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Locally finite groups in which every non-cyclic subgroup is self-centralizing

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Abstract

Locally finite groups having the property that every non-cyclic subgroup contains its centralizer are classified.

Keywords: Self-centralizing subgroup, Frobenius group, locally finite group

2010 MSC: 20F50, 20E34, 20D25

1. Introduction

A subgroup $H$ of a group $G$ is self-centralizing if the centralizer $C_G(H)$ is contained in $H$. In \cite{1} it has been remarked that a locally graded group in which all non-trivial subgroups are self-centralizing has to be finite; therefore it has to be either cyclic of prime order or non-abelian of order being the product of two different primes.

In this article, we consider the more extensive class $X$ of all groups in which every non-cyclic subgroup is self-centralizing. In what follows we use the term $X$-groups in order to denote groups in the class $X$. The study of properties of $X$-groups was initiated in \cite{1}. In particular, the first four authors determined the structure of finite $X$-groups which are either nilpotent, supersoluble or simple.

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In this paper, Theorem 2.1 gives a complete classification of finite \( X \)-groups. We remark that this result does not depend on classification of the finite simple groups rather only on the classification of groups with dihedral or semidihedral Sylow 2-subgroups. We also determine the infinite soluble \( X \)-groups, and the locally finite \( X \)-groups the results being presented in Theorems 3.6 and 3.7. It turns out that these latter groups are suitable finite extensions either of the infinite cyclic group or of a Prüfer \( p \)-group, \( \mathbb{Z}_{p^\infty} \), for some prime \( p \).

We follow [2] for basic group theoretical notation. In particular, we note that \( F^*(G) \) denotes the generalized Fitting subgroup of \( G \), that is the subgroup of \( G \) generated by all subnormal nilpotent or quasisimple subgroups of \( G \). The latter subgroups are the components of \( G \). We see from [2, Section 31] that distinct components commute. The fundamental property of the generalized Fitting subgroup that we shall use is that it contains its centralizer in \( G \) [2, (31.13)].

We denote the alternating group and symmetric group of degree \( n \) by \( \text{Alt}(n) \) and \( \text{Sym}(n) \) respectively. We use standard notation for the classical groups. The notation \( \text{Dih}(n) \) denotes the dihedral group of order \( n \) and \( Q_8 \) is the quaternion group of order 8. The term quaternion group will cover groups which are often called generalized quaternion groups. The cyclic group of order \( n \) is represented simply by \( n \), so for example \( \text{Dih}(12) \cong 2 \times \text{Dih}(6) \cong 2 \times \text{Sym}(3) \). Finally \( \text{Mat}(10) \) denotes the Mathieu group of degree 10. The Atlas [3] conventions are used for group extensions. Thus, for example, \( p^2:\text{SL}_2(p) \) denotes the split extension of an elementary abelian group of order \( p^2 \) by \( \text{SL}_2(p) \).

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2. Finite $\mathfrak{X}$-groups

In this section we determine all the finite groups belonging to the class $\mathfrak{X}$. The main result is the following.

**Theorem 2.1.** Let $G$ be a finite $\mathfrak{X}$-group. Then one of the following holds:

1. If $G$ is nilpotent, then either
   1.1 $G$ is cyclic;
   1.2 $G$ is elementary abelian of order $p^2$ for some prime $p$;
   1.3 $G$ is an extraspecial $p$-group of order $p^3$ for some odd prime $p$; or
   1.4 $G$ is a dihedral, semidihedral or quaternion 2-group.
2. If $G$ is supersoluble but not nilpotent, then, letting $p$ denote the largest prime divisor of $|G|$ and $P \in \text{Syl}_p(G)$, we have that $P$ is a normal subgroup of $G$ and one of the following holds:
   2.1 $P$ is cyclic and either
      2.1.1 $G \cong D \rtimes C$, where $C$ is cyclic, $D$ is cyclic and every non-trivial element of $D$ acts fixed point freely on $C$ (so $G$ is a Frobenius group);
      2.1.2 $G = D \rtimes C$, where $C$ is a cyclic group of odd order, $D$ is a quaternion group, and $C_G(D) = C \times D_0$ where $D_0$ is a cyclic subgroup of index 2 in $D$ with $G/D_0$ a dihedral group; or
      2.1.3 $G = D \rtimes C$, where $D$ is a cyclic $q$-group, $C$ is a cyclic $q'$-group (here $q$ denotes the smallest prime dividing the order of $G$), $1 < Z(G) < D$ and $G/Z(G)$ is a Frobenius group;
3. If $G$ is not supersoluble and $F^*(G)$ is nilpotent, then either (3.1) or (3.2) below holds.
   3.1 $F^*(G)$ is elementary abelian of order $p^2$, $F^*(G)$ is a minimal normal subgroup of $G$ and one of the following holds:
      3.1.1 $p = 2$ and $G \cong \text{Sym}(4)$ or $G \cong \text{Alt}(4)$; or
(3.1.2) $p$ is odd and $G = G_0 \rtimes N$ is a Frobenius group with Frobenius kernel $N$ and Frobenius complement $G_0$ which is itself an $X$-group. Furthermore, either

(3.1.2.1) $G_0$ is cyclic of order dividing $p^2 - 1$ but not dividing $p - 1$;
(3.1.2.2) $G_0$ is quaternion;
(3.1.2.3) $G_0$ is supersoluble as in (2.1.2) with $|C|$ dividing $p - \epsilon$ where $p \equiv \epsilon \pmod{4}$;
(3.1.2.4) $G_0$ is supersoluble as in (2.1.3) with $D$ a 2-group, $C_D(C)$ a non-trivial maximal subgroup of $D$ and $|C|$ odd dividing $p - 1$ or $p + 1$;
(3.1.2.5) $G_0 \cong \text{SL}_2(3)$;
(3.1.2.6) $G_0 \cong \text{SL}_2(3) \cdot 2$ and $p \equiv \pm 1 \pmod{8}$; or
(3.1.2.7) $G_0 \cong \text{SL}_2(5)$ and 60 divides $p^2 - 1$.

(3.2) $F^*(G)$ is extraspecial of order $p^3$ and one of the following holds:

(3.2.1) $G \cong \text{SL}_2(3)$ or $G \cong \text{SL}_2(3) \cdot 2$ (with quaternion Sylow 2-subgroups of order 16); or
(3.2.2) $G = K \rtimes N$ where $N$ is extraspecial of order $p^3$ and exponent $p$ with $p$ an odd prime, $K$ centralizes $Z(N)$ and is cyclic of odd order dividing $p + 1$. Furthermore, $G/Z(N)$ is a Frobenius group.

(4) If $F^*(G)$ is not nilpotent, then either

(4.1) $F^*(G) \cong \text{SL}_2(p)$ where $p$ is a Fermat prime, $|G/F^*(G)| \leq 2$ and $G$ has quaternion Sylow 2-subgroups; or
(4.2) $G \cong \text{PSL}_2(9), \text{Mat}(10)$ or $\text{PSL}_2(p)$ where $p$ is a Fermat or Mersenne prime.

Furthermore, all the groups listed above are $X$-groups.

We make a brief remark about the group $\text{SL}_2(3) \cdot 2$ and the groups appearing in part (4.1) of Theorem 2.1 in the case $G > F^*(G)$. To obtain such groups, take $F = \text{SL}_2(p^2)$, then the groups in question are isomorphic to the normalizer in $F$ of the subgroup isomorphic to $\text{SL}_2(p)$. We denote these groups by $\text{SL}_2(p) \cdot 2$. 

to indicate that the extension is not split (there are no elements of order 2 in the outer half of the group).

We shall repeatedly use the fact that if \( L \) is a subgroup of an \( \mathfrak{X} \)-group \( X \), then \( L \) is an \( \mathfrak{X} \)-group. Indeed, if \( H \leq L \) is non-cyclic, then \( C_L(H) \leq C_X(H) \leq H \).

The following elementary facts will facilitate our proof that the examples listed are indeed \( \mathfrak{X} \)-groups.

**Lemma 2.2.** The finite group \( X \) is an \( \mathfrak{X} \)-group if and only if \( C_X(x) \) is an \( \mathfrak{X} \)-group for all \( x \in X \) of prime order.

**Proof.** If \( X \) is an \( \mathfrak{X} \)-group, then, as \( X \) is subgroup closed, \( C_X(x) \) is an \( \mathfrak{X} \)-group for all \( x \in X \) of prime order (and hence of any order). Let \( H \leq X \) be non-cyclic. We shall show \( C_X(H) \leq H \). If \( C_X(H) = 1 \), then \( C_X(H) \leq H \) and we are done.

So assume \( x \in C_X(H) \) and \( x \neq 1 \). Then \( H \leq C_X(x) \) which is an \( \mathfrak{X} \)-group. Hence \( x \in C_{C_X(x)}(H) \leq H \). Therefore \( C_X(H) \leq H \), and \( X \) is an \( \mathfrak{X} \)-group.

**Lemma 2.3.** Suppose that \( X \) is a Frobenius group with kernel \( K \) and complement \( L \). If \( K \) and \( L \) are \( \mathfrak{X} \)-groups, then \( X \) is an \( \mathfrak{X} \)-group.

**Proof.** Let \( x \in X \) have prime order. Then, as \( K \) and \( L \) have coprime orders, \( x \in K \) or \( x \) is conjugate to an element of \( L \). But then, since \( X \) is a Frobenius group, either \( C_X(x) \leq K \) or \( C_X(x) \) is conjugate to a subgroup of \( L \). Since \( K \) and \( L \) are \( \mathfrak{X} \)-groups, \( C_X(x) \) is an \( \mathfrak{X} \)-group. Hence \( X \) is an \( \mathfrak{X} \)-group by Lemma 2.2.

The rest of this section is dedicated to the proof of Theorem 2.1; therefore \( G \)
always denotes a finite \( \mathfrak{X} \)-group. Parts (1) and (2) of Theorem 2.1 are already proved in [1] Theorems 2.2, 2.4, 3.2 and 3.4. However, our statement in (2.1.3) adds further detail which we now explain. So, for a moment, assume that \( G \) is supersoluble, \( q \) is the smallest prime dividing \( |G| \), \( D \) is a cyclic \( q \)-group and \( C \) is a cyclic \( q' \)-group. In addition, \( 1 \neq Z(G) = C_D(C) \). Assume that \( d \in D \setminus Z(G) \). Then, as \( d \notin Z(G) \), \( C \) is not centralized by \( d \). By coprime action, \( C = [C,d] \times C_C(d) \) and so \( Y = [C,d](d) \) is centralized by \( C_C(d) \). As \( Y \)
is non-abelian and $C_C(d) \cap Y = 1$, we deduce that $C_C(d) = 1$. Hence $G/Z(G)$ is a Frobenius group. This means that we can assume that (1) and (2) hold and, in particular, we assume that $G$ is not supersoluble.

The following lemma provides the basic case subdivision of our proof.

**Lemma 2.4.** One of the following holds:

(i) $F^*(G)$ is elementary abelian of order $p^2$ for some prime $p$.

(ii) $F^*(G)$ is extraspecial of order $p^3$ for some prime $p$.

(iii) $F^*(G)$ is quasisimple.

**Proof.** Suppose first that $F^*(G)$ is nilpotent. Then its structure is given in part (1) of Theorem 2.1. Suppose that $F^*(G)$ is cyclic. Since $C_G(F^*(G)) = F^*(G)$, we have $G/F^*(G)$ is isomorphic to a subgroup of $Aut(F^*(G))$. Because the automorphism group of a cyclic group is abelian, we have that $G$ is supersoluble. Therefore, by our assumption concerning $G$, $F^*(G)$ is not cyclic. Hence $F^*(G)$ is either elementary abelian of order $p^2$ for some prime $p$, is extraspecial of order $p^3$ for some odd prime $p$ or $F^*(G)$ is a dihedral, semidihedral or quaternion 2-group. Since the automorphism groups of dihedral, semidihedral and quaternion groups of order at least 16 are 2-groups, we deduce that when $p = 2$ and $F^*(G)$ is non-abelian, $F^*(G)$ is extraspecial. This proves the lemma when $F^*(G)$ is nilpotent.

If $F^*(G)$ is not nilpotent, then there exists a component $K \leq F^*(G)$. As $F^*(G) = C_{F^*(G)}(K)K$ and $K$ is non-abelian, we have $F^*(G) = K$ and this is case (iii). \qed

**Lemma 2.5.** Suppose that $p$ is a prime and $F^*(G)$ is extraspecial of order $p^3$. Then one of the following holds:

(i) $G \cong SL_2(3), G \cong SL_2(3):2$ (with quaternion Sylow 2-subgroups of order 16); or

(ii) $G = NK$ where $N$ is extraspecial of order $p^3$ of exponent $p$ with $p$ an odd prime, $K$ centralizes $Z(N)$ and is cyclic of odd order dividing $p + 1$. Furthermore, $G/Z(N)$ is a Frobenius group.
Proof. Let $N = F^*(G)$. We have that $N$ is extraspecial of order $p^3$ by assumption. Suppose first that $p = 2$, then we have $N \cong Q_8$ as the dihedral group of order 8 has no odd order automorphisms and $G$ is not a 2-group. Since $\text{Aut}(Q_8) \cong \text{Sym}(4)$, $G/Z(N)$ is isomorphic to a subgroup of $\text{Sym}(4)$ containing $\text{Alt}(4)$. If $G/Z(N) \cong \text{Alt}(4)$, then $G = NT \cong \text{SL}_2(3)$ where $T$ is a cyclic subgroup of order 3. When $G/Z(N) \cong \text{Sym}(4)$, taking $T \in \text{Syl}_3(G)$, we have $NT \cong \text{SL}_2(3)$, $N_G(T)$ has order 12 and $N_G(T)/Z(N) \cong \text{Sym}(3)$. Since $N_G(T)$ is an $S$-group and $N_G(T)$ is supersoluble, we see that $N_G(T)$ is a product $DT$ where $D$ is cyclic of order 4 by (2.1.3). Because the Sylow 2-subgroups of $G$ are either dihedral, semidihedral or quaternion and $D \not\leq N$, we see that $ND$ is quaternion. Thus $G \cong \text{SL}_2(3) \cdot 2$ as claimed in (i).

Assume that $p$ is odd. We know that the outer automorphism group of $N$ is isomorphic to a subgroup of $\text{GL}_2(p)$ and $C_{\text{Aut}(N)}(Z(N))/\text{Inn}(N)$ is isomorphic to a subgroup of $\text{SL}_2(p)$. Since $p$ is odd and the Sylow $p$-subgroups of $G$ are $\mathcal{X}$-groups, we have $N \in \text{Syl}_p(G)$ and $G/N$ is a $p'$-group by part (1) of Theorem 2.1. Set $Z = Z(N)$. Since $G/N$ and $N$ have coprime orders, the Schur Zassenhaus Theorem says that $G$ contains a complement $K$ to $N$. Set $K_1 = C_K(Z)$. Then $K_1$ commutes with $Z$ and so $K_1$ is cyclic. If $K_1 = 1$, then $|K|$ divides $p - 1$ and we find that $G$ is supersoluble, which is a contradiction. Hence $K_1 \neq 1$. Let $x \in K_1$. Then $[N, x]$ and $C_N(x)$ commute by the Three Subgroups Lemma. Hence $C_N(x)$ centralizes $[N, x]\langle x \rangle$ which is non-abelian. It follows that $[N, x] = N$ and $C_N(x) = Z$. If $\langle x \rangle$ does not act irreducibly on $N/Z$, then there exists $Z < N_1 < N$ which is $\langle x \rangle$-invariant. If $N_1$ is cyclic, then, as $\langle x \rangle$ centralizes $\Omega_1(N_1) = Z$, $\langle x \rangle$ centralizes $N_1 > Z$, a contradiction. If $N_1$ is elementary abelian, then, as $\langle x \rangle$ centralizes $Z$, $[N_1, \langle x \rangle]$ has order at most $p$ by Maschke’s Theorem. If $[N_1, \langle x \rangle] \neq 1$, then $[N_1, \langle x \rangle] \langle x \rangle$ is non-abelian and $Z$ centralizes $[N_1, \langle x \rangle] \langle x \rangle$, a contradiction. Hence $\langle x \rangle$ centralizes $N_1$ contrary to $C_N(\langle x \rangle) = Z$. We conclude that every element of $K_1$ acts irreducibly on $N/Z(N)$. In particular, since $K_1$ is isomorphic to a subgroup of $\text{SL}_2(p)$, we have that $K_1$ is cyclic of odd order dividing $p + 1$. Furthermore, as $K_1$ acts irreducibly on $N/Z(N)$, $N$ has exponent $p$. 

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By the definition of $K_1$, $|K/K_1|$ divides $|\text{Aut}(Z)| = p - 1$. Assume that $K \neq K_1$ and let $y \in K \setminus K_1$ have prime order $r$. Then $r$ does not divide $|K_1|$ and $Z(y)$ is non-abelian. Since $K_1$ centralizes $Z$, we have $C_{K_1}(y) = 1$. Let $w \in K_1$ have prime order $q$. Then $(y)\langle w \rangle$ is non-abelian and acts faithfully on $V = N/Z$. Therefore [2, 27.1] implies that $C_N(y) \neq 1$. As $C_N(y) \cap Z = 1$ and $C_N(y)$ centralizes $Z(y)$, we have a contradiction. Hence $K = K_1$. Finally, we note that $NK/Z(N)$ is a Frobenius group.

It remains to show that the groups listed are $\mathcal{X}$-groups. We consider the groups listed in (ii) and leave the groups in (i) to the reader. Assume that $H \leq G$ is non-cyclic. We shall show that $C_G(H) \leq H$. If $H \geq N$, then $C_G(H) \leq C_G(N) \leq N \leq H$ and we are done. Suppose that $H \leq N$. Then, as $N$ is extraspecial of exponent $p$, $H$ is elementary abelian of order $p^2$ and $C_N(H) = H$. Since $G/N$ is cyclic of odd order dividing $p + 1$, we see that $N_G(H) = N$ and so $C_G(H) = C_N(H) = H$ and we are done in this case. Suppose that $H \not\leq N$ and $N \not\leq H$. Let $h \in H \setminus N$. Then, as $|G/N|$ divides $p + 1$ and is odd, we either have $H \cap N = N$ or $H \cap N = Z$. So we must have $H \cap N = Z = Z(G)$. Now $H/Z \cong G/N$ is cyclic of order dividing $p + 1$ and so we get that $H$ is cyclic, a contradiction. Thus $G$ is an $\mathcal{X}$-group.

Lemma 2.6. Suppose that $N = F^*(G)$ is elementary abelian of order $p^2$. Then one of the following holds:

(i) $p = 2$, $G \cong \text{Sym}(4)$ or $\text{Alt}(4)$; or
(ii) $p$ is odd and $G = NG_0$ is a Frobenius group with Frobenius kernel $N$ and Frobenius complement $G_0$ which is itself an $\mathcal{X}$-group. Furthermore, either
   (a) $G_0$ is cyclic of order dividing $p^2 - 1$ but not dividing $p - 1$;
   (b) $G_0$ is quaternion;
   (c) $G_0$ is supersoluble as in part (2.1.2) of Theorem 2.1, with $|C|$ dividing $p - \epsilon$ where $p \equiv \epsilon \pmod{4}$;
   (d) $G_0$ is supersoluble as in part (2.1.3) of Theorem 2.1, with $D$ a 2-group, $C_D(C)$ a non-trivial maximal subgroup of $D$ and $|C|$ odd dividing $p - 1$ or $p + 1$;
(e) \(G_0 \cong \text{SL}_2(3)\);
(f) \(\text{SL}_2(3) \cdot 2\) and \(p \equiv \pm 1 \pmod{8}\); or
(g) \(G_0 \cong \text{SL}_2(5)\) and 60 divides \(p^2 - 1\).

Furthermore, all the groups listed are \(X\)-groups.

**Proof.** We have \(N\) has order \(p^2\), is elementary abelian and \(G/N\) is isomorphic to a subgroup of \(\text{GL}_2(p)\). If \(p = 2\), then we quickly obtain part (i). So assume that \(p\) is odd.

Suppose that \(p\) divides the order of \(G/N\). Let \(P \in \text{Syl}_p(G)\). Then \(P\) is extraspecial of order \(p^3\) and \(P\) is not normal in \(G\). Hence by [4, Theorem 2.8.4] there exists \(g \in G\) such that \(G \geq K = \langle P, P^g \rangle \cong p^2: \text{SL}_2(p)\). Let \(Z = Z(P)\), \(t\) be an involution in \(K\), \(K_0 = C_K(t)\) and \(P_0 = P \cap K_0\). Then, as \(t\) inverts \(N\), \(K_0 \cong \text{SL}_2(p)\), \(P_0\) has order \(p\) and centralizes \(Z(t)\), which is a contradiction as \(Z(t) \cong \text{Dih}(2p)\). Hence \(G/N\) is a \(p'\)-group.

Suppose that \(x \in G \setminus N\). If \(C_N(x) \neq 1\), then \(C_N(x)\) centralizes \([N, x]\langle x\rangle\) which is non-abelian, a contradiction. Thus \(C_N(x) = 1\) for all \(x \in G \setminus N\). It follows that \(G\) is a Frobenius group with Frobenius kernel \(N\). Let \(G_0\) be a Frobenius complement to \(N\). As \(G_0 \leq G\), \(G_0\) is an \(X\)-group. Recall that the Sylow 2-subgroups of \(G_0\) are either cyclic or quaternion and that the odd order Sylow subgroups of \(G_0\) are all cyclic [5, V.8.7].

Assume that \(N\) is not a minimal normal subgroup of \(G\). Then \(G/N\) is conjugate in \(\text{GL}_2(p)\) to a subgroup of the diagonal subgroup. Therefore \(G\) is supersoluble, which is a contradiction. Hence \(N\) is a minimal normal subgroup of \(G\) and \(G_0\) is isomorphic to an irreducible subgroup of \(\text{GL}_2(p)\). This completes the general description of the structure of \(G\). It remains to determine the structure of \(G_0\).

If \(G_0\) is nilpotent, then Theorem 2.1 (1) applies to give \(G_0\) is either quaternion or cyclic. In the latter case, as \(G_0\) acts irreducibly on \(N\) it is isomorphic to a subgroup of the multiplicative group of \(\text{GF}(p^2)\) and is not of order dividing \(p-1\). This gives the structures in (ii) (a) and (b).

If \(G_0\) is supersoluble, then the structure of \(G_0\) is described in part (2.1) of
Theorem 2.1 as $\text{GL}_2(p)$ contains no extraspecial subgroups of odd order. We adopt the notation from (2.1). By [5, V.8.18 c)], $Z(G_0) \neq 1$. Hence (2.1.1) cannot occur. Case (2.1.2) can occur and, as $C$ commutes with a cyclic subgroup of order at least 4 and $G_0$ is isomorphic to a subgroup of $\text{GL}_2(p)$, $|C|$ divides $p-1$ if $p \equiv 1 \pmod{4}$ and $|C|$ divides $p+1$ if $p \equiv 3 \pmod{4}$. In the situation described in part (2.1.3) of Theorem 2.1 the groups have no 2-dimensional faithful representations unless $q = 2$ and $C_D(C)$ has index 2. In this case $|C|$ is an odd divisor of $p-1$ or $p+1$.

Suppose that $G_0$ is not supersoluble. Refereing to Lemma 2.4 and using the fact that the Sylow subgroups of $G_0$ are either cyclic or quaternion, we have that $F^*(G_0)$ is either quaternion of order 8 or $F^*(G_0)$ is quasisimple. In the first case we obtain the structures described in parts (b), (e) and (f) from Lemma 2.5 where for part (f) we note that we require $\text{SL}_2(p)$ to have order divisible by 16.

If $F^*(G_0)$ is quasisimple, then Zassenhaus’s Theorem [5, Theorem 18.6, p. 204] gives $G_0 = WM$ where $W \cong \text{SL}_2(5)$ and $M$ is metacyclic. Since $G_0$ is an $\mathfrak{X}$-group, this means that $M \leq W$ and $G_0 \cong \text{SL}_2(5)$. Since $\text{SL}_2(5)$ is isomorphic to a subgroup of $\text{GL}_2(p)$ only when $p = 5$ or 60 divides $p^2 - 1$ and $p \neq 5$ part (g) holds.

That $\text{Sym}(4)$ and $\text{Alt}(4)$ are $\mathfrak{X}$-groups is easy to check. The groups listed in (ii) are $\mathfrak{X}$-groups by Lemma 2.3.

The finite simple $\mathfrak{X}$-groups are determined in [1]. We have to extend the arguments to the cases where $F^*(G)$ is simple or quasisimple. This is relatively elementary.

**Lemma 2.7.** Suppose that $F^*(G)$ is simple. Then $G \cong \text{SL}_2(4)$, $\text{PSL}_2(9)$, $\text{Mat}(10)$ or $\text{PSL}_2(p)$ where $p$ is a Fermat or Mersenne prime.

**Proof.** Set $H = F^*(G)$. As $\mathfrak{X}$ is subgroup closed, $H$ is an $\mathfrak{X}$-group and so $H$ is one of the groups listed in the statement by Theorem 3.7 of [1]. Hence we obtain $H \cong \text{SL}_2(4)$, $\text{PSL}_2(9)$ or $\text{PSL}_2(p)$ for $p$ a Fermat or Mersenne prime.

Suppose that $G > H$. If $H \cong \text{SL}_2(4)$, then $G \cong \text{Sym}(5)$ and the subgroup $2 \times \text{Sym}(3)$ witnesses the fact that $\text{Sym}(5)$ is not an $\mathfrak{X}$-group. Suppose $H \cong \text{PSL}_2(p)$ with $p = 5$ or $60 | p^2 - 1$. This is easy to check. If $H \cong \text{PSL}_2(p)$ with $p \neq 5$ and $p \nmid 60$, then $G \cong \text{PSL}_2(p)$ and the subgroup $2 \times \text{Sym}(3)$ witnesses the fact that $\text{PSL}_2(p)$ is not an $\mathfrak{X}$-group. If $H \cong \text{Mat}(10)$, then $G \cong \text{PSL}_2(9)$ and the subgroup $2 \times \text{Sym}(3)$ witnesses the fact that $\text{PSL}_2(9)$ is not an $\mathfrak{X}$-group. If $H \cong \text{PSL}_2(p)$ for a Mersenne prime $p$, then $G \cong \text{PSL}_2(p)$ and the subgroup $2 \times \text{Sym}(3)$ witnesses the fact that $\text{PSL}_2(p)$ is not an $\mathfrak{X}$-group. If $H \cong \text{PSL}_2(9)$, then $G \cong \text{PSL}_2(9)$ and the subgroup $2 \times \text{Sym}(3)$ witnesses the fact that $\text{PSL}_2(9)$ is not an $\mathfrak{X}$-group. If $H \cong \text{PSL}_2(p)$ for a Fermat prime $p$, then $G \cong \text{PSL}_2(p)$ and the subgroup $2 \times \text{Sym}(3)$ witnesses the fact that $\text{PSL}_2(p)$ is not an $\mathfrak{X}$-group.
PSL_2(9) \cong \text{Alt}(6). If G \geq K \cong \text{Sym}(6), then G contains Sym(5) which is impossible. Therefore G \cong PGL_2(9) or G \cong \text{Mat}(10). In the first case, G contains a subgroup \text{Dih}(20) \cong 2 \times \text{Dih}(10) which is impossible. Thus G \cong \text{Mat}(10) and this group is easily shown to satisfy the hypothesis as all the centralizer of elements of prime order are \text{X}-groups.

If H \cong PSL_2(p), p a Fermat or Mersenne prime, then G \cong PGL_2(p) and contains a dihedral group of order 2(p + 1) and one of order 2(p - 1). One of these is not a 2-group and this contradicts G being an \text{X}-group.

**Lemma 2.8.** Suppose that F^*(G) is quasisimple but not simple. Then F^*(G) \cong SL_2(p) where p is a Fermat prime, |G/H| \leq 2 and G has quaternion Sylow 2-subgroups.

*Proof.* Let H = F^*(G) and Z = Z(H). Since H centralizes Z, we have Z is cyclic. Let S \in \text{Syl}_2(H). If Z \not\leq S, then S must be cyclic. Since groups with a cyclic Sylow 2-subgroup have a normal 2-complement [2, 39.2], this is impossible. Hence Z \leq S. In particular, Z(G) \neq 1 as the central involution of H is central in G. It follows also that all the odd order Sylow subgroups of G are cyclic. By part (1) of Theorem 2.1, S is either abelian, dihedral, semidihedral or quaternion. If S is abelian, then S/Z is cyclic and again we have a contradiction. So S is non-abelian. Thus S/Z is dihedral (including elementary abelian of order 4). Hence H/Z \cong \text{Alt}(7) or PSL_2(q) for some odd prime power q [3, Theorem 16.3].

Since the odd order Sylow subgroups of G are cyclic, we deduce that H \cong SL_2(p) for some odd prime p. If p - 1 is not a power of 2, then H has a non-abelian subgroup of order pr where r is an odd prime divisor of p - 1 which is centralized by Z. Hence p is a Fermat prime.

Suppose that G > H with H \cong SL_2(p), p a Fermat prime. Note G/H has order 2. Let S \in \text{Syl}_2(G). Then S \cap H is a quaternion group. Suppose that S is not quaternion Then there is an involution t \in S \setminus H. By the Baer-Suzuki Theorem, there exists a dihedral group D of order 2r for some odd prime r which contains t. Since D and Z commute, this is impossible. Hence S is quaternion. This gives the structure described in the lemma.
It remains to demonstrate that the groups $\text{SL}_2(p)$ and $\text{SL}_2(p) \cdot 2$ with $p$ a Fermat prime are indeed $\mathcal{X}$-groups. Let $G$ denote one of these group, $H = F^*(G) \cong \text{SL}_2(p)$. Recall from the comments just after the statement of Theorem 2.1 that $G$ is isomorphic to a subgroup of $X = \text{SL}_2(p^2)$. Let $V$ be the natural $GF(p^2)$ representation of $X$ and thereby a representation of $G$. Assume that $L \leq G$ is non-cyclic. Since $H$ has no abelian subgroups which are not cyclic, $L$ is non-abelian and $L$ acts irreducibly on $V$. Schur’s Lemma implies that $C_X(L)$ consists of scalar matrices and so has order at most 2. If $L$ has even order, then as $G$ has quaternion Sylow 2-subgroups, $L \geq C_G(L)$. So suppose that $L$ has odd order. Then using Dickson’s Theorem [7, 260, page 285], as $p$ is a Fermat prime, we find that $L$ is cyclic, a contradiction. Thus $G$ is an $\mathcal{X}$-group.

Proof of Theorem 2.1 This follows from the combination of the lemmas in this section.

3. Locally finite $\mathcal{X}$-groups

It has been proved in [1, Theorem 2.2] that an infinite abelian group is in the class $\mathcal{X}$ if and only if it is either cyclic or isomorphic to $\mathbb{Z}_{p^\infty}$ (the Prüfer $p$-group) for some prime $p$. Moreover, Theorem 2.3 and Theorem 2.5 of [1] imply that every infinite nilpotent $\mathcal{X}$-group is abelian. We start this section by showing that some extensions of infinite abelian $\mathcal{X}$-groups provide further examples of infinite $\mathcal{X}$-groups.

Lemma 3.1. The infinite dihedral group belongs to the class $\mathcal{X}$.

Proof. Write $G = \langle a, y \mid y^2 = 1, a^y = a^{-1} \rangle$. Then for every non-cyclic subgroup $H$ of $G$ there exist non-zero integers $n$ and $m$ such that $a^n, a^m y \in H$. It easily follows that $C_G(H) = 1$.

Lemma 3.2. Let $G = A(y)$ where $A \cong \mathbb{Z}_{2^\infty}$ and $\langle y \rangle$ has order 2 or 4, with $y^2 \in A$ and $a^y = a^{-1}$, for all $a \in A$. Then $G$ belongs to the class $\mathcal{X}$. 
Proof. It is clear that $G/A$ has order 2, and $A$ is the Fitting subgroup of $G$. Also $C_G(A) = A$ and $Z(G)$ is the subgroup of order 2 of $A$. Let $H$ be a non-cyclic subgroup of $G$ with $H \neq A$. Then $H \not\subseteq A$ as every proper subgroup of $A$ is cyclic. Pick any element $h \in H \setminus A$. Then $G = A\langle h \rangle$ since $|G : A| = 2$. Therefore by the Dedekind modular law we get $H = C\langle h \rangle$, where $C = A \cap H > 1$ is finite.

Since $h = bv$ with $b \in A$ and $v \in \langle y \rangle \setminus A$, we get $a^h = a^{-1}$ for all $a \in A$. In particular, $C_A(h)$ has order 2 and $C_G(h)$ has order 4. Since $C$ has a unique involution and $h \in C_G(H)$, we conclude that $C_G(H) \leq H$ and so $G$ is an $\mathcal{X}$-group.

When $\langle y \rangle$ has order 2, the group $G = A \rtimes \langle y \rangle$ of Lemma 3.2 is a generalized dihedral group.

Let $p$ denote any odd prime. Then, by Hensel’s Theorem (see for instance [8], Theorem 127.5)), the group $Z_{p^\infty}$ has an automorphism of order $p - 1$, say $\phi$.

**Lemma 3.3.** The groups $G = Z_{p^\infty} \rtimes \langle \phi^j \rangle$ for $1 \leq j \leq p - 1$ are $\mathcal{X}$-groups.

Proof. As $\mathcal{X}$ is subgroup closed, it suffices to show that $G = Z_{p^\infty} \rtimes \langle \phi \rangle$ is an $\mathcal{X}$-group. Write the elements of $G$ in the form $ay$ with $a \in A \cong Z_{p^\infty}$ and $y \in \langle \phi \rangle$. Suppose there exist non-trivial elements $a, b \in A$ and $y \in \langle \phi \rangle$ such that $a^y = a$.

For a suitable non-negative integer $n$, the element $a^{p^n}$ has order $p$ and it is fixed by $y$. Then $y$ centralizes all elements of order $p$ in $A$, and therefore $y = 1$ by a result due to Baer (see, for instance, [9], Lemma 3.28)). This contradiction shows that $\langle \phi \rangle$ acts fixed point freely on $A$.

Let $H$ be any non-cyclic subgroup of $G$. Then, as $G/A$ is cyclic, $A \cap H \neq 1$. If $H = A$ then of course $C_G(H) = H$. Thus we can assume that there exist non-trivial elements $a, b \in A$ and $y \in \langle \phi \rangle$ such that $a, by \in H$. Let $g \in C_G(H)$.

If $g \in A$ then $1 = [g, by] = [g, y]$, so $g = 1$. Now let $g = cz$ with $c \in A$ and $1 \neq z \in \langle \phi \rangle$. Thus $1 = [cz, a] = [z, a]$, and $a = 1$, a contradiction. Therefore $C_G(H) \leq H$ for all non-cyclic subgroups $H$ of $G$, so $G$ is an $\mathcal{X}$-group.

**Lemma 3.4.** An infinite polycyclic group belongs to the class $\mathcal{X}$ if and only if it is either cyclic or dihedral.
Proof. Arguing as in the proof of Theorem 3.1 of [1], one can easily prove that every infinite polycyclic $X$-group is either cyclic or dihedral. On the other hand, the infinite dihedral group belongs to the class $X$ by Lemma 3.1. 

**Proposition 3.5.** A torsion-free soluble group belongs to the class $X$ if and only if it is cyclic.

Proof. Let $G$ be a torsion-free soluble $X$-group. Then every abelian subgroup of $G$ is cyclic, so $G$ satisfies the maximal condition on subgroups by a result due to Mal’cev (see, for instance, [10, 15.2.1]). Thus $G$ is polycyclic by [10, 5.4.12]. Therefore $G$ has to be cyclic. 

In next theorem we determine all infinite soluble $X$-groups.

**Theorem 3.6.** Let $G$ be an infinite soluble group. Then $G$ is an $X$-group if and only if one of the following holds:

(i) $G$ is cyclic;

(ii) $G \cong \mathbb{Z}_{p^\infty}$ for some prime $p$;

(iii) $G$ is dihedral;

(iv) $G = A(y)$ where $A \cong \mathbb{Z}_{2^\infty}$ and $\langle y \rangle$ has order 2 or 4, with $y^2 \in A$ and $a^y = a^{-1}$, for all $a \in A$;

(v) $G \cong A \rtimes D$, where $A \cong \mathbb{Z}_{p^\infty}$ and $1 \neq D \leq C_{p-1}$ for some odd prime $p$.

Proof. First let $G$ be an $X$-group. If $G$ is abelian then (i) or (ii) holds by [1, Theorem 2.2]. Assume $G$ is non-abelian, and let $A$ be the Fitting subgroup of $G$. Then $A \neq 1$ and $C_G(A) \leq A$ as $G$ is soluble. Let $N$ be a nilpotent normal subgroup of $G$. Then $N$ is finite, as, otherwise, using $N$ is self-centralizing and $G/Z(N)$ is a subgroup of $\text{Aut}(N)$, we obtain $G$ is finite, which is a contradiction. Thus [1, Theorems 2.3 and 2.5] imply that $N$ is abelian. In particular, as the product of any two normal nilpotent subgroups of $G$ is again a normal nilpotent subgroup by Fitting’s Theorem, we see that the generators of $A = F(G)$ commute. Hence $A$ is abelian. As $A$ is infinite and abelian, $A = C_G(A)$ is either infinite cyclic or isomorphic to $\mathbb{Z}_{p^\infty}$ for some prime $p$. In the former case clearly $G' \leq A$. 

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In the latter case, let $C$ be any proper subgroup of $A$. Thus $C$ is finite cyclic. Moreover $C$ is characteristic in $A$, so it is normal in $G$, and $G/C_G(C)$ is abelian since it is isomorphic to a subgroup of $\text{Aut}(C)$. It follows that $G' \leq C_G(C)$, and again $G' \leq C_G(A) = A$. Therefore $G/A$ is abelian.

If $A$ is infinite cyclic, then the argument in the proof of Theorem 3.1 of [1] shows that $G$ is dihedral. Thus (iii) holds.

Let $A \cong \mathbb{Z}_{p^\infty}$ for some prime $p$, and suppose there exists an element $x \in G$ of infinite order. Then $x \in G \setminus A$, and so there exists an element $y \in A$ such that $[x, y] \neq 1$. Then $\langle y \rangle$ is a finite normal subgroup of $G$, so conjugation by $x$ induces a non-trivial automorphism of $\langle y \rangle$. Since $\text{Aut}(\langle y \rangle)$ is finite, it follows that there is an integer $n$ such that $[x^n, y] = 1$. Now $y$ is a torsion element and $x^n$ has infinite order and so $\langle x^n, y \rangle$ is neither periodic nor torsion free and this contradicts [1, Theorems 2.2]. Therefore $G$ is periodic, and $G/A$ is isomorphic to a periodic subgroup of automorphisms of $\mathbb{Z}_{p^\infty}$.

It is well-known that $\text{Aut}(\mathbb{Z}_{p^\infty})$ is isomorphic to the multiplicative group of all $p$-adic units. It follows that periodic automorphisms of $\mathbb{Z}_{p^\infty}$ form a cyclic group having order 2 if $p = 2$, and order $p - 1$ if $p$ is odd (see, for instance, [11] for details). In the latter case (v) holds. Finally, let $p = 2$. Then $G/A = \langle y A \rangle$ has order 2, and $G = A \langle y \rangle$ with $y \notin A$ and $y^2 \in A$. Moreover $a^y = a^{-1}$, for all $a \in A$. If $y$ has order 2 then $G = A \rtimes \langle y \rangle$. Otherwise from $y^2 \in A$ it follows $y^2 = (y^2)^y = y^{-2}$, so $y$ has order 4. Therefore $G$ has the structure described in (iv).

On the other hand, Lemmas 3.1 – 3.3 show that the groups listed in (i) – (v) are $\mathcal{X}$-groups.

Finally, we determine all infinite locally finite $\mathcal{X}$-groups.

**Theorem 3.7.** Let $G$ be an infinite locally finite group. Then $G$ is an $\mathcal{X}$-group if and only if one of the following holds:

1. $G \cong \mathbb{Z}_{p^\infty}$ for some prime $p$;
2. $G = A \langle y \rangle$ where $A \cong \mathbb{Z}_{2^\infty}$ and $\langle y \rangle$ has order 2 or 4, with $y^2 \in A$ and $a^y = a^{-1}$, for all $a \in A$;
(iii) \( G \cong A \times D, \) where \( A \cong \mathbb{Z}_{p^\infty} \) and \( 1 \neq D \leq C_{p-1} \) for some odd prime \( p. \)

**Proof.** Any abelian subgroup of \( G \) is either finite or isomorphic to \( \mathbb{Z}_{p^\infty} \) for some prime \( p, \) so it satisfies the minimal condition on subgroups. Thus \( G \) is a Černikov group by a result due to Šunkov (see, for instance [10] page 436, I]). Hence there exists an abelian normal subgroup \( A \) of \( G \) such that \( A \cong \mathbb{Z}_{p^\infty} \) for some prime \( p, \) and \( G/A \) is finite. It follows that \( G \) is metabelian, arguing as in the proof of Theorem 3.6. Therefore the result follows from Theorem 3.6. \( \square \)

**Corollary 3.8.** Let \( G \) be an infinite locally nilpotent group. Then \( G \) is an \( X \)-group if and only if one of the following holds:

(i) \( G \) is cyclic;

(ii) \( G \cong \mathbb{Z}_{p^\infty} \) for some prime \( p; \)

(iii) \( G = A\langle y \rangle \) where \( A \cong \mathbb{Z}_{2^\infty} \) and \( \langle y \rangle \) has order 2 or 4, with \( y^2 \in A \) and \( a^y = a^{-1}, \) for all \( a \in A. \)

**Proof.** Suppose \( G \) is not abelian. Every finitely generated subgroup of \( G \) is nilpotent, so it is either abelian or finite. It easily follows that all torsion-free elements of \( G \) are central. Thus \( G \) is periodic (see [12] Proposition 1]). Therefore \( G \) is direct product of \( p \)-groups (see, for instance, [10] Proposition 12.1.1]). Actually only one prime can occur since \( G \) is an \( X \)-group, so \( G \) is a locally finite \( p \)-group. Thus the result follows by Theorem 3.7. \( \square \)

**References**


