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ON THE RELATION BETWEEN GRAPH DISTANCE AND EUCLIDEAN DISTANCE IN RANDOM GEOMETRIC GRAPHS

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Abstract. Given any two vertices u, v of a random geometric graph $\mathcal{G}(n, r)$, denote by $d_E(u, v)$ their Euclidean distance and by $d_G(u, v)$ their graph distance. The problem of finding upper bounds on $d_G(u, v)$ conditional on $d_E(u, v)$ that hold asymptotically almost surely has received quite a bit of attention in the literature. In this paper, we improve the known upper bounds for values of $r = \omega(\sqrt{\log n})$ (i.e. for r above the connectivity threshold). Our result also improves the best known estimates on the diameter of random geometric graphs. We also provide a lower bound on $d_G(u, v)$ conditional on $d_E(u, v)$.

Random geometric graphs, Graph distance, Euclidean distance, Diameter

1. Introduction

Given a positive integer n and a non-negative real function $r = r(n)$, a *random geometric graph* G on n vertices and *radius* r is defined as follows. The vertex set $V = V(G)$ is obtained by choosing n points independently and uniformly at random in the square $\mathcal{S}_n = [-\sqrt{n}/2, \sqrt{n}/2]^2$ (Note that, with probability 1, no point in $\mathcal{S}_n$ is chosen more than once, and thus we assume $|V| = n$). For notational purposes, we identify each vertex $v \in V$ with its corresponding geometric position $v = (x_v, y_v) \in \mathcal{S}_n$, where x_v and y_v denote the usual x- and y-coordinates in $\mathcal{S}_n$. For every two points $u, v \in \mathcal{S}_n$, we write $d_E(u, v)$ for their Euclidean distance. Finally, the edge set $E = E(G)$ is constructed by connecting each pair of vertices $u, v \in V$ by an edge if and only if $d_E(u, v) \leq r$. We denote this model of random geometric graphs by $\mathcal{G}(n, r)$, and use the notation $G \in \mathcal{G}(n, r)$ (or often simply $\mathcal{G}(n, r)$) to refer to a random outcome of this distribution. We will always assume that $r \leq \sqrt{2n}$, as for $r \geq \sqrt{2n}$ the graph obtained is always a clique.

Random geometric graphs were first introduced in a slightly different setting by Gilbert [3] to model the communications between radio stations. Since then, several closely related variants of these graphs have been widely used as a model for wireless communication, and have also been extensively studied from a mathematical point of view. The basic reference on random geometric graphs is the monograph by Penrose [10] (see [11] for a more recent survey).

The properties of $\mathcal{G}(n, r)$ are usually investigated from an asymptotic perspective, as n grows to infinity and $r = r(n)$. Throughout the paper, we use the following standard notation for the asymptotic behavior of sequences of non-negative numbers a_n and b_n: $a_n = O(b_n)$ if $\limsup_{n\to\infty} a_n/b_n \leq C < +\infty$; $a_n = \Omega(b_n)$ if $b_n = O(a_n)$; $a_n = \Theta(b_n)$ if $a_n = O(b_n)$ and $a_n = \Omega(b_n)$; $a_n = o(b_n)$ if $\lim_{n\to\infty} a_n/b_n = 0$; and $a_n = \omega(b_n)$ if $b_n = o(a_n)$. We also use $a_n \ll b_n$ and $b_n \gg a_n$ to denote $a_n = o(b_n)$. Finally, a sequence of events H_n holds *asymptotically almost surely* (a.a.s.) if $\lim_{n\to\infty} \Pr(H_n) = 1$.

It is well known that $r_c = \sqrt{\log n/\pi}$ is a sharp threshold function for the connectivity of a random geometric graph (see e.g. [4, 9]). This means that for every $\varepsilon > 0$, if $r \leq (1 - \varepsilon)r_c$, then $\mathcal{G}(n, r)$ is a.a.s. disconnected, whilst if $r \geq (1 + \varepsilon)r_c$, then it is a.a.s. connected.

Given a connected graph G, we define the *graph distance* between two vertices u and v, denoted by $d_G(u, v)$, as the number of edges on a shortest path from u to v. Observe first that any pair of vertices u and v must satisfy $d_G(u, v) \geq d_E(u, v)/r$ deterministically by the triangle inequality,

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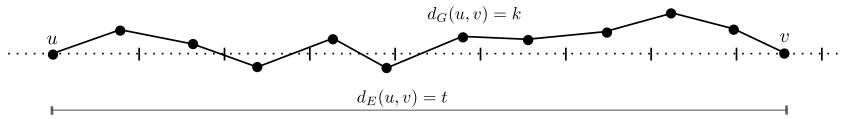

FIGURE 1. Graph distance vs. Euclidean distance between two points u and v in V

since each edge of a geometric graph has length at most r. The goal of this paper is to provide upper and lower bounds that hold a.a.s. for the graph distance of two vertices in terms of their Euclidean distance and in terms of r (see Figure 1).

Related work. This particular problem has risen quite a bit of interest in recent years. Given any two vertices $u, v \in V$, most of the work related to this problem has been devoted to study upper bounds on $d_G(u,v)$ in terms of $d_E(u,v)$ and r, that hold a.a.s. Ellis, Martin and Yan [2] showed that there exists some large constant K such that for every $r \geq (1+\varepsilon)r_c$, $G \in \mathcal{G}(n,r)$ satisfies a.a.s. the following property: for every $u, v \in V$ such that $d_E(u,v) > r$,

$$d_G(u,v) \leq K \cdot \frac{d_E(u,v)}{r} \,. \tag{1.1}$$

Their result is stated in the unit ball random geometric graph model, but it can be easily adapted into our setting. This result was extended by Bradonjic et al. [1] for the range of r for which $\mathcal{G}(n,r)$ has a giant component a.a.s., under the extra condition that $d_E(u,v) = \Omega(\log^{7/2} n/r^2)$. Friedrich, Sauerwald and Stauffer [6] improved this last result by showing that the result holds a.a.s. for every u and v satisfying $d_E(u,v) = \omega(\log n/r)$. They also proved that if $r = o(r_c)$, a linear upper bound of $d_G(u,v)$ in terms of $d_E(u,v)/r$ is no longer possible. In particular, a.a.s. there exist vertices u and v with $d_E(u,v) \leq 3r$ and $d_G(u,v) = \Omega(\log n/r^2)$.

The motivation for the study of this problem stems from the fact that these results provide upper bounds for the diameter of $G \in \mathcal{G}(n,r)$, denoted by $\mathrm{diam}(G)$, that hold a.a.s., and the runtime complexity of many algorithms can often be bounded from above in terms of the diameter of G. For a concrete example, we refer to the problem of broadcasting information (see [1, 6]).

One of the important achievements of our paper is to show that one can take the constant K for which (1.1) holds as $K = 1 + o(1)$ a.a.s., provided that $r = \omega(r_c)$. By the aforementioned result in [6], we know that the statement is false if $r = o(r_c)$.

A similar problem has been studied by Muthukrishnan and Pandurangan [8]. They proposed a new technique to study several problems on random geometric graphs — the so called *Bin-Covering* technique — which tries to cover the endpoints of a path by bins. They consider, among others, the problem of determining $D_G(u,v)$, which is the length of the shortest Euclidean path connecting u and v. Recently, Mehrabian and Wormald [7] studied a similar problem to the one in [8]. They deploy n points uniformly in $[0,1]^2$, and connect any pair of points with probability $p = p(n)$, independently of their distance. In this model, they determine the ratio of $D_G(u,v)$ and $d_E(u,v)$ as a function of p.

The following theorem is the main result of our paper.

Theorem 1. *Let $G \in \mathcal{G}(n,r)$ be a random geometric graph on n vertices and radius $0 < r \leq \sqrt{2n}$. A.a.s., for every pair of vertices $u, v \in V(G)$ with $d_E(u,v) > r$ (as otherwise the statement is trivial) we have:*

(i) *if $d_E(u,v) \geq \max\left\{12(\log n)^{3/2}/r,\, 21r\log n\right\}$, then*

$$d_G(u,v) \geq \left\lfloor \frac{d_E(u,v)}{r}\left(1 + \frac{1}{2\left(r d_E(u,v)\right)^{2/3}}\right)\right\rfloor \,;$$

(ii) *if* $r \geq 224\sqrt{\log n}$, *then*

$$d_G(u,v) \leq \left\lceil \frac{d_E(u,v)}{r}\left(1+\gamma r^{-4/3}\right)\right\rceil ,$$

where

$$\gamma = \gamma(u,v) = \max\left\{1358\left(\frac{3r\log n}{r+d_E(u,v)}\right)^{2/3},\ \frac{4\cdot 10^6\log^2 n}{r^{8/3}},\ 30000^{2/3}\right\} .$$

In order to prove (i), we first observe that all the short paths between two points must lie in a certain rectangle. Then we show that, by restricting the construction of the path on that rectangle, no very short path exists. For the proof of (ii) we proceed similarly. We restrict our problem to finding a path contained in a narrow strip. In this case, we show that a relatively short path can be constructed. We believe that the ideas in the proof can be easily extended to show the analogous result for d-dimensional random geometric graphs for all fixed $d \geq 2$.

Remark. *(1) Note that the condition* $d_E(u,v) \geq \max\{12(\log n)^{3/2}/r, 21r\log n\}$ *in the lower bound of (i) can be replaced by* $d_E(u,v) \geq 21r\log n$ *if* $r \geq \sqrt{4/7}(\log n)^{1/4}$, *and by* $d_E(u,v) \geq 12(\log n)^{3/2}/r$ *if* $r \leq \sqrt{4/7}(\log n)^{1/4}$. *We do not know whether this condition can be made less restrictive, besides improving the multiplicative constants involved (which we did not attempt to optimize).*

(2) Similarly, the constant 224 *in the condition* $r \geq 224\sqrt{\log n}$ *of (ii) (as well as those in the definition of* γ*) is not optimized either, and could be made slightly smaller. However, our method as is cannot be extended all the way down to* $r \geq \sqrt{\log n/\pi} = r_c$.

(3) Moreover, the error term in part (i) is $\left(2(rd_E(u,v))^{2/3}\right)^{-1} = O(1/\log n) = o(1)$.

(4) Finally, the error term in (ii) is

$$\gamma r^{-4/3} = \Theta\left(\max\left\{\left(\frac{\log n}{r^2+rd_E(u,v)}\right)^{2/3},\ \left(\frac{\sqrt{\log n}}{r}\right)^4,\ r^{-4/3}\right\}\right),$$

which is $o(1)$ *iff* $r = \omega(\sqrt{\log n}) = \omega(r_c)$. *Hence, for* $r = \omega(r_c)$, *statement (ii) implies that a.a.s.*

$$d_G(u,v) \leq \left\lceil (1+o(1))\,\frac{d_E(u,v)}{r}\right\rceil ,$$

thus improving the result in [2].

Theorem 1 gives an upper bound on the diameter as a corollary. First, observe that $d_E(u,v) \leq \sqrt{2n}$. From Theorem 10 in [2] for the particular case $d = 2$, one can deduce that if $r \geq (1+\varepsilon)r_c$ a.a.s.

$$\operatorname{diam}(G) \leq \frac{\sqrt{2n}}{r}\left(1+O\left(\sqrt{\frac{\log\log n}{\log n}}\right)\right) . \tag{1.2}$$

Directly from Theorem 1 we have that, for $r \geq 224\sqrt{\log n}$,

$$\operatorname{diam}(G) \leq \left\lceil \frac{\sqrt{2n}}{r}\left(1+\widehat{\gamma}r^{-4/3}\right)\right\rceil , \tag{1.3}$$

where

$$\widehat{\gamma} = \Theta\left(\left(\frac{r\log n}{\sqrt{n}}\right)^{2/3} + \frac{\log^2 n}{r^{8/3}} + 1\right).$$

(In fact, (1.3) holds for all $r \geq (1+\varepsilon)r_c$ as a consequence of (1.2)). From straightforward computations, one can check that (1.3) improves (1.2) provided that $r = \Omega\left(\frac{\log^{5/8} n}{(\log\log n)^{1/8}}\right)$.

On the other hand, for the lower bound on the diameter, observe the following: for any function ω growing arbitrarily slowly with n, we can a.a.s. find two vertices u and v, each at distance at most ω from one corner (opposite from each other) of the square $\mathcal{S}_n$. For such two vertices, we trivially (and deterministically) have

$$\operatorname{diam}(G) \geq d_G(u,v) \geq \left\lceil \frac{\sqrt{2n}\big(1-\Theta(\omega/\sqrt{n})\big)}{r} \right\rceil \tag{1.4}$$

Assuming that $\sqrt{\log^3 n/n} \ll r \ll \sqrt{n}/\log n$, our bound from Theorem 1 applied to these vertices gives that a.a.s.

$$\begin{aligned} \operatorname{diam}(G) \geq d_G(u,v) &\geq \left\lfloor \frac{\sqrt{2n}\big(1-\Theta(\omega/\sqrt{n})\big)}{r}\left(1+\Theta(r^{-2/3}n^{-1/3})\right) \right\rfloor \\ &\geq \left\lceil \frac{\sqrt{2n}\left(1-\Theta(\omega/\sqrt{n})+\Theta(r^{-2/3}n^{-1/3})\right)}{r} - 1 \right\rceil . \end{aligned} \tag{1.5}$$

Assuming the additional constraint $r \ll n^{1/10}$, we have that $r^{-2/3}n^{-1/3} \gg r/\sqrt{n}$, and also $r^{-2/3}n^{-1/3} \gg \omega/\sqrt{n}$ (for ω tending to infinity sufficiently slowly). In this case, our bound in (1.5) improves upon the trivial lower bound (1.4), and can be written as

$$\operatorname{diam}(G) \geq \left\lfloor \frac{\sqrt{2n}}{r}\left(1+\Theta(r^{-2/3}n^{-1/3})\right) \right\rfloor . \tag{1.6}$$

Note that this is a.a.s. still valid if we drop the constraint $r \gg \sqrt{\log^3 n/n}$, since for $r = O\left(\sqrt{\log^3 n/n}\right)$ the random geometric graph is a.a.s. disconnected (and has infinite diameter). Hence, by (1.3) and (1.6), we obtain the following corollary:

Corollary 2. *Let $G \in \mathcal{G}(n,r)$ be a random geometric graph on n vertices and radius $0 < r \leq \sqrt{2n}$. A.a.s. we have:*

(i) *if $r \geq (1+\varepsilon)r_c$, then*

$$\operatorname{diam}(G) \leq \left\lceil \frac{\sqrt{2n}}{r}\left(1+\widehat{\gamma}r^{-4/3}\right) \right\rceil ,$$

where

$$\widehat{\gamma} = \Theta\left(\left(\frac{r\log n}{\sqrt{n}}\right)^{2/3} + \frac{\log^2 n}{r^{8/3}} + 1\right).$$

(ii) *if $r \ll n^{1/10}$, then*

$$\operatorname{diam}(G) \geq \left\lfloor \frac{\sqrt{2n}}{r}\left(1+\Theta(r^{-2/3}n^{-1/3})\right) \right\rfloor .$$

2. Proof of Theorem 1

In order to simplify the proof of Theorem 1 we will make use of a technique known as de-Poissonization, which has many applications in geometric probability (see for ex. [10] for a detailed account of the subject). Here we sketch it.

Consider the following related model of a random geometric graph G with two distinguished vertices u, v. The vertex set of G is $V = V(G) = \{u,v\} \cup V'$, where the position of u and v is selected independently and uniformly at random in $\mathcal{S}_n$, and where V' is a set obtained from a homogeneous Poisson point process of intensity 1 in the square $\mathcal{S}_n$ of area n. Observe that V'

consists of N points in the square $\mathcal{S}_n$ chosen independently and uniformly at random, where N is a Poisson random variable of mean n. Exactly as we did for the model $\mathcal{G}(n,r)$, we connect $u_1, u_2 \in V$ by an edge if and only if $d_E(u_1,u_2) \le r$. We denote this new model by $\widetilde{\mathcal{G}}_{u,v}(n,r)$.

The main advantage of defining $V' = V \setminus \{u,v\}$ as a Poisson point process is motivated by the following two properties: the number of points of V' that lie in any region $A \subseteq \mathcal{S}_n$ of area a has a Poisson distribution with mean a; and the number of points of V' in disjoint regions of $\mathcal{S}_n$ are independently distributed. Moreover, conditional on $N = n-2$, the distribution of $\widetilde{\mathcal{G}}_{u,v}(n,r)$ is the one of $\mathcal{G}(n,r)$. Therefore, since $\Pr(N = n-2) = \Theta(1/\sqrt{n})$, any event holding in $\widetilde{\mathcal{G}}_{u,v}(n,r)$ with probability at least $1 - o(f_n)$ must hold in $\mathcal{G}(n,r)$ with probability at least $1 - o(f_n\sqrt{n})$. We make use of this property throughout the article, and do all the analysis for a graph $G \in \widetilde{\mathcal{G}}_{u,v}(n,r)$ or related models of Poisson point processes.

We will need the following concentration inequality for the sum of independently and identically distributed exponential random variables. For the sake of completeness we provide the proof here.

Lemma 3. *Let $X_1, \dots, X_m$ be independent exponential random variables of parameter $\lambda > 0$ (i.e. expectation $1/\lambda$) and let $X = X_1 + \cdots + X_m$. Then, for every $\varepsilon > 0$ we have*

$$\Pr\big(X \ge (1+\varepsilon)\mathbb{E}(X)\big) \le \left(\frac{1+\varepsilon}{e^{\varepsilon}}\right)^m,$$

and for any $0 < \varepsilon < 1$ we have

$$\Pr\big(X \le (1-\varepsilon)\mathbb{E}(X)\big) \le \big((1-\varepsilon)e^{\varepsilon}\big)^m \le e^{-\varepsilon^2 m/2}.$$

Proof. Let us prove the bound for the upper tail. The bound for the lower tail is proved in a similar way and its proof is omitted.

We have $\mathbb{E}X = m\mathbb{E}X_1 = m/\lambda$. By Markov's inequality, for every $0 < \beta < \lambda$ and every $\varepsilon > 0$ we have

$$\Pr(X \ge (1+\varepsilon)\mathbb{E}X) = \Pr(e^{\beta X} \ge e^{\beta(1+\varepsilon)m/\lambda}) \le \frac{\prod \mathbb{E}(e^{\beta X_i})}{e^{\beta(1+\varepsilon)m/\lambda}} = (\varphi_{X_1}(\beta))^m e^{-\beta(1+\varepsilon)m/\lambda},$$

where $\varphi_{X_1}(\beta) = \mathbb{E}(e^{\beta X_1}) = \frac{\lambda}{\lambda-\beta}$ is the moment-generating function of an exponentially distributed random variable with parameter λ. Thus,

$$\Pr(X \ge (1+\varepsilon)\mathbb{E}X) \le \left(\frac{\lambda}{\lambda-\beta}\right)^m e^{-\beta(1+\varepsilon)m/\lambda}$$

Now we set $\beta = \frac{\varepsilon}{1+\varepsilon}\lambda$ to obtain

$$\Pr(X \ge (1+\varepsilon)\mathbb{E}X) \le \left(\frac{1+\varepsilon}{e^{\varepsilon}}\right)^m.$$

□ □

2.1. **Proof of statement (i).** In this subsection we prove the lower bound in Theorem 1. For every $t \ge 0$, we introduce the following model of infinite random geometric graphs. The vertex set is constructed by adding two vertices $u = (0,0)$ and $v = (t,0)$ to a homogeneous Poisson point process of intensity 1 in the infinite plane $\mathbb{R}^2$. We denote this new model by $\widetilde{\mathcal{G}}^{\infty}_{u,v}(r,t)$.

The main task in the sequel is to show that, for any $t \ge \max\big\{12(\log n)^{3/2}/r,\, 21r\log n\big\}$, the lower bound in part (i) of Theorem 1 holds with probability at least $1 - o(n^{-5/2})$ in $\widetilde{\mathcal{G}}^{\infty}_{u,v}(r,t)$, for the distinguished vertices $u = (0,0)$ and $v = (t,0)$. Combining this with an appropriate de-Poissonization argument will allow us to conclude the desired result for $\mathcal{G}(n,r)$.

Our next lemma shows that short paths connecting u and v are contained in small strips. The lemma is stated in the more general context of a deterministic geometric graph $G = (V,E)$ of radius r, where the vertex set V is an arbitrary subset of points in $\mathbb{R}^2$ (containing u and v) and edges

connect (as usual) every pair of vertices at Euclidean distance at most r. For a given $r > 0$, for every $k \in \mathbb{N}$ and $\alpha > 0$, consider the rectangle

$$R(k,\alpha) = \left[\frac{-\alpha^2}{kr}, kr\right] \times [-\alpha, \alpha] \ .$$

Lemma 4. *Given any $r, t, \alpha > 0$ and any $k \in \mathbb{N}$ satisfying $t \geq kr - \frac{2\alpha^2}{kr}$, let G be a geometric graph of radius r in $\mathbb{R}^2$, and suppose that $u = (0,0)$ and $v = (t,0)$ are two vertices of G. Then all paths of length at most k from u to v are contained in $R(k,\alpha)$.*

Proof. If there is no path of length at most k from u to v, the statement of the lemma is trivially true. Thus, we suppose that $P = (u = z_0, z_1, \ldots, z_\ell = v)$ is a path of length $\ell \leq k$, where $z_i = (x_i, y_i)$ for every $0 \leq i \leq \ell$. Also, note that it suffices to prove the lemma for α satisfying $t = kr - \frac{2\alpha^2}{kr}$, since this trivially implies the statement for larger α. In particular, we have $\alpha^2 < (kr)^2/2$.

Write $x^+ = \max\{x_i : 0 \leq i \leq \ell\}$ and $x^- = \min\{x_i : 0 \leq i \leq \ell\}$. It is clear that $x^+ \leq \ell r \leq kr$ since every edge has length at most r. Let $0 \leq j \leq \ell$ be such that $x_j = x^-$ and observe that $x^- \leq x_0 = 0$. Then

$$kr \geq d_E(u, z_j) + d_E(z_j, v) \geq -x^- + (t - x^-) \geq kr - 2\left(x^- + \frac{\alpha^2}{kr}\right),$$

and we obtain $x^- \geq -\alpha^2/(kr)$.

Now write $y^+ = \max\{y_i : 0 \leq i \leq \ell\}$ and $y^- = \min\{y_i : 0 \leq i \leq \ell\}$. We will show that $y^+ \leq \alpha$ and that $y^- \geq -\alpha$. For every $0 \leq i \leq \ell$, we have

$$kr \geq d_E(u, z_i) + d_E(z_i, v) = \sqrt{x_i^2 + y_i^2} + \sqrt{(t - x_i)^2 + y_i^2} \geq \sqrt{t^2 + 4y_i^2} \ ,$$

where we used the fact that the left-hand side of the last inequality is minimized at $x_i = t/2$. Using that $t \geq kr - \frac{2\alpha^2}{kr}$, we obtain

$$(kr)^2 \geq t^2 + 4y_i^2 \geq \left(kr - \frac{2\alpha^2}{kr}\right)^2 + 4y_i^2.$$

Thus, for every $0 \leq i \leq \ell$, we have $|y_i| \leq \alpha\sqrt{1 - \alpha^2/(kr)^2} \leq \alpha$, and so in particular $-\alpha \leq y^- \leq y^+ \leq \alpha$. Using the bounds on x^+, x^-, y^+ and y^-, we conclude that P is contained in $R(k,\alpha)$. □ □

Proposition 5. *For every $t > r$, let $G \in \widetilde{\mathcal{G}}^{\infty}_{u,v}(r,t)$ be a random geometric graph on $\mathbb{R}^2$ with additional vertices $u = (0,0)$ and $v = (t,0)$. Then, for every $0 < \delta < 2^{-1/3}$, we have that*

$$\Pr\left(d_G(u,v) \leq \left\lfloor \frac{t}{r}\left(1 + \frac{\delta}{(tr)^{2/3}}\right)\right\rfloor\right)$$
$$\leq \frac{(1+o(1))t}{r}\exp\left(-\sqrt{\delta/2}(tr)^{2/3}\right) + \exp\left(-(1 - \sqrt{2\delta^3} - o(1))^2\frac{t}{2r}\right). \tag{2.1}$$

Proof. We first set

$$k = \left\lfloor \frac{t}{r}\left(1 + \frac{\delta}{(tr)^{2/3}}\right)\right\rfloor \quad \text{and} \quad \alpha = \sqrt{\frac{\delta}{2}}\left(\frac{k^3 r^2}{t}\right)^{1/3} .$$

Observe that since $t > r$, we have $k \geq 1$. Let A_1 the event that $d_G(u,v) \leq k$; that is, there exists a path P in $\widetilde{\mathcal{G}}^{\infty}_{u,v}(r,t)$ from u to v of length at most k. Our goal is to show that the probability of A_1 is small.

Let x_1 be the largest x-coordinate of the vertices inside the rectangle $R_1 = [0,r] \times [-\alpha, \alpha]$ (possibly $x_1 = 0$ if u is the only vertex in R_1). Define the random variable $a_1 = r - x_1$. We proceed similarly

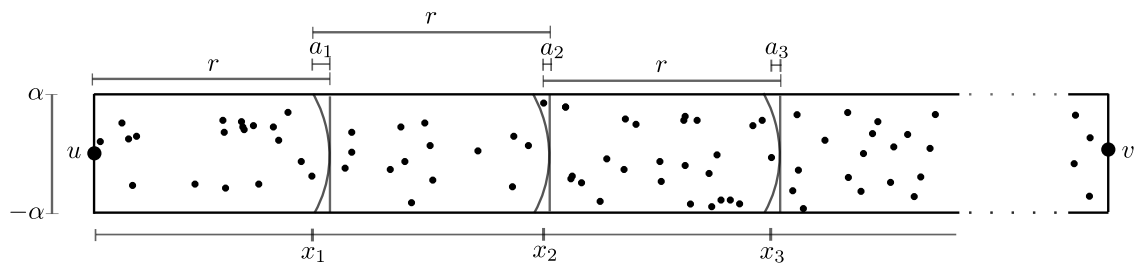

FIGURE 2. Example of some values of x_i and their corresponding a_i ($i \in \{1,2,3\}$).

for every $2 \le i \le k$. We define x_i as follows: if $R_i = (x_{i-1} + a_{i-1}, x_{i-1} + r] \times [-\alpha, \alpha]$ is non-empty, let x_i be the largest x-coordinate of the vertices inside R_i; otherwise, set $x_i = x_{i-1} + a_{i-1}$ (see Figure 2). Define then also $a_i = x_{i-1} + r - x_i$.

Claim: If A_1 holds, then $t \le x_k$.
Proof of the claim. Suppose that A_1 holds, let $P = (u = z_0, z_1, \dots, z_\ell = v)$ be one such path and for every $0 \le i \le \ell$ and let $\hat{x}_i$ be the x-coordinate of z_i. We will prove by induction on i that we have $\hat{x}_i \le x_i$. In particular, this implies $t = \hat{x}_\ell \le x_\ell \le x_k$, and proves the claim.

Observe that

$$t \ge \frac{kr}{\left(1 + \frac{\delta}{(tr)^{2/3}}\right)} \ge kr\left(1 - \frac{\delta}{(tr)^{2/3}}\right) = kr - \frac{2\alpha^2}{kr}\,. \tag{2.2}$$

Thus, we can use Lemma 4 to show that the path P is contained in the strip $\mathbb{R} \times [-\alpha, \alpha]$. Moreover, we must have $\hat{x}_1 - \hat{x}_0 \le r$ (since $u = z_0$ and z_1 are adjacent vertices). Therefore, our choice of x_1 and the fact that $z_1 \in \mathbb{R} \times [-\alpha, \alpha]$ imply that $\hat{x}_1 \le x_1$. So, the statement holds for $i = 1$. Now we inductively assume that $\hat{x}_{i-1} \le x_{i-1}$. We must have $\hat{x}_i \le \hat{x}_{i-1} + r$ (since z_{i-1} and z_i are adjacent vertices), and therefore $\hat{x}_i \le x_{i-1} + r$. Similarly as before, since $z_i \in \mathbb{R} \times [-\alpha, \alpha]$ and by the choice of x_i, we conclude that $\hat{x}_i \le x_i$, as desired. This completes the proof of the claim.

Thus, it suffices to show that $x_k \ge t$ with very small probability. We first study the random variables a_i. Define $a_0 = 0$. By the choice of x_i, for every $1 \le i \le k$ we have that $0 \le a_i \le r - a_{i-1}$. Recall that $R_i = (x_{i-1} + a_{i-1}, x_{i-1} + r] \times [-\alpha, \alpha]$. Since $G \in \widetilde{\mathcal{G}}^{\infty}_{u,v}(r,t)$, the number of vertices from V inside a region of $\mathbb{R}^2$ is a Poisson random variable with mean equal to the area of that region. So, for every $2 \le i \le k$ we have

$$\Pr(a_i \ge \beta) = \begin{cases} \Pr((x_{i-1} + r - \beta, x_{i-1} + r] \times [-\alpha, \alpha] \text{ empty}) = e^{-2\alpha\beta} & \text{if } 0 \le \beta \le r - a_{i-1} \\ 0 & \text{if } \beta > r - a_{i-1}. \end{cases} \tag{2.3}$$

Thus, a_i is stochastically dominated by an exponentially distributed random variable $\tilde{a}_i$ of parameter 2α. We assume that a_i and $\tilde{a}_i$ are coupled together in the same probability space, so that $a_i = \min\{\tilde{a}_i, r - a_{i-1}\} \le \tilde{a}_i$.

Moreover, since the regions $R_1, R_2, \dots, R_k$ that define the random variables a_i are disjoint, the joint distribution of $a_1, a_2, \dots, a_k$ is stochastically dominated by the joint distribution of $\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_k$, that is, the distribution of k i.i.d. exponentially distributed random variables of parameter 2α.

Define

$$a = \sum_{i=1}^{k} a_i \qquad \text{and} \qquad \tilde{a} = \sum_{i=1}^{k} \tilde{a}_i.$$

Expanding recursively from the relations $x_i = x_{i-1} + r - a_i$ and $x_1 = r - a_1$, we get

$$x_k = \sum_{i=1}^{k} (r - a_i) = kr - a.$$

Let us consider the event A_2 defined by $\tilde{a}_i \leq r/2$ for all $1 \leq i \leq k$. Since we aim to bound the probability that x_k is large (or equivalently, that a is small), we cannot use the fact that the joint distribution of the a_i's is stochastically dominated by the ones of $\tilde{a}_i$'s. Nevertheless, note that conditional on A_2, we have $a_i = \tilde{a}_i$ for all $1 \leq i \leq k$; if $a_{i-1} \leq r/2$ then $a_i \leq r/2 \leq r - a_{i-1}$, and from (2.3), $a_i = \tilde{a}_i$. In other words, for every $\beta \geq 0$

$$\Pr(a \leq \beta, A_2) = \Pr(\tilde{a} \leq \beta, A_2) \ .$$

Since each $\tilde{a}_i$ is exponentially distributed with parameter 2α and stochastically dominates a_i, we can bound the probability that A_2 does not occur:

$$\Pr(\overline{A_2}) \leq \sum_{i=1}^{k} \Pr\left(a_i > r/2\right) \leq \sum_{i=1}^{k} \Pr\left(\tilde{a}_i > r/2\right) = ke^{-\alpha r} \ . \tag{2.4}$$

Therefore, using the bound on t given in (2.2), we have

$$\begin{aligned} \Pr(x_k \geq t) &\leq \Pr(\overline{A_2}) + \Pr(x_k \geq t, A_2) \leq ke^{-\alpha r} + \Pr(kr - a > t, A_2) \\ &\leq ke^{-\alpha r} + \Pr\left(a \leq \frac{2\alpha^2}{kr}, A_2\right) = ke^{-\alpha r} + \Pr\left(\tilde{a} \leq \frac{2\alpha^2}{kr}, A_2\right) \\ &\leq ke^{-\alpha r} + \Pr\left(\tilde{a} \leq \frac{2\alpha^2}{kr}\right) \ . \end{aligned} \tag{2.5}$$

Thus it remains to give a good upper bound on the lower tail of $\tilde{a}$. Notice that $\mathbb{E}(\tilde{a}) = \sum_{i=1}^{k} \mathbb{E}(\tilde{a}_i) = \frac{k}{2\alpha}$. We use the definition of k and α, as well as Lemma 3 with $\varepsilon = \left(1 - \sqrt{2}\delta^{3/2} - o(1)\right)$ to show

$$\begin{aligned} \Pr\left(\tilde{a} \leq \frac{2\alpha^2}{kr}\right) &= \Pr\left(\tilde{a} \leq \frac{4\alpha^3}{k^2 r} \cdot \mathbb{E}(\tilde{a})\right) \\ &\leq \Pr\left(\tilde{a} \leq \left(\sqrt{2}\delta^{3/2} + o(1)\right) \mathbb{E}(\tilde{a})\right) \\ &\leq e^{-\varepsilon^2 k/2} \ . \end{aligned} \tag{2.6}$$

Finally, we use (2.5), (2.6) and the definition of k, α and ε to obtain

$$\Pr\left(x_k \geq t\right) \leq \frac{(1 + o(1))t}{r} \exp\left(-\sqrt{\delta/2}(tr)^{2/3}\right) + \exp\left(-(1 - \sqrt{2\delta^3} - o(1))^2 \frac{t}{2r}\right) \ .$$

Since the event A_1 implies $x_k \geq t$, $\Pr(A_1) \leq \Pr(x_k \geq t)$ and the proposition follows. □ □

Proposition 6. *Let $\widetilde{\mathcal{G}}^{\infty}_{u,v}(r,t)$ be a random geometric graph in $\mathbb{R}^2$ with two additional distinguished vertices $u = (0,0)$ and $v = (t,0)$ such that*

$$t = d_E(u,v) \geq \max\left\{12(\log n)^{3/2}/r, 21r\log n\right\}. \tag{2.7}$$

Then we have

$$d_G(u,v) \leq \left\lfloor \frac{t}{r}\left(1 + \frac{1}{2(rt)^{2/3}}\right)\right\rfloor \ ,$$

with probability at most $o(n^{-5/2})$.

Proof. Set $\delta = 1/2$. Since $t \geq 12(\log n)^{3/2}/r$, we have

$$\sqrt{\delta/2}(tr)^{2/3} - \log\left((1-o(1))t/r\right) > \frac{5}{2}\log n\ ,$$

and since $t \geq 21r\log n$,

$$\left(1-\sqrt{2\delta^3}-o(1)\right)^2 \frac{t}{2r} > \frac{5}{2}\log n\ .$$

By Proposition 5, this implies that

$$\Pr\left(d_G(u,v) \leq \left\lfloor \frac{t}{r}\left(1+\frac{1}{2(rt)^{2/3}}\right)\right\rfloor\right) = o(n^{-5/2})\ .$$

□ □

The same conclusion in Proposition 5 must be true (for $t \leq \sqrt{2n}$) if we restrict $\widetilde{\mathcal{G}}^{\infty}_{u,v}(r,t)$ to any arbitrary square $\widehat{\mathcal{S}}_n$ of area n containing $u=(0,0)$ and $v=(t,0)$ (i.e. we consider the subgraph induced by the vertices lying inside of that square), since the graph distance between u and v can only increase when doing so. Moreover, by rotating and mapping an appropriate square $\widehat{\mathcal{S}}_n$ to $\mathcal{S}_n = [-\sqrt{n}/2, \sqrt{n}/2]^2$, we conclude that statement (i) in Theorem 1 holds in $\widetilde{\mathcal{G}}_{u,v}(n,r)$ with probability $1-o(n^{-5/2})$. Hence, in view of the de-Poissonization argument described in the beginning of Section 2, this same property holds in $\mathcal{G}(n,r)$ with probability $1-o(n^{-2})$, for a given pair of vertices u,v. The statement follows by taking a union bound over all at most n^2 pairs of vertices.

2.2. **Proof of statement (ii).** In this subsection we complete the proof of Theorem 1. To derive the bound on the upper tail on the graph distance between $u,v \in V$, we first assume that $u=(0,0)$ and $v=(t,0)$ (for some $0<t\leq\sqrt{2n}$), and analize $\widetilde{\mathcal{G}}^{\infty}_{u,v}(r,t)$ restricted to a suitable rectangle. Our goal is to find a path P from u to v inside of that rectangle that gives an appropriate upper bound on $d_G(u,v)$. Then, we will use similar ideas to those at the end of Subsection 2.1 to derive the desired conclusion about $\mathcal{G}(n,r)$.

For every measurable set $S \subseteq \mathbb{R}^2$ containing u and v, let $\widetilde{\mathcal{G}}_{S,u,v}(r,t)$ denote the random geometric graph obtained as the subgraph of $\widetilde{\mathcal{G}}^{\infty}_{u,v}(r,t)$ induced by the vertices contained in S. Observe that $\widetilde{\mathcal{G}}_{S,u,v}(r,t)$ can also be constructed by taking as the vertex set a Poisson point process of intensity 1 in S, adding the vertices $u=(0,0)$ and $v=(t,0)$, and connecting any two vertices by an edge if they are at Euclidean distance at most r.

For every $0<\alpha\leq r$, we define the rectangle

$$S(t,\alpha) = [0,t]\times[0,\alpha].$$

(The precise value of α will be specified later; it will be different to the one given in the previous subsection.) Given α and r, we write $\rho = r - \frac{\alpha^2}{r}$. Then, for every point $z=(x_z,y_z)\in S$, we define the rectangle

$$S_z = S_z(\alpha) := [x_z, x_z+\rho]\times[0,\alpha]\ .$$

We need the following auxiliary lemma.

Lemma 7. *Let $t>0$ and $0<\alpha\leq r$. Then, for every pair of points $z\in S(t,\alpha)$ and $z'\in S_z(\alpha)$, we have $d_E(z,z')\leq r$ (see Figure 3).*

Proof. It is enough to show that the upper-left corner $z_1=(x_z,\alpha)$ and the bottom-right corner $z_2=(x_z+\rho,0)$ of $S_z(\alpha)$ satisfy $d_E(z_1,z_2)\leq r$. Then all the points inside $S_z(\alpha)$ lie at distance at most r, and in particular $d_E(z,z')\leq r$.

We have

$$(d_E(z_1,z_2))^2 = \rho^2+\alpha^2 = r^2-\alpha^2\left(1-(\alpha/r)^2\right)\leq r^2,$$

and the lemma follows. □ □

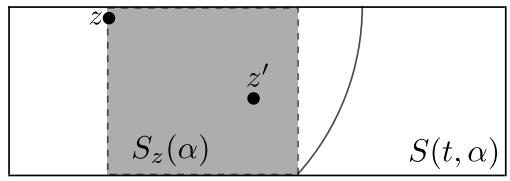

FIGURE 3. The rectangle S_z

Our next task is to bound the graph distance between u and v in $\widetilde{\mathcal{G}}_{S(t,\alpha),u,v}(r,t)$ by finding a path of length at most $\lceil \frac{t}{r}\left(1+\delta r^{-4/3}\right)\rceil$ from u to v, for some δ that will be made precise in the following proposition.

Proposition 8. *Let $F>0$ and $J>3(F+1)$ be constants and define $g(x)=x-\log(1+x)$. For every $J\le\delta\le Fr^{4/3}$, there exists an α such that the following holds: fix $t\ge 0$ and consider $\widetilde{\mathcal{G}}_{S,u,v}(r,t)$ to be a random geometric graph with $u=(0,0)$, $v=(t,0)$ in the rectangle $S=S(t,\alpha)$. Then we have*

$$\Pr\left(d_G(u,v)>\left\lceil\frac{t}{r}\left(1+\delta r^{-4/3}\right)\right\rceil\right)\le n\exp\left(-\frac{(F+1)\delta^{1/2}r^{4/3}}{2J^{3/2}}\right)+\exp\left(-g\Big((\delta/J)^{3/2}\Big)\frac{t}{r}\right).$$

Proof. Let us first define some parameters that will be useful in our analysis. Set $C=J^{-3/2}$ and let B be an arbitrary positive constant satisfying

$$B^2+2C/B<1/(F+1). \tag{2.8}$$

Some elementary analysis shows that such B must exist. In fact, the equation $B^2+2C/B=1/(F+1)$ has exactly two positive solutions B_1 and B_2 for any $0<C=J^{-3/2}<\frac{1}{(3(F+1))^{3/2}}$, and any $0<B_1<B<B_2<1/\sqrt{F+1}$ satisfies (2.8).

Fix some δ with $J\le\delta\le Fr^{4/3}$, and set

$$\alpha=B\delta^{1/2}r^{1/3}. \tag{2.9}$$

In order to use Lemma 7, let us first show that $\alpha\le r$. Since $\delta\le Fr^{4/3}$ by hypothesis of the proposition, we have

$$\alpha\le(B\sqrt{F})r, \tag{2.10}$$

and $B\sqrt{F}<1$, since $B<1/\sqrt{F+1}$. Moreover, we have

$$\rho=r-\alpha^2/r\ge(1-B^2F)r. \tag{2.11}$$

Let us consider the integer $k=\lceil\frac{t}{r}(1+\delta r^{-4/3})\rceil$ and let A_1 be the event that $d_G(u,v)\le k$; that is, there exists a path $P=(u=z_0,z_1,\dots,z_\ell,v)$ in $\widetilde{\mathcal{G}}_{S,u,v}(r,t)$ from u to v of length at most k. Such a path will only use vertices inside $S=S(t,\alpha)$, but due to some technical considerations in the argument, we extend the Poisson point process of our probability space to the semi-infinite strip $S(\alpha)=[0,\infty)\times[0,\alpha]$. Our goal is to show that the probability of A_1 is large.

As we did in the proof of Proposition 5, now we define random variables x_i and a_i for every $i\ge 1$. Set $x_0=0$ and $a_0=0$. For each $i\ge 1$, consider the rectangle $R_i=R_i(\alpha):=(x_{i-1}+\rho/2,x_{i-1}+\rho]\times[0,\alpha]$. If R_i contains at least a vertex, let z_i be the vertex with largest x-coordinate inside R_i. In such a case, define x_i to be the x-coordinate of z_i and $a_i=x_{i-1}+\rho-x_i$. Otherwise, we stop the process.

Let $\tau=\min\{i\ge 1: R_i \text{ contains no points}\}$ be the stopping time of the process.

Claim: Conditional on $\tau \geq k$, if $x_{k-1} + \rho \geq t$, then A_1 holds.
Proof of the claim. Assume that $\tau \geq k$ and that $x_{k-1} + \rho \geq t$. Observe that for every $i < k$, we have $0 \leq a_i \leq \rho/2$. Moreover, by construction of the process, for every $1 \leq i < k$, we have $z_i \in R_i \subseteq S_{z_{i-1}}$ and, since $\alpha \leq r$, Lemma 7 implies that z_i is adjacent to z_{i-1}. Thus, the vertices $z_0, z_1, \ldots, z_{k-1}$ form a path. In particular,

$$x_1 \geq \rho/2. \tag{2.12}$$

Since $x_{k-1} + \rho \geq t$, we know that there exists a value $\ell \leq k-1$ such that $x_{\ell-1} + \rho \geq t$ and also $x_{\ell-1} \leq t$, and thus, by Lemma 7, z_ℓ and v are connected by an edge. The path $P = (u = z_0, z_1 \ldots, z_\ell, v)$ has length $\ell + 1 \leq k$, connects u and v and is *fully contained* in S. Therefore, A_1 is satisfied, which completes the proof of the claim.

It suffices to show that we have with high probability $\tau \geq k$, and that conditional on it, with high probability $x_{k-1} + \rho \geq t$.

For every $0 \leq i < \tau$, let $A_2^{(i)}$ be the event that $a_j \leq \rho/2$ for every $1 \leq j \leq i$ and let $A_2 = A_2^{(k-1)}$ be the event that $\tau \geq k$. Conditional on A_2, we have that the regions $R_1, \ldots, R_{k-1}$ are disjoint. Hence, we deduce that conditional on A_2, the joint distribution of $a_1, \ldots, a_{k-1}$ is the same as the joint distribution of $\tilde{a}_1, \ldots, \tilde{a}_{k-1}$, with $\tilde{a}_1, \ldots, \tilde{a}_{k-1}$ being $k-1$ independent exponentially distributed random variables with parameter α. In particular, conditional only on $A_2^{(i-1)}$, we also have that a_i is stochastically dominated by $\tilde{a}_i$, and hence,

$$\Pr(\overline{A_2}) = \sum_{i=1}^{k-1} \Pr\left(a_i \geq \rho/2 \mid A_2^{(i-1)}\right) \leq \sum_{i=1}^{k-1} \Pr(\tilde{a}_i \geq \rho/2) \ .$$

Since $\alpha\rho/2 \geq (1 - B^2F)\alpha r/2 = (1 - B^2F)B\delta^{1/2}r^{4/3}/2$, we have

$$\Pr(\tilde{a}_i \geq \rho/2) = e^{-\alpha\rho/2} \leq e^{-(1-B^2F)B\delta^{1/2}r^{4/3}/2} \ .$$

and that

$$\Pr(\tau < k) = \Pr(\overline{A_2}) \leq ne^{-(1-B^2F)B\delta^{1/2}r^{4/3}/2} \ . \tag{2.13}$$

Also, if we let $a = \sum_{i=1}^{k-1} a_i$ and $\tilde{a} = \sum_{i=1}^{k-1} \tilde{a}_i$, conditional on A_2 (or in other words, on $\tau \geq k$), by the same argument, for every $\beta \geq 0$, we have

$$\Pr(a \geq \beta,\ A_2) = \Pr(\tilde{a} \geq \beta,\ A_2) \ . \tag{2.14}$$

Observe that now $\mathbb{E}(\tilde{a}) = \frac{k-1}{\alpha}$. Let A_3 be the event that $\tilde{a} \leq (1 + C\delta^{3/2})\frac{k-1}{\alpha}$. We first show that A_3 implies the event $\{k\rho - \tilde{a} > t\}$. Conditional on A_3, using the definition of α, the fact that $\delta^{-3/2} \leq C$ and that $\delta \leq Fr^{4/3}$, we have

$$\begin{aligned}
k\rho - \tilde{a} > k\rho - \frac{\left(1 + C\delta^{3/2}\right)(k-1)}{\alpha} &\geq kr\Big(1 - \frac{\alpha^2}{r^2} - \frac{(1 + C\delta^{3/2})}{\alpha r}\Big) \\
&\geq t(1 + \delta r^{-4/3})\left(1 - \delta r^{-4/3}\Big(B^2 + \frac{(\delta^{-3/2} + C)}{B}\Big)\right) \\
&\geq t(1 + \delta r^{-4/3})\left(1 - \delta r^{-4/3}\Big(B^2 + \frac{2C}{B}\Big)\right) \\
&= t\left[1 + \delta r^{-4/3}\left(1 - \big(\delta r^{-4/3} + 1\big)\Big(B^2 + \frac{2C}{B}\Big)\right)\right] \\
&\geq t\left[1 + \delta r^{-4/3}\left(1 - (F+1)\Big(B^2 + \frac{2C}{B}\Big)\right)\right] > t.
\end{aligned} \tag{2.15}$$

Now, we can use (2.14) and the upper-tail bound in Lemma 3 to prove

$$\Pr(\overline{A_3}) = \Pr\left(\tilde{a} \geq (1 + C\delta^{3/2})\frac{k-1}{\alpha}\right) \leq e^{-g((\delta/J)^{3/2})(k-1)} \leq e^{-g\left((\delta/J)^{3/2}\right)(\lceil t/r\rceil - 1)} . \tag{2.16}$$

By expanding the definition of x_{k-1}, we can write $x_{k-1} = (k-1)\rho - a$. Thus, using (2.13), (2.14), (2.15) and (2.16) we obtain

$$\begin{aligned}
\Pr(\{\tau < k\} \cup \{x_{k-1} + \rho \leq t\}) &= \Pr(\overline{A_2}) + \Pr(x_{k-1} + \rho \leq t, A_2) \\
&\leq ne^{-(1-B^2F)B\delta^{1/2}r^{4/3}/2} + \Pr(k\rho - a \leq t, A_2) \\
&\leq ne^{-(1-B^2F)B\delta^{1/2}r^{4/3}/2} + \Pr(k\rho - \tilde{a} \leq t) \\
&\leq ne^{-(1-B^2F)B\delta^{1/2}r^{4/3}/2} + \Pr(\overline{A_3}) + \Pr(k\rho - \tilde{a} \leq t, A_3) \\
&\leq ne^{-(1-B^2F)B\delta^{1/2}r^{4/3}/2} + e^{-g\left((\delta/J)^{3/2}\right)(\lceil t/r\rceil - 1)} . \qquad (2.17)
\end{aligned}$$

Moreover, by the properties of B and the definition of C, we have $(1-B^2F)B > (1-B^2(F+1))B > 2C(F+1) = 2(F+1)J^{-3/2}$. Thus,

$$\Pr(x_{k-1} + \rho \leq t) \leq ne^{-\frac{(F+1)\delta^{1/2}r^{4/3}}{J^{3/2}}} + e^{-g\left((\delta/J)^{3/2}\right)(\lceil t/r\rceil - 1)},$$

concluding the proof of the proposition. □ □

Remark. *Observe the trade-off between δ and the success probability in the proof of Proposition 8: for a given value of δ, we set $\alpha = \Theta(\sqrt{\delta}r^{1/3})$. That is, for a given radius r, the smaller δ, the smaller α. Proposition 8 computes the probability that a path using vertices only within a strip of width α can be found. Clearly, the smaller δ, the straighter a path has to be, and the smaller the rectangle in which we have to find a path has to be, therefore making also α smaller. On the other hand, for smaller α, the probability of indeed finding a path in such a small strip also gets smaller.*

Proposition 9. *Given $t > 0$ and the vertices $u = (0,0)$ and $v = (t,0)$, let $\gamma = \gamma(t)$ be defined as in the statement of Theorem 1. Let $\widetilde{\mathcal{G}}_{S,u,v}(r,t)$ be a random geometric graph in the rectangle $S = S(t,\alpha)$, with additional vertices u and v. Suppose that $r \geq 224\sqrt{\log n}$. Then, we have*

$$d_G(u,v) > \left\lceil \frac{t}{r}\left(1 + \gamma r^{-4/3}\right)\right\rceil ,$$

with probability at most $o(n^{-5/2})$.

Proof. First, observe that, if $t \leq r$, then $d_G(u,v) = 1$ with probability 1, and the statement of the proposition holds trivially. Thus, we assume henceforth that $t > r$. Set $B = 0.01/(2.02\sqrt{2})$, $C = 10^{-4}$, $F = 1$, $D = 4 \cdot 10^6$, $E = 1358$ and $J = 10^{8/3}$. Set

$$\gamma' = \max\left\{E\left(\frac{\log n}{\lceil t/r\rceil - 1}\right)^{2/3}, D\frac{\log^2 n}{r^{8/3}}, 3^{2/3}J\right\} .$$

Note that $\gamma' \leq \gamma$ for γ as given in Theorem 1: indeed, the second and the third term are equal, and for the first term, for $t > r$, we have that $(3/(1+t/r)) > (1/(\lceil t/r\rceil - 1))$ holds. Therefore, it suffices to apply Proposition 8 with $\delta = \gamma'$. It is straightforward to check that the restrictions (2.8) and $J > 3(F+1)$, required in Proposition 8, hold. We also need to show that $J \leq \gamma' \leq Fr^{4/3}$. Notice that $\frac{D\log^2 n}{r^{8/3}} \leq Fr^{4/3}$, since $r \geq 224\sqrt{\log n} \geq D^{1/4}\sqrt{\log n}$; also $E\left(\frac{\log n}{\lceil t/r\rceil - 1}\right)^{2/3} \leq Fr^{4/3}$, since $\lceil t/r\rceil - 1 \geq 1$, and since $r^2 \geq E^{3/2}\log n$, which follows from our assumption of $r \geq 224\sqrt{\log n}$; and finally $3^{2/3}J \leq Fr^{4/3}$ since $r = \Omega(\sqrt{\log n})$. Moreover, $\gamma' \geq 3^{2/3}J \geq J$.

Note that this choice of constants combined with (2.10) and (2.11) implies

$$\alpha \leq \frac{0.01}{2.02\sqrt{2}} r \leq r/3 \qquad \text{and} \qquad \rho \geq 8r/9 \geq 8\alpha/3. \tag{2.18}$$

We will now apply (2.17) in the proof of Proposition 8 with this given δ, in order to show that $\Pr(d_G(u,v) > k) = o(n^{-5/2})$. On the one hand, $\delta \geq \frac{D \log^2 n}{r^{8/3}}$ implies

$$\begin{aligned}\frac{(1-B^2F)B\delta^{1/2}r^{4/3}}{2} - \log n &\geq \frac{(1-B^2F)BD^{1/2}\log n}{2} - \log n \\ &> \frac{7.0009}{2}\log n - \log n = \frac{5.0009}{2}\log n\ .\end{aligned}$$

On the other hand, $\delta \geq 3^{2/3}J$ and $\delta \geq E\left(\frac{\log n}{\lceil t/r\rceil - 1}\right)^{2/3}$ imply

$$g\left((\delta/J)^{3/2}\right)(\lceil t/r\rceil - 1) > \frac{(\delta/J)^{3/2}}{2}(\lceil t/r\rceil - 1) \geq \frac{1}{2}CE^{3/2}\log n > \frac{5.004}{2}\log n,$$

where we have used that $g(x) \geq x/2$ if $x \geq 3$, and that $C = J^{-3/2}$.

Therefore, $\Pr(d_G(u,v) > k) \leq n^{-5.0009/2} + n^{-5.004/2} = o(n^{-5/2})$. □ □

Corollary 10. *Statement (ii) in Theorem 1 is true.*

Proof. We will use an argument similar to that at the end of Subsection 2.1 to relate the models $\widetilde{\mathcal{G}}_{S(t,\alpha),u,v}(r,t)$ and $\widetilde{\mathcal{G}}_{u,v}(n,r)$. However, such endeavour entails extra difficulties. Given two vertices $u, v \in \mathcal{S}_n = [-\sqrt{n}/2, \sqrt{n}/2]^2$ at Euclidean distance $t > 0$, there are exactly two isometries that map them to $(0,0)$ and $(t,0)$, denoted by π^+ and π^-. Unfortunately, the preimage of the rectangle $S(t,\alpha)$ under such isometries may not be entirely contained in the square $\mathcal{S}_n$. In order to overcome this obstacle, we just need to show that the internal vertices of the path from $(0,0)$ to $(t,0)$ that we built in the proof of Proposition 8 are contained in a smaller rectangle whose preimage under either π^+ or π^- is contained in $\mathcal{S}_n$. This will be enough for us to conclude the existence (with sufficiently high probability) of a path in $\widetilde{\mathcal{G}}_{u,v}(n,r)$ between u and v of the desired length.

Recall the definition of α given in (2.9). Observe that from (2.12) together with (2.18), $x_1 \geq \rho/2 > 4\alpha/3$ with probability at least $1 - o(n^{-5/2})$. In particular, this event implies that z_1 is outside of the square $[0, 1.01\alpha] \times [0,\alpha]$. If z_ℓ (the last internal vertex of the path P found) is outside $[t - 1.01\alpha, t] \times [0,\alpha]$, we obtain a path connecting u and v of length $\ell + 1 \leq k$ with all its internal vertices in $R := [1.01\alpha, t - 1.01\alpha] \times [0,\alpha]$. Otherwise, suppose that z_ℓ lies in $[t - 1.01\alpha, t] \times [0,\alpha]$. Then, also with probability $1 - o(n^{-5/2})$, we can find some point $\hat{z}_\ell$ in $[t - 1.01\alpha - r/2, t - 1.01\alpha) \times [0,\alpha]$: indeed, since $\rho \leq r$, the region in which we want $\hat{z}_\ell$ is bigger than the regions S_i in the proof of Proposition 8 and Proposition 9. We now can use Lemma 7 to show that z_ℓ can be replace by $\hat{z}_\ell$ in P. Observe that $\hat{z}_\ell$ is connected to v, since $1.01\alpha + r/2 \leq \rho$, and also $\hat{z}_\ell$ is connected to $z_{\ell-1}$, since its x-coordinate $\hat{x}_\ell$ satisfies $|\hat{x}_\ell - x_{\ell-1}| \leq \max\{\rho, r/2 - \rho/2\} \leq \rho$. Thus, we can replace z_ℓ with $\hat{z}_\ell$, and obtain a new path connecting u and v of length $\ell + 1 \leq k$ with all its internal vertices in R. We will show that either $\pi^+(R)$ or $\pi^-(R)$ is always contained in $\mathcal{S}_n$. We first introduce some definitions.

Consider two points $u = (x_u, y_u)$ and $v = (x_v, y_v)$ in $\mathcal{S}_n$. By symmetry we may assume that $x_u < x_v$ and $y_u \leq y_v$. Let β be the angle of the vector $\vec{uv}$ with respect to the horizontal axis. Again by symmetry, we may consider $\beta \in [0, \pi/4]$.

We consider now two rectangles of dimensions $\alpha \times t$ placed on each side of the segment uv. Let R^+ be the rectangle to the left of $\vec{uv}$ (that is, $R^+ = \pi^+(R)$), and let R^- be the rectangle to the right of $\vec{uv}$ (also, $R^- = \pi^-(R)$). We will show that at least one of these rectangles contains a copy of R fully contained in $\mathcal{S}_n$. This choice will determine which of the isometries, π^+ or π^-, map R inside $\mathcal{S}_n$.

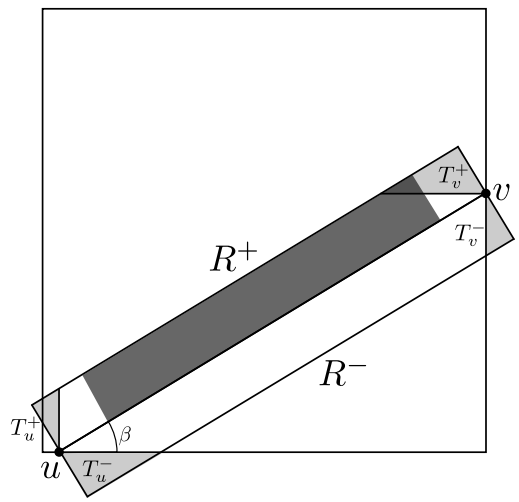

FIGURE 4. The black area represents the copy of R contained in $\mathcal{S}_n$

Notice that the intersection of R^+ and R^- with each of the halfplanes $x \leq x_u$, $x \geq x_v$, $y \leq y_u$ and $y \geq y_v$ gives 4 triangles. We call them T_u^+, T_v^-, T_u^- and T_v^+ respectively. All these triangles are right-angled, and denote by t_u^+, t_v^-, t_u^- and t_v^+ the side of the corresponding triangle that it is parallel to the segment uv. Notice that $|t_u^+| = |t_v^-|$ and $|t_u^-| = |t_v^+|$. Call a triangle T_w^*, with $w \in \{u, v\}$ and $* \in \{+, -\}$, *safe* if $|t_w^*| \leq 1.01\alpha$. Note that if T_u^+ and T_v^+ are safe or fully contained in the square, then R^+ contains the desired rectangle R, and analogously for R^-.

Since we assumed that $\beta \leq \pi/4$, we have $|t_u^+| = |t_v^-| = \alpha|\tan\beta| \leq 1.01\alpha$. Thus, T_u^+ and T_v^- are safe. If $y_u = y_v$, that is $\beta = 0$, it is clear that either R^+ or R^- contain the desired copy of R. Thus, we may assume that $\beta > 0$.

We can also assume that both u and v are on the boundary of $\mathcal{S}_n$, as otherwise we extend the line segment uv to the boundary of the square, and the original rectangles are contained in the new ones.

Recall that T_u^+ and T_v^- are safe. If $y_v < \sqrt{n}/2 - \alpha$, then T_v^+ is completely contained in the square, and hence R^+ satisfies the conditions. Similarly, if $y_u > -\sqrt{n}/2 + \alpha$, R^- satisfies the conditions. Thus, assume that $y_v \geq \sqrt{n}/2 - \alpha$ and $y_u \leq -\sqrt{n}/2 + \alpha$. We will show that R^- contains the desired copy of R: as before, T_v^- is safe, so it remains to consider T_u^-. We have $|t_u^-| = \alpha \tan\left(\frac{\pi}{2} - \beta\right)$. For $0 < \beta \leq \pi/4$, $\tan\left(\frac{\pi}{2} - \beta\right)$ is decreasing in β, and it therefore suffices to show that T_u^- is safe for the smallest possible value of β. Notice that the minimal angle β under our assumptions on y_u and y_v is obtained for $u = (-\sqrt{n}/2, -\sqrt{n}/2 + \alpha)$ and $v = (\sqrt{n}/2, \sqrt{n}/2 - \alpha)$, and thus $\beta \geq \arctan\left(\frac{\sqrt{n}-2\alpha}{\sqrt{n}}\right)$, or equivalently $\tan\left(\frac{\pi}{2} - \beta\right) \leq \frac{\sqrt{n}}{\sqrt{n}-2\alpha}$. In this case,

$$|t_u^-| \leq \alpha \cdot \frac{\sqrt{n}}{\sqrt{n} - 2\alpha} = \alpha\left(1 + \frac{2\alpha}{\sqrt{n} - 2\alpha}\right) \leq 1.01\alpha,$$

where the last inequality follows from the fact that $\alpha \leq \frac{0.01}{2.02\sqrt{2}} r \leq \frac{0.01}{2.02}\sqrt{n}$ since we assumed $r \leq \sqrt{2n}$ (see also (2.18)), and therefore $\frac{2\alpha}{\sqrt{n}-2\alpha} \leq 0.01$.

Again, by de-Poissonizing $\widetilde{\mathcal{G}}_{u,v}(n, r)$, we can use Proposition 9 to show that for given u and v in $G \in \mathcal{G}(n, r)$, statement (ii) in Theorem 1 holds with probability at least $1 - o(n^{-2})$. By taking a union bound over all at most n^2 possible pairs of vertices, statement (ii) in Theorem 1 follows. □ □

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