q \in \mathsf{Prm}$ with $q \neq p$, put $f_q := p^n + qh^n \in \mathfrak{m}^n$. Then f_q is primitive by construction and is in S by Eisenstein's criterion (applied to the prime q). □

Proof of Theorem 6.5. For $\mathfrak{m} \in \mathsf{max\text{-}Spec}(\mathscr{A})$ and $n \geq 1$, there exists $f \in \mathfrak{m}^n \cap S$ such that $f \neq \pm v$ by Lemma 6.7 applied to $\mathfrak{m} \cap \mathbb{Z}[v] \in \mathsf{max\text{-}Spec}(\mathbb{Z}[v])$. By the hypothesis, we have $I|_{v=\theta} = J|_{v=\theta}$ for a root $\theta \in \overline{\mathbb{Q}} \setminus \{0\}$ of f, whence $I + \mathscr{A} f = \pi_\theta^{-1}(I|_{v=\theta}) = \pi_\theta^{-1}(J|_{v=\theta}) = J + \mathscr{A} f$. Since $\mathscr{A} f \subseteq \mathfrak{m}^n$, it follows that $I + \mathfrak{m}^n = J + \mathfrak{m}^n$. By Lemma 6.6, we have $I = J$. □

Remark 6.8. We learned Theorem 6.5 from Hiraku Kawanoue. His proof yields the existence of $f \in S$ such that $I + \mathscr{A} f \neq J + \mathscr{A} f$ for ideals $I \neq J \subseteq \mathscr{A}$ and can be applied when we replace $\mathbb{Z}$ by any unique factorization domain R which has infinitely many prime elements modulo $R^\times$. In order to keep this section short, we adapted the proof to one sufficient for Proposition 6.4. While the above proof depends on the description of $\mathsf{max\text{-}Spec}(\mathbb{Z}[v])$ and does not allow the indicated generalization, it shares the same spirit with Kawanoue's.

Index of notation

The following index gives references to subsections where symbols are defined:

$\mathfrak{S}_n$	symmetric group	1.1
$\mathcal{H}_n(\mathbb{F}; q)$	Hecke algebra	1.1
$\eta_\ell \in k_\ell$	a primitive ℓ-th root of unity in a field	1.1
$\mathsf{Mod}(A)$	the category of finite-dimensional left A-modules	1.1
$\mathsf{PC}(D)$	projective cover of D	1.1
C_A	Cartan matrix of an algebra A	1.1
$\equiv_R$	unimodular equivalence of matrices	1.2
ℓ_k	$\ell/(\ell, k)$	1.2
$I_\ell^v(\lambda), J_\ell^v(\lambda)$	Laurent polynomials in Definition 1.8	1.4
$\mathsf{max\text{-}Spec}(R)$	the set of maximal ideals of a ring R	1.7.1
$\mathsf{Mat}_\ell(R)$, $\mathsf{Mat}_S(R)$	matrix algebra	1.7.2
1_S	identity matrix	1.7.2
$\mathsf{diag}(\{r_s \mid s \in S\})$	diagonal matrix	1.7.2
$\bigoplus_i M_i$	block-diagonal matrix	1.7.2
ν_p	p-adic valuation	1.7.3
$\mathbb{N}$	the set of nonnegative integers	1.7.4
Prm	the set of prime numbers	1.7.4
n_Π	Π-part of n	1.7.4
Π', p'	the complements of Π, $\{p\}$ in Prm	1.7.4
(a, b)	greatest common divisor of a and b	1.7.4
$\mathbf{k}$	the function field $\mathbb{Q}(v)$	1.7.5
$\mathscr{A}$	the ring of Laurent polynomials $\mathbb{Z}[v, v^{-1}]$	1.7.5

bar	bar-involution $v \mapsto v^{-1}$ on $\mathbf{k}$	1.7.5
Infl_t	the inflation map $v \mapsto v^t$ on $\mathscr{A}$	1.7.5
$[n]_m$	quantum integer	1.7.5
$[n]_m!$	quantum factorial	1.7.5
$\equiv_G$	conjugacy relation in a group G	1.7.6
$g_p, g_{p'}$	p-part and p'-part of g	1.7.6
$m_k(\lambda)$	multiplicity of k in a partition λ	1.7.7
$\ell(\lambda)$	length of a partition λ	1.7.7
$\lvert\lambda\rvert$	size of a partition λ	1.7.7
Par, $\mathsf{Par}(n)$	set of partitions	1.7.7
$\mathsf{CRP}_s(n)$	the set of s-class regular partitions of n	1.7.7
$\mathsf{RP}_s(n)$	the set of s-regular partitions of n	1.7.7
$\mathsf{Par}_m(n)$	the set of m-multipartitions of n	1.7.7
$\mathsf{Par}_p(n,\nu)$	the set of partitions of n with "p'-part" ν	1.7.7
$\mathsf{Pow}_p(n)$	the set of p-power partitions of n	1.7.7
$\lambda + \mu$	sum of two partitions	1.7.7
Λ	ring of symmetric functions	2.1
χ_V	character of $\mathfrak{S}_n$ afforded by module V	2.1
p_μ, p_k	power sum symmetric functions	2.1
C_μ	conjugacy class corresponding to a partition μ	2.1
z_μ	order of centralizer of an element of C_μ	2.1
ch	isometry between a Grothendieck group and symmetric functions	2.1
$\mathfrak{S}_\lambda$	parabolic subgroup of $\mathfrak{S}_n$	2.1
$\mathsf{triv}_{\mathfrak{S}_\lambda}$	trivial representation of $\mathfrak{S}_\lambda$	2.1
M_n	table of permutation characters of $\mathfrak{S}_n$	2.1
h_μ	complete symmetric function	2.1
m_μ	monomial symmetric function	2.1
$\mathcal{M}_{\lambda,\mu}$	a certain set of size $M_{\lambda,\mu}$	2.1
$N_n^{(p)}$	"p-local" submatrix of M_n	2.2
$L_n^{(p)}$	a certain block-diagonal matrix	2.2
$a_\theta^{(p)}(n)$	a rational number from Definition 2.8	2.2
Sym^m	symmetric power functor	2.3
$\mathsf{Mult}_m(\ell)$	the set of weakly increasing m-tuples of elements of $\{1,\dots,\ell\}$	2.3
$\Omega_{\ell,d}$	a set of tuples, which is in bijection with $\mathsf{Par}_\ell(d)$	2.3
$S^d(A)$	matrix in Definition 2.13	2.3
$\Lambda_\ell = \bigotimes_{t=1}^{\ell} \Lambda^{(t)}$	ℓ-colored ring of symmetric functions	2.3
$m_\mu^{(t)}$, $h_\mu^{(t)}$, $p_\mu^{(t)}$	images of m_μ, h_μ, p_μ in $\Lambda^{(t)}$	2.3
$M_{\ell,d}$, $K_{\ell,d}$	transition matrices in Definition 2.14	2.3
$\mathcal{P}$, $\mathcal{P}^\vee$	weight lattice and its dual	3.1
Π, $\Pi^\vee$	sets of simple roots and corresponding coroots	3.1
Q^+	positive part of the root lattice	3.1
$\mathcal{P}^+$	set of dominant integral weights	3.1
Λ_i	a dominant integral weight	3.1
$W = W(X)$	Weyl group	3.1
$U_v = U_v(X)$	quantum group	3.1
U_v^+, U_v^0, U_v^-	subalgebras in the triangular decomposition of U_v	3.1
$V(\lambda)$	highest weight module	3.1
1_λ	highest weight vector	3.1
$\langle\cdot,\cdot\rangle_{\mathsf{QSh}}$, $\langle\cdot,\cdot\rangle_{\mathsf{RSh}}$	versions of Shapovalov form from Proposition 3.2	3.1
$P(\lambda)$	the set of weights of $V(\lambda)$	3.1
$V(\lambda)_\mu$	μ-weight space of $V(\lambda)$	3.1
$(U_v^-)^{\mathscr{A}}$	an $\mathscr{A}$-lattice in U_v^-	3.1
$V(\lambda)^{\mathscr{A}}$, $V(\lambda)_\nu^{\mathscr{A}}$	$\mathscr{A}$-lattices in $V(\lambda)$, $V(\lambda)_\nu$	3.1

$\mathsf{QSh}^{\mathsf{M}}_{\lambda,\mu}$, $\mathsf{RSh}^{\mathsf{M}}_{\lambda,\mu}$	Gram matrices of Shapovalov forms, see Definition 3.3	3.1
$\doteqdot$	an equivalence relation on matrices	3.1
$\widehat{X}$	extended Cartan matrix of X	3.2
δ	null-root	3.2
$\mathsf{Mod}_{\mathsf{gr}}(A)$	category of finite-dimensional graded A-modules	3.3
M_n	graded n-component of a module M	3.3
$M\langle k\rangle$	graded module M with grading shifted down by k	3.3
$\mathcal{S}(A)$	set of representatives of simple graded A-modules	3.3
C^v_A	graded Cartan matrix of A	3.3
$\mathsf{Proj}_{\mathsf{gr}}(A)$	category of projective graded A-modules	3.3
$\mathsf{Bl}_\ell(n)$	the set of pairs (ρ, d) where ρ is an ℓ-core and $d \in \mathbb{N}$	3.3
Cok_T	cokernel of the map given by a matrix T	4.1
$\equiv'_R$	unimodular pseudo-equivalence of matrices, see Definition 4.1	4.1
$\equiv^F_R$	Fitting equivalence of matrices, see Definition 4.1	4.1
$\mathsf{Fitt}_d(T)$	d-th Fitting ideal of a matrix T	4.1
Φ_n, Ψ_n	cyclotomic polynomial and its scaled version	4.2
$\rho^{(p)}_z$	function from Definition 4.5	4.2
$\varphi_{s,n}$	a bijection from s-regular to s-class regular partitions	4.2
β_M	auto-bijection of Par from Definition 4.6	4.2
$g^{(\ell,p)}_{k,t}$, $f^{(\ell)}_{k,t}$, $I^v_{\ell,p}(\lambda)$	certain products of quantum integers, see Definition 4.7	4.2
$\mathcal{F}^{(\ell,p)}_{k,t,z}$, $\mathcal{G}^{(\ell,p)}_{k,t,z}$	certain sets of integers related to $f^{(\ell)}_{k,t}$, $g^{(\ell,p)}_{k,t}$	4.2
$\lambda^{<r}, \lambda^{\geq r}, \overline{\lambda}^r$	p-power partitions from Definition 5.6	5.2

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School of Mathematics, University of Birmingham, Edgbaston, Birmingham B15 2TT, UK
E-mail address: a.evseev@bham.ac.uk

Graduate School of Mathematical Sciences, University of Tokyo, Komaba, Meguro, Tokyo, 153-8914, Japan
E-mail address: tshun@kurims.kyoto-u.ac.jp