ON TREEWIDTH AND RELATED PARAMETERS OF RANDOM GEOMETRIC GRAPHS

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Abstract. We give asymptotically exact values for the treewidth \( tw(G) \) of a random geometric graph \( G \in G(n,r) \) in \([0,\sqrt{n}]^2\). More precisely, let \( r_c \) denote the threshold radius for the appearance of the giant component in \( G(n,r) \). We then show that for any constant \( 0 < r < r_c \), \( tw(G) = \Theta\left(\frac{\log n}{\log \log n}\right) \), and for \( c \) being sufficiently large, and \( r = r(n) \geq c \), \( tw(G) = \Theta(r\sqrt{n}) \). Our proofs show that for the corresponding values of \( r \) the same asymptotic bounds also hold for the pathwidth and the treedepth of a random geometric graph.

Key words. random geometric graphs, treewidth, treedepth, pathwidth

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1. Introduction. Let \( V \) be a set of \( n \) points in the square \( S_n = [0, \sqrt{n}]^2 \) and \( r = r(n) \) a nonnegative real number. This choice of the square is only for convenience; by suitable scaling we could have chosen the square \([0, 1]^2\) and all the results would be still valid. We will identify each point with its position, that is, \( v \in V \) refers also to the geometrical position of \( v \) in \( S_n \).

The geometric graph \( G \) of \( V \) with radius \( r \) is the graph constructed by connecting two points of \( V \) if their Euclidean distance in \( S_n \) is smaller than \( r \). For any two points \( u, v \in S_n \) we will denote by \( \text{dist}_E(u,v) \) their Euclidean distance and by \( \text{dist}_G(u,v) \) their distance in the graph \( G \).

Then we define \( G(n,r) \) as the probability space of the geometric graphs of order \( n \) with radius \( r \). A graph \( G \) chosen uniformly at random from \( G(n,r) \) will be called a random geometric graph and will be denoted by \( G \in G(n,r) \). Note that with probability one, no two vertices of \( G \in G(n,r) \) are placed in the same position.

Starting with the seminal paper of Gilbert [8], random geometric graphs have in recent decades received a lot of attention as a model for large communication networks such as sensor networks. Network agents are represented by the vertices of the graph, and direct connectivity is represented by edges. For applications of random geometric graphs, we refer to [11, Chapter 3], and for a survey of many theoretical results, we refer to Penrose’s monograph [22].

All our stated results are asymptotic as \( n \to \infty \). We use the usual notation \( a.a.s. \) to denote \textit{asymptotically almost surely}, i.e., with probability \( 1 - o(1) \). It is well known that the property of the existence of a giant component of order \( \Theta(n) \) undergoes a sharp threshold in \( G(n,r) \) (see, e.g., [9]), that is, there exists a constant value \( r_c \)
such that for any $\varepsilon > 0$, a.a.s. the largest component of $G \in \mathcal{G}(n, r_c - \varepsilon)$ is of order $O(\log n)$, whereas in $G \in \mathcal{G}(n, r_c + \varepsilon)$, a single component of order $\Theta(n)$ is present, while the others have order $O(\log n)$ (see [22, Chapter 10]). The exact value of $r_c$ is not yet determined, but it is known that $c^- \leq r_c \leq c^+$, where $c^- \approx 0.834$ and $c^+ \approx 1.836$ (see [22, p. 189]). Moreover, simulation studies suggest that the exact value of $r_c \approx 1.2$ (see again [22, p. 189]).

Since random geometric graphs have been heavily used for modeling communication networks, it is natural to analyze the expected complexity of different algorithms applied to this class. Courcelle’s theorem [5] states that any problem that can be expressed in monadic second order logic can be solved in linear time for the class of graphs with bounded treewidth. This motivates the study of this parameter and other tree-like parameters on random geometric graphs. In this paper, we study the behavior of the treewidth and the treedepth on random geometric graphs.

The treewidth was introduced independently by Halin in [10] and by Robertson and Seymour in [26].

For a graph $G = (V, E)$ on $n$ vertices, we call $(T, W)$ a tree decomposition of $G$, where $W$ is a set of vertex subsets $W_1, \ldots, W_s \subseteq V$, called bags, and $T$ is a forest with vertices in $W$, such that

1. $\bigcup_{1 \leq i \leq s} W_i = V$,
2. for any $e = uv \in E$ there exists a set $W_i \in W$ such that $u, v \in W_i$,
3. for any $v \in V$, the subgraph induced by the $W_i \ni v$ is connected as a subgraph of $T$.

The width of a tree-decomposition is $w(T, W) = \max_{1 \leq i \leq s} |W_i| - 1$, and the treewidth of a graph $G$ can be defined as

$$\text{tw}(G) = \min_{(T, W)} w(T, W).$$

Observe that if $G$ is a graph with connected components $H_1, \ldots, H_m$, then

(1) $\text{tw}(G) = \max_{1 \leq i \leq m} \text{tw}(H_i)$.

The concept of treedepth has been introduced under different names in the literature. In this paper we follow the definition given by Nešetřil and Ossona de Mendez as a tree-like parameter in the scope of homomorphism theory, where it provides an alternative definition of bounded expansion classes [19]. For the sake of completeness, we note that the treedepth is also equivalent to the height of an elimination tree (used, for instance, in the parallel Cholesky decomposition [24]). Furthermore, analogous definitions can be found using the terminology of rank function [18], vertex ranking number (or ordered coloring) [7], or weak coloring number [12].

We now give the precise definition of treedepth. Let $T$ be a rooted tree. The height of $T$ is defined as the number of vertices of the longest rooted path. The closure of $T$ is the graph that has the same set of vertices and a pair of vertices is connected by an edge if one is an ancestor of the other in $T$. We say that the tree $T$ is an elimination tree of a connected graph $G$ if $G$ is a subgraph of the closure of $T$. The treedepth of a connected graph $G$, $\text{td}(G)$, is defined to be the minimum height of an elimination tree of $G$.

The definition of treedepth can also be extended to nonconnected graphs. If $G$ is a graph with connected components $H_1, \ldots, H_m$, then

(2) $\text{td}(G) = \max_{1 \leq i \leq m} \text{td}(H_i)$.
Hence, if $S \subset V(G)$ separates $G$ into two subsets $A$ and $B$, we have

\begin{equation}
\text{td}(G) \leq |S| + \max\{\text{td}(A), \text{td}(B)\}.
\end{equation}

Observe that if $H$ is a subgraph of $G$, then

\begin{equation}
\text{td}(H) \leq \text{td}(G) \text{ and } \text{tw}(H) \leq \text{tw}(G).
\end{equation}

Both parameters are closely connected: while the treewidth of a graph $G$ is a parameter that measures the similarity between $G$ and the class of trees in general, the treedepth of $G$ measures how close $G$ is to a star. In other words, the treedepth also takes into account the diameter of the tree we are comparing the graph with. The two parameters are related by the inequalities

$$\text{tw}(G) \leq \text{td}(G) \leq (\text{tw}(G) + 1) \log_2 n,$$

both bounds being sharp (see [19]). Note also that $\text{tw}(G) \geq \omega(G) - 1$, where $\omega(G)$ denotes the size of the largest clique in $G$.

**Results of the paper.** In this paper we study the values of $\text{tw}(G)$ and $\text{td}(G)$ of a random geometric graph $G \in \mathcal{G}(n, r)$ for different values of $r = r(n)$. In particular, we prove the following two main theorems.

**Theorem 1.1.** Let $0 < r < r_c$ and let $G \in \mathcal{G}(n, r)$. Then, a.a.s., $\text{tw}(G) = \Theta(\frac{\log n}{\log \log n})$, and also a.a.s., $\text{td}(G) = \Theta(\frac{\log n}{\log \log n})$.

**Theorem 1.2.** Let $c$ be a sufficiently large constant. Let $r = r(n) \geq c$ and $G \in \mathcal{G}(n, r)$, a.a.s. $\text{tw}(G) = \Theta(r\sqrt{n})$ and $\text{td}(G) = \Theta(r\sqrt{n})$.

**Remark 1.3.** For $G \in \mathcal{G}(n, r)$ with $r$ constant, but $r \geq c$, by the results of [6], many problems such as Steiner Tree, Feedback Vertex Set, Connected Vertex Cover can be solved in time $O(poly(n)3^{\sqrt{n}})$, while others like Connected Dominating Set, Connected Feedback Vertex Set, Min Cycle Cover, Longest Path, Longest Cycle, Graph Metric Travelling Salesman Problem can be solved in time $O(poly(n)4^{\sqrt{n}})$.

**Remark 1.4.** Other width parameters that are sandwiched between the treewidth and the treedepth clearly then also have the same asymptotic behavior in $\mathcal{G}(n, r)$. For instance, the pathwidth of a graph, introduced by Robertson and Seymour [25], measures the similarity between a graph and a path. Since the pathwidth is well known to be bounded from below by the treewidth and bounded from above by the treedepth (see [27, Theorems 5.3 and 5.11]), the former theorems imply that for those values of $r = r(n)$ the pathwidth of the graph is of the same order.

**Remark 1.5.** Whereas intuitively it might be clear that around the threshold of the existence of a giant component there should be a jump for parameters like treewidth or treedepth in $\mathcal{G}(n, r)$, the orders of magnitude of these parameters are not so obvious (for us). Moreover, we point out that there are differences between $\mathcal{G}(n, r)$ and $\mathcal{G}(n, p)$: it is known that in the Erdős–Rényi random graph model $\mathcal{G}(n, p)$, as soon as the giant component appears, the graph has linear treewidth (see [14]). In contrast to this, Theorem 1.2 shows that a random geometric graph with a linear number of edges containing a giant component only has treewidth $\Theta(\sqrt{n})$. This different behavior of the two models can be explained by their different expansion properties and the connection between balanced separators and treewidth (see Lemma 4.3 below). Classical random graphs have very good expansion properties, and thus it is
difficult to find small separators of large sets of vertices. The geometric properties of the model $\mathcal{G}(n, r)$ imply the lack of large expanders. For this reason, in the latter case one can construct a tree decomposition with smaller bags. On the other hand, in the subcritical regime (with a linear number of edges, but before the existence of a giant component) the treedepth of $\mathcal{G}(n, p)$ is $O(\log \log n)$ (see [23]), whereas by Theorem 1.1, for random geometric graphs it is already $\Theta(\log \log n)$. (In fact, a lower bound of this order is very easy, since the largest clique is of that order, and an upper bound of $O(\log n)$ is also easy, since $O(\log n)$ is an upper bound for the size of the largest component.) Furthermore, in this range, in classical random graphs the treewidth is bounded by a constant (see [23]), whereas our theorems show that in $\mathcal{G}(n, r)$ both treewidth and treedepth are asymptotically of the same order for a wide range of parameters $r$. The fact that for random geometric graphs the treedepth and treewidth are always asymptotically of the same order implies that $\mathcal{G}(n, r)$ is more similar to a star–shaped tree than to a path–shaped tree, which in general is not true for random graphs.

Poissonization. In order to simplify calculations, we will use the well-known idea of Poissonization (see [22, section 1.7]): let $V$ be a set of points obtained as a homogeneous Poisson point process $\mathcal{G}(P_1, r)$ of intensity 1 in $S_n$. In other words, $V$ consists of $N$ points in the square $S_n$ chosen independently and uniformly at random, where $N$ is a Poisson random variable of mean $n$. Exactly as in $\mathcal{G}(n, r)$, two points $u, v \in V$ are connected by an edge if their Euclidean distance in $S_n$ is at most $r$. The main advantage of the Poisson point process is that the number of points of $V$ that lie in any region $A \subseteq S_n$ of area $a$ has a Poisson distribution with mean $a$; and the number of points of $V$ in disjoint regions of $S_n$ are independently distributed. Moreover, by conditioning $\mathcal{G}(P_1, r)$ upon the event $N = n$, we recover the original distribution of $\mathcal{G}(n, r)$. Therefore, since $\Pr(N = n) = \Theta(1/\sqrt{n})$, any event holding in $\mathcal{G}(P_1, r)$ with probability at least $1 - o(f_n)$ must hold in $\mathcal{G}(n, r)$ with probability at least $1 - o(f_n \sqrt{n})$. In particular, an event holding with probability $1 - o(n^{-1/2})$ in $\mathcal{G}(P_1, r)$ holds a.a.s. in $\mathcal{G}(n, r)$. We make use of this property throughout the article and perform the proofs of Theorems 1.1 and 1.2 for a graph $G \in \mathcal{G}(P_1, r)$.

The paper is organized as follows. In section 2 we define the cell graph of a geometric graph and give some properties of it. The proof of Theorem 1.1 is presented in section 3. Whereas the lower bound follows from a standard argument using the clique number of $\mathcal{G}(n, r)$, the proof of the upper bound is more involved. In section 4 we continue by proving Theorem 1.2. Finally, in section 5 we conclude by mentioning some open problems.

2. Properties of deterministic geometric graphs.

2.1. The cell graph of a geometric graph. For any constant $\ell > 0$, we tessellate $S_n$ into squares of sidelength $\ell$ called cells. For the sake of simplicity of presentation, we assume that $\sqrt{n}/\ell$ is an integer for the values of $\ell$ considered in this paper. We use this tessellation to construct the cell graph $C_G(\ell)$ of $G$: each nonempty cell will be represented by a vertex and two different vertices of $C_G(\ell)$ will be joined if there exist two points of $G$ in the corresponding cells that share an edge. (See Figure 1, where the tessellation is omitted for clarity.)

From now on, unless otherwise stated, we will call points the vertices of the geometric graph $G$ and use the word vertex for the cells of $C_G(\ell)$. The cell-graph $C_G(\ell)$ simplifies the original geometric graph $G$ while preserving the same structure. For any subgraph $H$ of $G$ we will denote its cell graph by $C_H(\ell)$.
Remark 2.1. Notice that $C_H(\ell)$ is always a subgraph of $C_G(\ell)$. Observe that, for any $\ell \leq r/\sqrt{2}$, each nonempty cell contains points from exactly one connected component of $G$, since all the points inside a cell are connected. Thus, if $\ell \leq r/\sqrt{2}$, there exists a natural bijection between the connected components of $G$ and the connected components of $C_G(\ell)$.

We need another auxiliary graph, the grid graph $L_{a,b}^k$, defined as follows: its vertex set is $V(L_{a,b}^k) = \{(i,j) : 1 \leq i \leq a, 1 \leq j \leq b\}$, and $(i,j)(i',j') \in E(L_{a,b}^k)$ if and only if $(i,j) \neq (i',j')$ and $\max\{|i-i'|,|j-j'\| \leq k$. Note that by construction, for a geometric graph $G$ in $S_n$ with radius $r$ we have the following relation (as subgraphs):

\[
C_G(\ell) \subseteq L_{\sqrt{n}/\ell,\sqrt{n}/\ell}^{r/\ell}.
\]

The following lemma bounds the maximal number of different connected subgraphs of a given size in $L_{a,b}^k$.

**Lemma 2.2.** The number of connected subgraphs of size $s$ in $L_{a,b}^k$ is at most $O(ab(2k+1)^4s)$.

**Proof.** A connected subgraph is determined by a root $v$ and any of its spanning trees, rooted at $v$. Observe that there are $ab$ many ways to choose $v \in V(L_{a,b}^k)$. Moreover, the degree of a vertex in $L_{a,b}^k$ is at most $(2k+1)^2$, since for any cell $(i,j)$ there are at most $(2k+1)^2$ cells $(i',j')$ such that $\max\{|i-i'|,|j-j'\| \leq k$.

One can construct at most $((2k+1)^2)^{2s-3} \leq (2k+1)^4s$ walks of length $2s-2$ that have both start and end points at $v$. In particular, these walks contain all the possible spanning trees rooted at $v$ since a spanning tree has $s-1$ edges and each edge is traversed twice. Thus, the lemma follows.

**Remark 2.3.** Lemma 2.2 is certainly not tight. For the same problem on the integer lattice (each cell is connected to the four closest ones) the asymptotic growth is $poly(s)\lambda^s$. However the exact value of $\lambda$ is not yet known. The best known lower and upper bounds for $\lambda$ are 4.0025 and 4.5685, respectively (see [3, 2]).

The following proposition bounds the treedepth of a strong product of a graph and a clique.

Given two graphs $G_1$ and $G_2$, the strong product $G = G_1 \boxtimes G_2$ is defined
as $V(G) = V(G_1) \times V(G_2)$ and $(u_1, u_2)(v_1, v_2) \in E(G)$ iff for $i = 1, 2$, either $u_i = v_i$ or $u_i v_i \in E(G_i)$. Denote by $K_t$ the complete graph on $t$ vertices.

**Lemma 2.4.** Let $G = G_1 \boxtimes K_t$. Then

$$\text{td}(G) \leq t \cdot \text{td}(G_1).$$

**Proof.** Let $T_1$ be a tree of height $\text{td}(G_1)$ that embeds $G_1$ in its closure. Note also that $K_t$ is contained in the closure of a rooted path of order $t$, $P_t$. Observe that $T_1 \boxtimes P_t$ is not a tree, but it contains a tree $T$, in whose closure $T_1 \boxtimes P_t$ is contained (see Figure 2). Indeed, $T$ can be constructed in the following way: each vertex $u \in V(T_1)$ is replaced by a path of order $t$ (call these new vertices $u_1, \ldots, u_t$), and if there is an edge $uv \in E(T_1)$, such that $u$ is ancestor of $v$, then in $T$, $u_t$ is connected by an edge to $v_1$ (the depth of $v_1$ in $T$ is exactly one more than the depth of $u_t$); see Figure 2. Note that $T$ is a tree and its closure contains $G$ as a subgraph. Since each vertex of $G_1$ is replaced by $t$ vertices, $\text{td}(G) \leq t \cdot \text{td}(G_1)$. \hfill \Box

Observe also that for a geometric graph $G$,

$$G \subseteq C_G(\ell) \boxtimes K_t,$$

where $t$ is the maximum number of points inside a cell of the tessellation of length $\ell$.

Since we can express the treedepth of $G$ in terms of the treedepth of its cell graph and the latter one is a subgraph of $L_{a,b}^k$, the following proposition will be useful.

**Proposition 2.5.** Let $L_{a,b}^k$ the grid graph defined as above and suppose that $a \leq b$. Then

$$\text{td}(L_{a,b}^k) \leq O(ka \log b).$$

**Proof.** We describe an elimination tree for $L_{a,b}^k$ in a recursive way. First, note that $\text{td}(L_{a,b}^k) = O(ka)$, since the treedepth of a graph is always smaller than its order. Let us compute now the treedepth of $L_{a,b}^k$. By removing the central copy of $L_{a,k}^k$ in
To decompose $L^k_{a,b}$, we disconnect the original graph and get two copies of $L^k_{a,(b-k)/2}$. Applying this recursively and using (3), we obtain

$$td(L^k_{a,b}) \leq O(ka) + td(L^k_{a,(b-k)/2}) \leq O(ka) + \cdots + O(ka) + td(L^k_{a,k})$$

$$= O(ka \log b). \quad \square$$

The following proposition will be very useful in the proof of Theorem 1.1 but can be applied to any sparse geometric graph.

**Proposition 2.6.** Let $H$ be a geometric graph of order $m$ such that there are no more than $t$ points inside each cell of length $\ell = r/\sqrt{2}$.

Then, we have

$$td(H) = O \left( \max \left\{ \frac{m}{\log m}, t(\log m)^3 \right\} \right).$$

**Proof.** Throughout this proof all cells will have length $\ell = r/\sqrt{2}$. Notice that by Remark 2.1 the connected components of the cell graph $C_H(\ell)$ are in one to one correspondence with the connected components in $H$. Thus, we may assume that $H$ is connected. We will show an upper bound on $td(H)$ by providing an elimination scheme for $C_H$ which then induces an elimination scheme for $H$.

Fix a vertex $v \in V(C_H)$ corresponding to a cell of the tessellation. For any integer $d \geq 0$, denote by $V_d$ the set of vertices in the cell graph, which are at $L_\infty$ distance $d$ in the underlying grid graph from $v$ (see Figure 3).

Analogously, we define $P_d$ to be the set of points of $H$ inside the cells of $V_d$.

For the sake of convenience, we define

$$K = \frac{m}{(\log m)^2}.$$  

The idea of the proof is to find a separator $S$ of $H$ that contains at most $O(K)$ points. This separator will split the graph into some smaller subgraphs. Using (3) and applying the same procedure recursively to the remaining parts, we will get an upper bound on $td(H)$. 

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**Fig. 3. Decomposition of $C_H$.**
Let $f$ be the largest integer for which
\[ \sum_{d=0}^{f-1} |P_d| \leq \frac{m}{2}. \]

Let $f_1$ be the largest integer for which $f_1 \leq f$ and $|P_{f_1}| \leq K$ and $f_2$ be the smallest integer for which $f_2 \geq f$ and $|P_{f_2}| \leq K$. Since $H$ contains $m$ points, $f_2 - f_1 \leq \frac{m}{K} = (\log m)^2$.

Given a graph $G$ and $S \subset V(G)$, we will denote by $G[S]$ the subgraph of $G$ induced by $S$. We decompose of $C_H$ into the following subgraphs (see Figure 3):

\[ C_S = C_H[V_{f_1} \cup V_{f_2}], \quad C_A = C_H \left[ \bigcup_{d=0}^{f_1-1} V_d \right], \]
\[ C_L = C_H \left[ \bigcup_{d=f_1+1}^{f_2-1} V_d \right], \quad C_B = C_H \left[ \bigcup_{d \geq f_2+1} V_d \right], \]

and we define accordingly

\[ H_S = H[P_{f_1} \cup P_{f_2}], \quad H_A = H \left[ \bigcup_{d=0}^{f_1-1} P_d \right], \]
\[ H_L = H \left[ \bigcup_{d=f_1+1}^{f_2-1} P_d \right], \quad H_B = H \left[ \bigcup_{d \geq f_2+1} P_d \right]. \]

In the case $|P_f| \leq K$, we have $f_1 = f_2$ and $C_L$ and $H_L$ are graphs on zero vertices. Thus, suppose that this is not the case, and focus on $C_L$.

Since $\ell = r/\sqrt{2}$, by (5) we know that $C_L$ is a subgraph of at most four copies of $L_{a,b}^2$ (see Figure 3), where $a = (\log m)^2$ and $b = m$, since $f_2 - f_1 \leq (\log m)^2$ and $|P_d| \leq m$ for any $d$. By (3) and Proposition 2.5, we get
\[ \text{td}(C_L) \leq O(4a) + \text{td}(L_{a,b}^2) = O \left( (\log m)^3 \right). \]

Moreover, $H_L \subseteq C_L \oslash K_t$. Hence, by Lemma 2.4,
\[ \text{td}(H_L) = O \left( t(\log m)^3 \right). \]

By (3), now applied to $H$ and the separator $S = V_{f_1} \cup V_{f_2}$, we have
\[ \text{td}(H) \leq |S| + \max\{\text{td}(H_A), \text{td}(H_L), \text{td}(H_B)\} \leq 2K + \max\{\text{td}(H_A), O \left( t(\log m)^3 \right) , \text{td}(H_B)\}, \]

since $|S| \leq 2K$ by definition of $f_1$ and $f_2$.

We recursively repeat this procedure for the two subgraphs $H_A$ and $H_B$. By the choice of $f$ in (7), both subgraphs contain at most $m/2$ points. Hence, the recursion depth of our procedure is at most $\log_2 m = O(\log m)$. This implies that
\[ \text{td}(H) = O \left( \max \left\{ K \log m, t(\log m)^3 \right\} \right) = O \left( \max \left\{ \frac{m}{\log m}, t(\log m)^3 \right\} \right). \]
2.2. Separators and cells. During the rest of the section we will consider a tessellation of length \( \ell = r/4 \).

Given \( S \subseteq S_n \) a set of positive measure, we denote by \( \text{vol}(S) \) the area of \( S \) and by \( \partial S \) its boundary in the euclidean topology. We also use \( \text{vol}(\partial S) \) to refer to the length of \( \partial S \). We only consider sets \( S \) that are finite unions of discs, so that the length of the boundary is well defined.

For any set \( A \subseteq V(H) \), let \( \mathcal{A} = \{ x \in S_n : \text{min}_{v \in A} \text{dist}_E(x, v) \leq \frac{r}{2} \} \subseteq S_n \), and notice that \( \partial \mathcal{A} = \{ x \in S_n : \text{min}_{v \in A} \text{dist}_E(x, v) = \frac{r}{2} \} \).

We will use the fact that for any connected set of positive measure \( S \subseteq S_n \), we have

\[
\text{dist}_E(u, v) \leq \frac{r}{2\sqrt{2}}.
\]

Also, we make use of the following isoperimetric inequality (see [20, Theorem 1.6.1]):

\[
\text{vol}(\partial S) \geq \Omega \left( \sqrt{\text{vol}(S)} \right).
\]

This inequality can be extended to a nonconnected set \( S \) as follows: suppose that \( S \) is a union of disjoint connected sets \( S_1, \ldots, S_m \). Then, for each \( i = 1, \ldots, m \), we have \( \text{vol}(\partial S_i) = \Omega(\sqrt{\text{vol}(S_i)}) \), and thus

\[
\text{vol}(\partial S) = \sum_{i=1}^{m} \text{vol}(\partial S_i) = \sum_{i=1}^{m} \Omega \left( \sqrt{\text{vol}(S_i)} \right) = \Omega \left( \sqrt{\text{vol}(S)} \right),
\]

where the last inequality follows from concavity of the square root function, that is, for any \( x, y \geq 0 \), we have \( \sqrt{x} + \sqrt{y} \geq \sqrt{x + y} \).

Denote by \( \hat{S}_n \) the interior of \( S_n \). We have the following lemma.

**Lemma 2.7.** Let \( S \subseteq S_n \) be a measurable connected set. Then,

\[
\text{vol}\left( \partial S \cap \hat{S}_n \right) = \Omega(\text{min}\{\text{vol}(\partial S), \text{vol}(\partial(S_n \setminus S))\}).
\]

**Proof.** Consider the complement of \( S \), \( U = S_n \setminus S \). Let \( \mathcal{U}_1, \ldots, \mathcal{U}_m \) denote the disjoint connected sets of \( U \).

Let us focus on \( \mathcal{U}_i \) for some \( i \in [m] \). Let \( V_i = S_n \setminus \mathcal{U}_i \) denote its complement. We will show that \( \text{vol}(\partial \mathcal{U}_i \cap \hat{S}_n) = \Omega(\text{min}\{\text{vol}(\partial \mathcal{U}_i), \text{vol}(\partial V_i)\}) \). Since \( \mathcal{U}_i \) and \( V_i \) are connected sets that partition \( S_n \), if \( \partial \mathcal{U}_i \cap \hat{S}_n = \partial \mathcal{U}_i \), then we are done. Otherwise, there exist two points \( x \) and \( y \) in \( \partial \mathcal{U}_i \cap \partial S_n \) such that \( \partial \mathcal{U}_i = C_1 \cup C_2 \), where \( C_1 \) is a connected simple curve with endpoints \( x \) and \( y \), \( C_1 \subseteq \partial S_n \) and \( C_2 \cap \partial S_n = \{ x, y \} \). Let \( C_3 = \partial S_n \setminus C_1 \) and notice that \( \partial V_i = C_2 \cup C_3 \) and that \( C_1 \cup C_3 = \partial S_n \).

Let \( W_i = U_i \) if \( \text{vol}(C_1) \leq \text{vol}(C_2) \) and \( W_i = V_i \) otherwise. This implies that \( \text{vol}(C_2) \geq \|x - y\|_2 = \Omega(\text{min}\{\text{vol}(C_1), \text{vol}(C_3)\}) \). Using that \( \text{vol}(\partial W_i \cap \hat{S}_n) = \text{vol}(C_2) = \Omega(\text{min}\{\text{vol}(\partial U_i), \text{vol}(\partial V_i)\}) \).

Since each point in \( \partial S_n \) belongs to at most one set \( U_i \), there is at most one set \( U_i \) such that \( \text{vol}(\partial U_i) \geq \text{vol}(\partial V_i) \). If this is not the case, then we have \( \text{vol}(\partial S \cap \hat{S}_n) = \)
The following lemma shows that for any separator $S$ of a geometric graph $H$, we can find a large number of cells of length $\ell = r/4$, whose points are entirely contained in $S$ (see also Figure 4, left).

**Lemma 2.8.** Let $H$ be a connected geometric graph of order $m$ and $S \subset V(H)$ be a separator of $H$. Fix a connected component $H_1$ of $H \setminus S$ and denote by $A = V(H_1)$. Consider a tessellation with side length $\ell = r/4$. There exists a set of cells $D_S$ of size $d_S$, such that all points inside $D_S$ belong to $S$ and $d_S = \Omega(\frac{r}{4} \sqrt{\min\{\text{vol}(A), \text{vol}(S \setminus A)\}})$.

**Proof.** Define $B = V(H) \setminus (S \cup A)$, that is, $B$ is the set of vertices of $H$ that are contained neither in $S$ nor in $A$.

Observe that for any pair of points $v \in A$ and $w \in B$, we have $\text{dist}_E(v, w) \geq r$, since $v$ and $w$ belong to different connected components of $H \setminus S$. Let $\mathcal{C} = \partial A \cap \partial S_n$ denote the boundary of $A$. By definition, all points in $\mathcal{C}$ lie at distance exactly $r/2$ from some point in $A$. Thus, they lie at distance at least $r/2$ from any point in $B$.

Let $D_S$ be the union of cells that have nonempty intersection with $\mathcal{C}$. Let us point out that some of these cells may not contain any point of $V(H)$. Let us now show that $d_S = \Omega(\frac{r}{4} \sqrt{\min\{\text{vol}(A), \text{vol}(S_n \setminus A)\}})$. 

$$
\sum_{i=1}^{m} \text{vol}(\partial U_i \cap \hat{S}_n) = \sum_{i=1}^{m} \Omega(\text{vol}(\partial U_i)) = \Omega(\text{vol}(\partial U)).
$$

Otherwise,

$$
\text{vol}(\partial S \cap \hat{S}_n) = \text{vol}(\partial U \cap \hat{S}_n) = \sum_{i=1}^{m} \text{vol}(\partial U_i \cap \hat{S}_n) = \sum_{i=1}^{m} \Omega(\min\{\text{vol}(\partial U_i), \text{vol}(\partial V_i)\})
$$

$$
= \Omega \left( \text{vol}(\partial V_{i^*}) + \sum_{i \neq i^*} \text{vol}(\partial U_i) \right) = \Omega(\text{vol}(\partial S)),
$$

where the last equality follows from

$$
\text{vol}(\partial V_{i^*}) + \sum_{i \neq i^*} \text{vol}(\partial U_i) = \text{vol}(S) + \sum_{i \neq i^*} \text{vol}(\partial U_i \cap \partial S_n).
$$
By Lemma 2.7 and using the isoperimetric inequality in (10), we have that
\[
\text{vol}(C) = \text{vol}\left(\partial A \cap S_n\right)
= \Omega\left(\min\{\text{vol}(\partial A), \text{vol}(\partial(S_n \setminus A))\}\right)
= \Omega\left(\sqrt{\min\{\text{vol}(A), \text{vol}(S_n \setminus A)\}}\right).
\]

For any cell \(D \in D_S\) we denote by \(C_D = C \cap D\) the restriction of \(C\) to \(D\). We will show that the length of \(C_D\) is not too large by projecting the elements of \(C_D\) onto \(\partial D\), in such a way that the length of \(C_D\) does not decrease by too much.

Let \(p : C_D \to \partial D\) the application that sends an element \(c \in C_D \subset C\) being at distance \(r/2\) from a point \(v \in A\) to the intersection of \(\partial D\) and the segment that joins \(c\) and \(v\) (see Figure 4, right). In the case where there is more than one point of \(A\) at the same distance from \(c\), \(p(c)\) chooses one of them arbitrarily.

Note that \(p\) is injective, since no two elements of \(C_D\) can have the same image: indeed, suppose that there exist two different \(c, c' \in C_D\) with corresponding points \(v, v' \in A\) such that \(p(c) = p(c')\). Then, the segments \(cv\) and \(c'v'\) would intersect at \(p(c)\), and either \(\text{dist}_{E}(c, v') < r/2\) or \(\text{dist}_{E}(c', v) < r/2\) holds, contradicting the definition of \(C\).

Let us show that the application does not contract \(C_D\) too much. Recall that \(\text{dist}_{E}(c, v) = r/2\). Since \(c, p(c) \in D\), by (9) we have \(\text{dist}_{E}(c, p(c)) \leq \frac{r}{2\sqrt{2}}\), and therefore \(\text{dist}_{E}(p(c), v) \geq \frac{\sqrt{2} - 1}{2\sqrt{2}}r\) by the triangle inequality.

A simple geometric argument shows that
\[
\text{vol}(p(S)) \geq \frac{\pi}{2\sqrt{2}} \text{vol}(S).
\]

Since \(p\) is injective and \(\text{vol}(\partial D) = 4\ell = r\),
\[
\text{vol}(C_D) = O(\text{vol}(\partial D)) = O(r).
\]

Using this upper bound for all cells \(D \in D_S\), we obtain
\[
d_S \geq \frac{\text{vol}(C)}{\max_{D \in D_S} \text{vol}(C_D)} = \Omega\left(\frac{1}{r^{1/2}}\right).
\]

Moreover, all points contained in \(D_S\) belong to \(S\): by (9), any point \(u\) contained in \(D_S\) lies at distance at most \(r/(2\sqrt{2})\) from some element \(c \in C\). However, all points of \(A \cup B\) lie at distance at least \(r/2\) from all the elements of \(C\). Thus, \(u \notin A \cup B\), implying that \(u \notin S\).

We finish with some properties of the tessellation when choosing \(\ell = r/4\).

**Lemma 2.9.** Let \(H\) be a geometric graph with connected components \(H_1, \ldots, H_t\). Define \(A_i = V(H_i)\) and consider a tessellation with \(\ell = r/4\). Then, for any cell \(D\) we have the following:

1. If there exists a point \(v \in A_i\) such that \(v \in D\), then \(D \subset A_i\).
2. There are at most 24 curves \(C_i = \partial A_i\) that intersect the cell.

**Proof.** For the first part, by (9), for any \(u \in D\),
\[
\text{dist}_{E}(u, v) < \frac{r}{2},
\]
and thus \(u \in A_i\).
For the second part, observe that if $C_i$ intersects $D$, then there must exist a point of $v \in A_i$ at distance at most $r/2$ from some point in $D$. There are at most 24 cells satisfying this criterion, namely, the ones in the first and second neighborhood of $D$. Since all points of a cell belong to the same component (they are all connected), there are at most 24 different curves $C_i$ intersecting $D$.

Combining Lemmas 2.8 and 2.9 we obtain the following corollary.

**Corollary 2.10.** Let $H$ be a connected geometric graph of order $m$ and $S \subset V(H)$ be a separator of $H$. Let $H_1, H_2, \ldots$ denote the connected components of $H \setminus S$ and let $I$ index a collection of them. Let $A = V(\bigcup_{i \in I} H_i)$. Consider a tessellation with side length $\ell = r/4$. There exists a set of cells $D_S$ of size $d_S$, such that all points inside $D_S$ belong to $S$ and

$$d_S = \Omega \left( r^{-1} \sqrt{\min\{\text{vol}(A), \text{vol}(S_n \setminus A)\}} \right).$$

**Proof.** For every $i \in I$, we let $A_i = V(H_i)$. By Lemma 2.8, there exists a set of cells $D_{S_{A_i}}$ containing only points in $S$ of size

$$d_{S_{A_i}} = \Omega \left( r^{-1} \sqrt{\min\{\text{vol}(A_i), \text{vol}(S_n \setminus A_i)\}} \right).$$

Now, by Lemma 2.9 part 2, for every cell $D$, there are at most 24 connected components $H_j$ of $H \setminus S$, such that $\partial A_j$ intersects $D$. Hence, if $D_S = \bigcup_{i \in I} D_{S_{A_i}}$, then $D_S$ has size

$$d_S \geq \frac{1}{24} \sum_{i \in I} d_{S_{A_i}} = \sum_{i \in I} \Omega \left( r^{-1} \sqrt{\min\{\text{vol}(A_i), \text{vol}(S_n \setminus A_i)\}} \right) = \Omega \left( r^{-1} \sqrt{\min\{\text{vol}(A), \text{vol}(S_n \setminus A)\}} \right),$$

where the last equality follows from $A = \bigcup_{i \in I} A_i$ and from the concavity of the square root function (see also (11)).

### 3. Subcritical regime

In this section we compute the treedepth of a random geometric graph with $0 < r < r_c$, that is, below the existence of a giant component. By [22, Theorem 10.3], a.a.s. the order of each component is at most $O(\log n)$. In fact, by looking at [22, Theorem 10.3], it is easily seen that with probability at least $1 - o(n^{-3/2})$ the order of each component is $O(\log n)$.

We will use the following result several times: McDiarmid in [16] proved that for any $r = \Theta(1)$ and $G \in \mathcal{G}(n, r)$, a.a.s.

$$\omega(G) = \Theta \left( \frac{\log n}{\log \log n} \right).$$

In fact, by looking at the proof of [16, Lemma 5.3], by choosing (in the notation of the proof given there) $k_1 = k_1(r)$ to be sufficiently large and $k_2 = k_2(r)$ to be sufficiently small, we can also easily see that with probability at least $1 - o(n^{-1/2})$ we have

$$\omega(G) = \Theta \left( \frac{\log n}{\log \log n} \right),$$
and by looking at [16, Lemmas 4.4 and 5.3], the same result holds for $G(P_1, r)$ as well. (In fact, for Lemma 5.3, either the number of points of $G(P_1, r)$ is not in the set \( \{ n - C \sqrt{n \log n}, n + C \sqrt{n \log n} \} \) for $C$ large enough, which happens with probability $o(n^{-1/2})$, or the respective lower and upper bounds for the number of points can be used in the calculations of Lemma 5.3, again by choosing $k_1$ large enough and $k_2$ small enough.)

By (2), the order of the largest connected component implies a coarse upper bound, namely,

\[
\text{td}(G) = O(\log n) .
\]

In order to find a better upper bound, more work is needed. First, we need the following simple lemma, whose proof is included for completeness.

**Lemma 3.1.** Let $X$ be a random variable that follows a Poisson distribution with parameter $\lambda$. Then, for any $k \geq 2\lambda$,

\[
\Pr(X \geq k) \leq 2 \Pr(X = k).
\]

**Proof.** We have

\[
\Pr(X \geq k) = \sum_{i \geq k} \Pr(X = i) = \sum_{i \geq k} e^{-\lambda} \frac{\lambda^i}{i!}
\]

\[
= e^{-\lambda} \frac{\lambda^k}{k!} \left( 1 + \frac{\lambda}{k+1} + \frac{\lambda^2}{(k+1)(k+2)} + \cdots \right)
\]

\[
\leq e^{-\lambda} \frac{\lambda^k}{k!} \sum_{i \geq 0} \left( \frac{\lambda}{k} \right)^i = e^{-\lambda} \frac{\lambda^k}{k!} \cdot \frac{1}{1 - \lambda/k}
\]

\[
\leq 2e^{-\lambda} \frac{\lambda^k}{k!} = 2 \Pr(X = k),
\]

where the last inequality follows from the assumption $k \geq 2\lambda$. \(\square\)

Let $\nu = \nu(r)$ be a sufficiently large constant. For the sake of convenience, we define

\[
T_{\text{max}} = \frac{\nu \log n}{\log \log n} \quad \text{and} \quad T = \frac{\sqrt{2} \log n}{\log \log n}.
\]

From now on, we consider in this section the cell graph $C_G(\ell)$ of $G \in G(P_1, r)$ with $\ell = r/\sqrt{2}$ and write simply $C_G$ for $C_G(\ell)$. Notice that all points inside a cell of $C_G$ form a clique. Hence, by (14), by choosing $\nu = \nu(r)$ sufficiently large, each cell contains less than $T_{\text{max}}$ points a.a.s. For this particular tessellation, we call a cell **sparse** if it contains less than $T$ points, and **dense** otherwise.

**Proposition 3.2.** Let $0 < r < r_c$ and let $G \in G(P_1, r)$. With probability at least $1 - o(n^{-1/2})$, every connected component $H$ of $G$ contains at most $O(T_{\text{max}})$ points in dense cells.

**Proof.** For any connected component $H$ of $G$ we will show that the probability that the number of points in dense cells of $H$ is at least $2T_{\text{max}}$ is $o(n^{-3/2})$. Since there are clearly at most $n$ connected components in $G$, by taking a union bound over all them, with probability $1 - o(n^{-1/2})$ no component will have more than $2T_{\text{max}}$ points in dense cells.

Let $A_i$ be the number of points in the cell $i$. Since we are using a Poisson point process of intensity 1, $A_i$ follows a Poisson distribution with parameter $\lambda = r^2/2$. Denote by $p = \Pr(A_i \geq T)$ the probability that cell $A_i$ is dense.
By Lemma 3.1,

\[
(15) \quad p \geq \Pr(A_i = T) = \left(1 - O\left(T^{-1}\right)\right) \frac{e^{-\lambda}}{\sqrt{2\pi T}} \left(\frac{e\lambda}{T}\right)^T,
\]

\[
(16) \quad p = \Pr(A_i \geq T) \leq 2 \Pr(A_i = T) \leq 2 \frac{e^{-\lambda}}{\sqrt{2\pi T}} \left(\frac{e\lambda}{T}\right)^T,
\]

where we have used Stirling’s formula \( T! = (1 + O(T^{-1}))\sqrt{2\pi T} = \left(\frac{T}{e}\right)^T \).

To count the number of points lying in dense cells, we define the following random variable for each cell \( i \in V(C_G) \):

\[
Y_i = \begin{cases} 
  t & \text{if } i \text{ is dense and has } t \text{ points inside,} \\
  0 & \text{otherwise.} 
\end{cases}
\]

Our aim is to show that \( Y_H = \sum_{i \in V(C_H)} Y_i \) is at most \( O(T_{\max}) \).

Notice that the probability that the cell \( i \) is sparse is \( 1 - p \), while the probability of having \( T + j \) points is

\[
\Pr(A_i = T + j) = \left(1 - O\left((T + j)^{-1}\right)\right) \frac{e^{-\lambda}}{\sqrt{2\pi(T + j)}} \left(\frac{e\lambda}{T + j}\right)^{T+j} \leq \left(\frac{e\lambda}{T}\right)^T \frac{e^{-\lambda}}{\sqrt{2\pi T}} \left(\frac{e\lambda}{T}\right)^j
\]

for any integer \( j \geq 0 \). Using (16) we have

\[
\Pr(A_i = T + j) \leq 2p \left(\frac{e\lambda}{T}\right)^j.
\]

These observations lead to the definition of the following independent random variable \( R_i \) for each cell \( i \in V(C_G) \):

\[
R_i = \begin{cases} 
  0 & \text{with probability } 1 - 2p, \\
  T + j & \text{with probability } 2p \left(\frac{e\lambda}{T}\right)^j \text{ for any } j \geq 1, \\
  T & \text{with probability } 2p \left(1 - \frac{e\lambda}{T - e\lambda}\right).
\end{cases}
\]

First, observe that \( R_i \) is a probability distribution. The random variables \( Y_i \) and \( R_i \) have similar distributions. In particular, each variable \( R_i \) stochastically dominates the corresponding random variable \( Y_i \). Analogously, we define \( R = \sum_{i \in V(C_H)} R_i \).

Then,

\[
(17) \quad \Pr(R \geq j) \geq \Pr(Y \geq j)
\]

for any \( j \geq 0 \). In particular, this also hold if \( j = O(T_{\max}) \).

Therefore, it is enough to compute an upper bound for \( \Pr(R > 2T_{\max}) \). Clearly, since \( r < r_c \), and all connected components are of order \( O(\log n) \) with probability at least \( 1 - o(n^{-3/2}) \), with the same probability in the cell graph \( C_G \) the graph diameter of each component \( C_H \) is at most \( K \log n \) for some sufficiently large constant \( K = K(r) \).

For the case where the graph diameter is bigger than \( K \log n \), \( \Pr(R > 2T_{\max}) \) can be easily bounded by \( o(n^{-3/2}) \). For the case where it is smaller than \( K \log n \), we observe the following: given a cell from \( C_H \), all points that belong to \( H \) are contained in the
box of cells of size $(2K \log n + 1) \times (2K \log n + 1)$ centered on the first cell. Let $\eta > 0$ such that $(2K \log n + 1)^2 \leq \eta \log^2 n$.

Hence we have

\begin{equation}
\Pr(R > 2T_{\text{max}}) \leq o \left(n^{-3/2}\right) + \sum_{m=1}^{2(2K \log n + 1)^2} \sum_{S \in \binom{\eta \log^2 n}{m}} \sum_{c_i \in S} \Pr\left(\bigcap_{i \in S} R_i = c_i\right),
\end{equation}

where $m$ counts the number of dense cells in the distribution given by the $R_i$, $S$ is the set of dense cells, and $c_i$ is the number of points inside the dense cell $i \in S$. There are at most $\eta^m (\log n)^{2m}$ ways to choose the set $S$ of size $m$ and at most $(T_{\text{max}})^m < (\log n)^m$ possible values for the $c_i$.

Recall that the variables $R_i$ are independent and that $\Pr(R_i = T + j) = 2p(\lambda_p/T)^j$ for any $j \geq 1$. Therefore,

\[ \Pr\left(\bigcap_{i \in S} R_i = c_i\right) = \prod_{i=1}^m 2p \left(\frac{e\lambda}{T}\right)^{c_i - T}. \]

On the one hand, if $m \leq 2\sqrt{\log n}$, using (15),

\[ \prod_{i=1}^m 2p \left(\frac{e\lambda}{T}\right)^{c_i - T} \leq \prod_{i=1}^m \frac{4}{\sqrt{2\pi T}} \left(\frac{e\lambda}{T}\right)^{c_i} \leq \prod_{i=1}^m \left(\frac{e\lambda}{T}\right)^{c_i} \leq \left(2e\sqrt{2\pi T} p\right)^{\sum_{j \geq 0} c_j} \leq \left(2e\sqrt{2\pi T} p\right)^{2\sqrt{\log n}}. \]

On the other hand, if $m = 2\sqrt{\log n} + j$ for some integer $j \geq 0$,

\[ \prod_{i=1}^m 2p \left(\frac{e\lambda}{T}\right)^{c_i - T} \leq (2p)^m = (2p)^{2\sqrt{\log n}} (2p)^j. \]

Therefore, by splitting the second part of (18) into two sums, we obtain

\[ \Pr(R > 2T_{\text{max}}) \leq o \left(n^{-3/2}\right) + \sum_{m=1}^{2\sqrt{\log n}} \eta^m (\log n)^{3m} \left(2e\sqrt{2\pi T} p\right)^{2\sqrt{\log n}} + (2\eta (\log n)^3 p)^{2\sqrt{\log n}} \sum_{j \geq 1} (2\eta (\log n)^3 p)^j. \]

From the bounds on $p$ in (15) and (16), one can derive that $\eta (\log n)^3 p < 1/2$, and the infinite sum of the second term above is bounded from above by one. Thus,

\[ \Pr(R > 2T_{\text{max}}) \leq o \left(n^{-3/2}\right) + \left(2\sqrt{\log n} \left(\frac{\eta (\log n)^3 p \left(2e\sqrt{2\pi T} + 2\right)}{2\sqrt{\log n}}\right)^{2\sqrt{\log n}} \right) \leq o(n^{-3/2}) + \exp \left\{ \log \log n/2 + 2\sqrt{\log n} (3 \log \log n + \log p + O(\log T)) \right\}. \]
Moreover, by (16), we also have $p \leq \frac{2e^{-\lambda}}{\sqrt{2\pi T}} \left( \frac{e \lambda}{T} \right)^T$, and hence $\log p \leq -(1 + o(1))T \log T \leq -\sqrt{\log n}$. Thus,

\begin{equation}
\Pr(R > 2T_{\text{max}}) < o\left(n^{-3/2}\right) + \exp\{-(1 + o(1))2\log n\} = o\left(n^{-3/2}\right).
\end{equation}

By (17), this also implies that $\Pr(Y > 2T_{\text{max}}) = o(n^{-3/2})$, and by taking a union bound over all components, this implies that the probability of having a connected component with more than $2T_{\text{max}}$ points inside dense cells is $o(n^{-1/2})$.

\textbf{Proof of Theorem 1.1.} The lower bound on $\text{tw}(G)$ follows easily from (14), which yields

$$\text{td}(G) \geq \text{tw}(G) \geq \omega(G) - 1 = \Omega\left(\frac{\log n}{\log \log n}\right).$$

For the upper bound, we construct an elimination tree for $G$. By (2) it suffices to bound from above the treedepth of each connected component. Let $H$ be a connected component of $G$.

From Proposition 3.2, there are at most $O(T_{\text{max}})$ points in dense cells of $H$. We temporarily remove all these points and add them at the end. Let $H'$ be the subgraph of $H$ that remains after removing the points in the dense cells.

Observe that now, by definition of sparse, every cell of $C_{H'}$ contains at most $T$ points. Denoting by $m = |V(H')|$, by Proposition 2.6 we have

$$\text{td}(H') = O\left(\max\left\{\frac{m}{\log m}, T(\log m)^3\right\}\right).$$

Since, with probability at least $1 - o(n^{-3/2})$, $m = O(\log n)$, we have that for every component $H$ of $G$, $\text{td}(H') = O(T_{\text{max}})$ with probability at least $1 - o(n^{-1/2})$.

Recall that adding a new point to $H$ can increase the treedepth by at most one unit. Thus, $\text{td}(H) \leq \text{td}(H') + O(T_{\text{max}}) = O(T_{\text{max}})$, and therefore, using (1), we have

$$\text{td}(G) = O\left(\frac{\log n}{\log \log n}\right)$$

with probability at least $1 - o(n^{-1/2})$. \hfill \Box

\section{Supercritical regime.}

Fix now $r = r(n) \geq c$ for some sufficiently large constant $c$. Recall that for any subset $S \subseteq S_n = [0, \sqrt{n}]^2$ of positive measure, we denote by $\text{vol}(S)$ the area of $S$. We need the following standard lemma (which is a simple application of Chernoff bounds for Poisson variables, see, for example, [1, Theorem A.1.15]).

\textbf{Lemma 4.1.} For any $S \subseteq S_n$ and any $\delta > 0$, let $|S|$ denote the number of points inside $S$. Then, we have the following:

1. With probability at least $1 - (e^{\delta}(1 + \delta)^{-(1+\delta)})^{\text{vol}(S)} \geq 1 - e^{-\frac{\delta^2}{2}}\text{vol}(S)$, $|S| \leq (1 + \delta)\text{vol}(S)$.

2. With probability at least $1 - e^{-\frac{\delta^2}{2}}\text{vol}(S)$, $|S| \geq (1 - \delta)\text{vol}(S)$.

We will use this lemma to show that there exist separating sets with few points, and consequently, give an upper bound on $\text{td}(G)$.

\textbf{Proposition 4.2.} Let $c$ be a sufficiently large constant, let $r = r(n) \geq c$, and let $G \in \mathcal{G}(P_1, r)$. With probability $1 - e^{-\Omega(r\sqrt{n})}$, $\text{td}(G) \leq O(r\sqrt{n})$. 

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Proof. Consider the tessellation of $S_n$ into square cells of side length $\ell = r$. Denote by $D(i,j)$ the $j$th cell in the $i$th row, where $1 \leq i, j \leq a = \sqrt{n}/r$.

Define $X_1^j = \left( \bigcup_{i=1}^{a} D(a/2,i) \right) \cup \left( \bigcup_{i=1}^{a} D(i,a/2) \right),$ 

and consider the set $Y_1^j \subset V(G)$, containing the points inside $X_1^j$. Observe that $Y_1^j$ is a separator, since $\ell = r$, and it splits the graph into four subgraphs $G_1^j$, $G_2^j$, $G_3^j$, and $G_4^j$, each one consisting of a (possibly empty) union of vertex-disjoint connected components.

By (3), we have $\text{td}(G) \leq |Y_1^j| + \max_{1 \leq j \leq 4} \{ \text{td}(G_j^j) \}$.

We then define analogously the sets $X_2^j$, for all $G_2^j$, and using (3), we continue iteratively. Let $t$ denote the step where all sets $X_t^j$ have size one (see Figure 5).

The treedepth of $G$ will be bounded from above by the maximum number of points inside any of the possible sets of cells

$X_{j_1j_2 \ldots j_t} = X_1^{j_1} \cup X_2^{j_2} \cup \ldots \cup X_t^{j_t},$

where $1 \leq j_i \leq 4^{i-1}$.

Observe that $|X_t^{j_t}| \leq a2^{-(i-2)}$. The sets $X_{j_1j_2 \ldots j_t} = X_1^{j_1} \cup X_2^{j_2} \cup \ldots \cup X_t^{j_t}$ are not disjoint, but they all have the same size

$|X_{j_1j_2 \ldots j_t}| = \sum_{i=1}^{t} |X_i^{j_i}| \leq \sum_{i=1}^{t} a2^{-(i-2)} \leq 4a.$

Let $Y_{j_1j_2 \ldots j_t}$ denote the set of points in $X_{j_1j_2 \ldots j_t}$. Thus, $|Y_{j_1j_2 \ldots j_t}|$ is a random variable following a Poisson distribution with mean at most $4ar^2$.

By part 1 of Lemma 4.1 applied with $\delta = 1$,

$\Pr(|Y_{j_1j_2 \ldots j_t}| \geq 8ar^2) < e^{-4ar^2/3} = e^{-\Omega(r\sqrt{n})}.$
Moreover, there are at most
\[ \prod_{i=1}^{t} \frac{4^{i-1}}{i} = e^{O(t^2)} \]
sets of the form \( X_{j_1,j_2,...,j_t} \). Observe also that, by construction, \( t = O(\log a) = O(\log n) \).

Now, by a union bound over all sets,
\[ \Pr \left( \exists j_1, j_2, \ldots, j_k : |Y_{j_1,j_2,...,j_k}| > 8ar^2 \right) \leq e^{O(\log^2 n) - \Omega(r\sqrt{n})} = e^{-\Omega(r\sqrt{n})}. \]

Thus, we have that the treedepth of \( G \) is at most
\[ \text{td}(G) \leq 8ar^2 = O(r\sqrt{n}) \]
with probability at least \( 1 - e^{-\Omega(r\sqrt{n})} \), finishing the proof.

For a lower bound on \( \text{tw}(G) \), we need the following link between the treewidth of a graph and the existence of a vertex separator with special properties. A vertex partition \( V = (A, S, B) \) is a balanced \( k \)-partition if \( |S| = k + 1 \), \( S \) separates \( A \) and \( B \), and \( \frac{1}{4} (n - k - 1) \leq |A|, |B| \leq \frac{3}{4} (n - k - 1) \). In this case, \( S \) is also called a balanced separator.

The following result connecting balanced partitions and treewidth is due to Kloks [13].

**Lemma 4.3** (see [13]). Let \( G \) be a graph on \( n \) vertices, and suppose that \( \text{tw}(G) \leq k \) for some \( n \geq k - 4 \). Then \( G \) has a balanced \( k \)-partition.

From now on and until the end of the section, we consider the tessellation of \( S_n \) into square cells of side length \( \ell = r/4 \).

Recall that for any set \( A \subset V(H) \), we define \( \mathcal{A} = \{ x \in S_n : \min_{v \in A} \text{dist}_E(x, v) \leq r/2 \} \). Observe that in a geometric graph, no direct relation exists between the size of \( A \) and the volume of \( \mathcal{A} \). In the case of a random geometric graph and for a set \( A \) of linear size, however, \( \text{vol}(\mathcal{A}) \) can be bounded from below using the size of \( A \), as the following lemma shows.

**Lemma 4.4.** Let \( c \) be a sufficiently large constant and let \( r = r(n) \geq c \). Let \( G \in \mathcal{G}(P_1, r) \) and let \( \alpha \in (0, 1) \). Then there exists \( \beta = \beta(\alpha) > 0 \), such that with probability \( 1 - e^{-\Omega(n)} \), for any set \( A \subseteq V(G) \) with \( |A| \geq \alpha n \), we have
\[ \text{vol}(\mathcal{A}) \geq \beta n. \]

**Proof.** Let \( \lambda = r^2/16 \). Set \( m = m(\alpha) \) to be the smallest constant such that \( m\lambda \) is integer,
\[ \frac{e^{-1}}{m!} \left( \frac{m^2}{m - 1} + \frac{m}{(m - 1)^2} \right) \leq \frac{\alpha}{4} \quad \text{and} \quad m \geq 4e, \]
which exists for any \( \alpha > 0 \), since the left-hand side of the first condition tends to zero, when \( m \to +\infty \).

Recall that the number of points inside a cell \( D \) follows a Poisson distribution with mean \( \lambda \). Suppose that \( D \) contains \( t \geq 0 \) points. Define then \( Z_D \) to be the random variable
\[ Z_D = \begin{cases} t & \text{if } t \geq m\lambda, \\ 0 & \text{otherwise}, \end{cases} \]
and let \( Z = \sum Z_D \) be the sum of these random variables over all cells of the tessellation.
We may consider \( r \geq \sqrt{32} \), since by hypothesis \( r \geq c \) for some \( c \) large enough. This implies that \( \lambda \geq 2 \). Using that for every \( N \geq 1 \), \( \sqrt{2\pi N^2} e^{-N} \leq N! \leq e\sqrt{2\pi N^2} + \frac{1}{2} e^{-N} \), we obtain

\[
\Pr(Z_D = m\lambda) = e^{-\lambda} \frac{\lambda^{m\lambda}}{(m\lambda)!} \\
\leq e^{-\lambda} \frac{1}{\sqrt{2\pi}} \left( \frac{e}{m} \right)^{m\lambda} \frac{1}{\sqrt{m\lambda}} \\
\leq e^{-2} \frac{1}{\sqrt{2\pi}} \left( \frac{e}{\lfloor m \rfloor} \right)^{\lfloor m \rfloor \lambda} \frac{1}{\sqrt{\lfloor m \rfloor \lambda}} \\
\leq e^{-1} \frac{1}{\lfloor m \rfloor!}.
\]

Also, for any \( i \geq 1 \),

\[
\Pr(Z_D = m\lambda + i) = e^{-\lambda} \frac{\lambda^{m\lambda+i}}{(m\lambda+i)!} \\
= e^{-\lambda} \frac{\lambda^{m\lambda+(i-1)}}{(m\lambda+(i-1))!} \cdot \frac{\lambda}{m\lambda + i} \\
\leq \frac{1}{m} \Pr(Z_D = m\lambda + (i-1)).
\]

Hence,

\[
E(Z_D) = \sum_{t \geq m\lambda} t \Pr(Z_D = t) \\
\leq e^{-1} \frac{1}{\lfloor m \rfloor!} \sum_{i \geq 0} (m\lambda + i) m^{-i} \\
\leq e^{-1} \frac{1}{\lfloor m \rfloor!} \left( \frac{m^2\lambda}{m-1} + \frac{m}{(m-1)^2} \right) \\
\leq \frac{\alpha\lambda}{4},
\]

where the last inequality follows from the definition of \( m \). Since \( \lambda = r^2/16 \) and there are \( 16n/r^2 \) cells in the tessellation, we have

\[
E(Z) \leq \frac{\alpha n}{4}.
\]

By Hoeffding bounds for unbounded random variables (the precise version we use here is [4, Theorem 1]), applied with \( X_D = \epsilon_D = Z_D \), and thus \( S = T = Z \), \( Y = Po(\lambda) \), \( m_b = m = E(Z_D) \) for any \( k \), and \( b = m\lambda - 1 \), so that \( m(b) = m \) and the measure \( \mu[m] \) is exactly our probability distribution of \( Z_D \), and \( x = 2E(Z) \)

\[
\Pr(Z > 2E(Z)) < \inf_{h \leq x} e^{-h2E(Z)} E(e^{hZ}) \leq e^{-2E(Z)} E(e^Z).
\]

Now, observe that

\[
e^{2E(Z_D)} \geq e^{2m\lambda} \Pr(Z_D = m\lambda) \geq e^{(2m-1)\lambda} \frac{\lambda^{m\lambda}}{(m\lambda)!}
\]
and

\[ \mathbb{E}(e^{Z_D}) = \Pr(Z_D = 0) + \sum_{i \geq 0} e^{m\lambda + i} \Pr(Z_D = m\lambda + i) \leq 1 + e^{(m-1)\lambda} \frac{\lambda m^{\lambda}}{(m\lambda)!} \sum_{i \geq 0} \left( \frac{e}{m} \right)^i. \]

Since by assumption on \( m, \frac{e}{m} \leq \frac{1}{4} \), we have

\[ \mathbb{E}(e^{Z_D}) \leq 1 + \frac{4}{3} \frac{\lambda m^{\lambda}}{(m\lambda)!} e^{(m-1)\lambda} \leq 3 \frac{\lambda m^{\lambda}}{(m\lambda)!} e^{(m-1)\lambda}. \]

The random variables \( Z_D \) are mutually independent. Thus,

\[ e^{2\mathbb{E}(Z)} = \prod e^{2\mathbb{E}(Z_D)} \geq \left( \frac{\lambda m^{\lambda}}{(m\lambda)!} e^{(2m-1)\lambda} \right)^{16n} \]

and

\[ \mathbb{E}(e^{Z}) \leq \left( \frac{3}{2} \frac{\lambda m^{\lambda}}{(m\lambda)!} e^{(m-1)\lambda} \right)^{16n}, \]

and therefore

\[ \Pr(Z > 2\mathbb{E}(Z)) \leq e^{-2\mathbb{E}(Z)} \mathbb{E}(e^{Z}) \leq \left( \frac{3}{2} e^{-m\lambda} \right)^{16n} = e^{-\Omega(n)}. \]

Thus, with probability at least \( 1 - e^{-\Omega(n)} \), there are at most \( \alpha n/2 \) points of \( G \) contained in cells with at least \( m\lambda \) points, and thus with the same probability there are at least \( \alpha n/2 \) points of \( A \) contained in cells with less than \( m\lambda \) points.

Therefore, with this probability, there are at least

\[ \frac{\alpha n}{2m\lambda} = \frac{8\alpha n}{mr^2} \]

different cells \( D \) that contain at least one point from \( A \). By part 1 of Lemma 2.9, \( D \subset A \), and

\[ \text{vol}(A) \geq \frac{8\alpha n}{mr^2} \cdot \text{vol}(D) = \beta n \]

with probability at least \( 1 - e^{-\Omega(n)} \).

Using the previous lemmata, we are able to provide a lower bound for \( \text{tw}(G) \).

**Theorem 4.5.** Let \( c \) be a sufficiently large constant, and let \( r = r(n) \geq c \). Let also \( G \in \mathcal{G}(P_1, r) \). Then, \( \text{tw}(G) = \Omega(r \sqrt{n}) \) with probability at least \( 1 - e^{-\Omega(r\sqrt{n})} \).

Before proving the theorem we sketch its proof. We are going to show that any balanced separator \( S \) of the giant component contains many points. Observe that if \( \text{vol}(S) \) is large, then the probability of containing few points is exponentially small. We show that in general, any such separator has a large volume. Here we strongly use the condition that \( S \) is balanced. The conclusion will then follow by taking a union bound over all potential separators.

**Proof.** Fix \( \gamma > 0 \) to be a sufficiently small constant. Let \( H \) be the largest component of \( G \). Note that for \( r \geq c \) with \( c \) sufficiently large, by [21, Theorem 3.3],

\[ |V(H)| = \Omega(n) \]
with probability at least $1 - e^{-\Omega(n)}$. We will for now assume deterministically that $|V(H)| = \Omega(n)$ holds and only in the end add the probability $e^{-\Omega(n)}$ that $|V(H)| = o(n)$ holds. By choosing $c$ sufficiently large, to simplify calculations, we may even assume $|V(H)| \geq 0.9n$. We will show that there exists no balanced separator of size $\gamma r \sqrt{n}$ for $H$. Then, by Lemma 4.3, this implies that $\text{tw}(H) \geq \gamma r \sqrt{n} = \Omega(r \sqrt{n})$, and by (1), $\text{tw}(G) \geq \text{tw}(H) = \Omega(r \sqrt{n})$.

For any balanced separator $S \subset V(H)$ of $H$, denote by $t$ be the number of connected components of the graph induced by $S$ and let $S_1, \ldots, S_t$ denote the subsets inducing connected components within $H$. We may assume that $S$ is minimal, and hence each component of $S$ contains at least one point of $H$. Therefore we can assume that $t \leq \gamma r \sqrt{n}$, as otherwise there is nothing to prove. We may assume that $r \leq 2 \sqrt{n}$, since for $r = 2 \sqrt{n}$, $G(P_1, r)$ is already the complete graph. If $S$ is a balanced separator of size at most $\gamma r \sqrt{n} \leq 2 \gamma n$, there exist two not necessarily connected sets $A, B \subset V(H)$ of size $\Omega(\frac{1 - \gamma}{\sqrt{\gamma}})|V(H)| \leq |A|, |B| \leq \frac{2(1 - \gamma^2)}{\gamma^3} |V(H)|$, such that $H \setminus S$ contains no edges from $A$ to $B$.

Since $\gamma$ is a sufficiently small constant and $|V(H)| \geq 0.9n$, $|A|, |B| \geq n/4$. By Lemma 4.4, with probability at least $1 - e^{-\Omega(n)}$, for all balanced separators $S$, $\text{vol}(A)$ and $\text{vol}(B)$ are linear in $n$. In particular, if $\beta = \beta(1/4)$ is the constant provided by Lemma 4.4 for $\alpha = 1/4$, we have

$$
\beta n \leq \text{vol}(A) \leq (1 - \beta)n
$$

with probability at least $1 - e^{-\Omega(n)}$.

Since $\text{vol}(A), \text{vol}(S_n \setminus A) \geq \beta n$, by Corollary 2.10, there exists $\eta = \eta(\beta) > 0$ such that with probability at least $1 - e^{-\Omega(n)}$, for each balanced separator $S$ there is a set of cells $D_S$ of size

$$
d_s = \Omega\left(\frac{r^{-1} \sqrt{\text{vol}(A)}}{n}\right) \geq \frac{\eta \sqrt{n}}{r},
$$

such that all points inside $D_S$ belong to $S$. Recall that some cells in $D_S$ may not contain any point. We will assume this deterministically for now and add the failure probability at the very end.

Now it suffices to show that with high probability, for any balanced separator $S$, with $D_S$ denoting the set of cells provided by Corollary 2.10 of size at least $\eta \sqrt{n}/r$, there are at least $\gamma r \sqrt{n}$ points contained in $D_S$. Denote by $Y_{D_S}$ the random variable counting the number of points inside such a set $D_S$. Since $\text{vol}(D_S) = \frac{r^2}{16} d_s$, by part 2 of Lemma 4.1 applied with $\delta = 1/2$, we obtain

$$
\Pr\left(Y_{D_S} < \frac{r^2}{32} d_s\right) \leq e^{-\frac{r^2}{16} ds}.
$$

We will choose $\gamma$ small enough such that $\gamma \leq \eta/32$. We will show that with high probability every balanced separator that occupies at least $\eta \sqrt{n}/r$ cells contains at least $\frac{r^2}{32} d_s \geq \gamma r \sqrt{n}$ points. We will do it by combining the inequality in (22) with a union bound over all separators $S$ together with the corresponding sets of cells $D_S$ of size $d_S \geq \eta \sqrt{n}/r$.

By definition of the cell graph, $D_S$ has at most $t$ connected components. (Some connected components of the graph induced by $S$ can merge in $D_S$.) We will assume that $D_S$ has exactly $t$ connected components denoted by $D_{S_1}, \ldots, D_{S_t}$ and with sizes $d_{S_1}, \ldots, d_{S_t}$.

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We have

\begin{equation}
\Pr(\exists S \text{ balanced, } d_S \geq \eta \sqrt{n}/r, |S| \leq \gamma r \sqrt{n})
\leq \sum_{d \geq \eta \sqrt{n}/r} \sum_{t \leq \gamma r \sqrt{n}} \sum_{d_{S_1} + \cdots + d_{S_t} = d}
\Pr(\exists S \text{ balanced with } t \text{ components, } d_S = d, |S| \leq \gamma r \sqrt{n}).
\end{equation}

Denote by $C_H$, by $C_A$, and by $C_B$ the set of cells that contain at least one point of $H$, $A$, and $B$, respectively. Recall that $\beta n \leq \text{vol}(A), \text{vol}(B) \leq (1 - \beta)n$. By Lemma 4.4, there exists an $\epsilon > 0$ such that $|C_A|, |C_B| \geq \epsilon n/2r^2$ with probability at least $1 - e^{-\Omega(n)}$. Since $C_A$ and $C_B$ are disjoint, and $C_A \cup C_B \subseteq C_H \setminus D_S$, we have $|C_H \setminus D_S| \geq \epsilon n/r^2$ with probability at least $1 - e^{-\Omega(n)}$. Once more, we will assume this deterministically for now and add the failure probability at the very end. Let $\nu$ be a small constant.

Our aim for the rest of the proof is to show that each summand of (23) can be bounded from above by an exponentially small term. We will do it by splitting the proof into four cases:

- **Case 1**, $r > 32 \sqrt{\log n}$. Observe that $t \leq d$, since $d_{S_j} \geq 1$ by definition. Since by assumption $r \geq c$ for $c$ sufficiently large, we may assume that $r \geq 4$. Then, by setting $a = b = 4 \sqrt{n}/r \leq \sqrt{n}$, $k = 4$, and $s = d_{S_j}$ in Lemma 2.2, we conclude that there are at most $n^t 9^{d_{S_1} + \cdots + d_{S_t}} \leq n^t e^{9d}$ ways to construct possible sets of cells $D_S$ corresponding to all balanced separators $S$ with $t$ components. Using $t$ nonnegative numbers, there are at most $d^t \leq n^t$ ways to add up to $d$, and thus the right-hand side of (23) can be bounded from above by

\begin{equation}
\sum_{d \geq \eta \sqrt{n}/r} \sum_{t \leq \gamma r \sqrt{n}} n^t e^{9d} e^{-r^2 d/128}.
\end{equation}

Since $n^t e^{9d} e^{-r^2 d/128} \leq e^{2 \log n + O(1) - r^2 d/256} \leq e^{-r^2 d/256} = e^{-\Omega(r \sqrt{n})}$.

- **Case 2**, $c \leq r \leq 32 \sqrt{\log n}$ and $d \geq \sqrt{n}(\log n)^{3/2}/r$. We start with (24) as before. Note that for $c$ sufficiently large, since $c \leq r$, $e^{9d} < e^{r^2 d/256}$. Note also that $e^{-r^2 d} \geq e^{\sqrt{n}(\log n)^{3/2}} \geq e^{c\sqrt{n}(\log n)^{3/2}}$. Thus, since $t \leq \gamma r \sqrt{n}$, we have $n^t = e^{2t \log n} \leq e^{2 \gamma r \sqrt{n} \log n} \leq e^{64 \gamma \sqrt{n}(\log n)^{3/2}} \leq e^{\frac{d^2}{2}} r^2 d$. Provided that $c$ is a large enough constant, we obtain

\begin{equation}
\sum_{d \geq \eta \sqrt{n}/r} \sum_{t \leq \gamma r \sqrt{n}} n^t e^{9d} e^{-r^2 d/128} \leq e^{-r^2 d/512} = e^{-\Omega(r \sqrt{n})}.
\end{equation}

- **Case 3**, $c \leq r \leq 32 \sqrt{\log n}$, and $t \leq \frac{\sqrt{n}}{\log n}$. Once more, we start with (24). If $\nu$ is small enough, since $d \geq \frac{\nu \sqrt{n}}{\log n}$, we have $n^t < e^{2\nu \sqrt{n}} < e^{r^2 d/512}$. If $c$ is sufficiently large, we have $e^{9d} < e^{r^2 d/512}$. Thus, in such case the summand in (24), and therefore also the right-hand side of (23), is bounded from above by $e^{-r^2 d/256} = e^{-\Omega(r \sqrt{n})}$.

- **Case 4**, $c \leq r \leq 32 \sqrt{\log n}$, $t > \frac{\sqrt{n}}{\log n}$, and $d < \sqrt{n}(\log n)^{3/2}/r$. Since this is the most complex case, we will first highlight the main steps of the argument:

1. We first show that most cells of $C_H \setminus D_S$ are in components of the cell graph that are relatively large.
2. We then show that these large components occupy a positive fraction of $S_n$, implying that they participate in many cells of $D_S$. We will restrict the counting on the number of points in $S$ to the boundaries of these components.

3. Although there are not many large components, each of them may induce many connected components in $D_S$. We will restrict the counting on the number of points in $S$ to the boundaries of these components. We next fill in these holes to obtain a connected set of cells corresponding to the boundary for each large component. In order to avoid having a unique filled component that spans all $S_n$, we exclude the component of $H$ having the largest intersection with the boundary of $S_n$. (In fact we already exclude it at step (3).) We will use the fact that the boundaries of the filled components are connected to count the number of possible boundaries via counting lattice animals.

4. Finally, to show that there are many points inside $D_S$, we restrict the boundaries of the filled components to $S_n$ again. Since all the considered components have a relatively small intersection with the boundary of $S_n$, using Lemma 2.7, we show that the number of boundary cells inside $S_n$ is of the same order as the total number of boundary cells, which is of order at least $\sqrt{n}/r$.

Let us now go into detail. We start with step (1) and show that almost all cells of $C_H \setminus D_S$ are contained in components of the cell graph of order at least $\sqrt{\frac{n \log n}{\nu r}}$ (called large components). Assume that this is not the case. Recall that $|C_H \setminus D_S| \geq \frac{\varepsilon n}{r^2}$, and assume that there exists an $\varepsilon' \leq \varepsilon$ such that at least $\varepsilon'n/r^2$ cells of $C_H \setminus D_S$ are in components of order at most $\sqrt{\frac{n \log n}{\nu^r}}$. As in the concavity argument of (11), $d$ is minimized if there are at most $\varepsilon'r\sqrt{n}$ components of order $\sqrt{\frac{n \log n}{\nu^r}}$. Recalling that each cell has area $\Theta(r^2)$, by Corollary 2.10 we obtain

$$d = \Omega \left( r^{-1} \sqrt{\frac{n \log n}{\nu^r}} \cdot \frac{\varepsilon'r\sqrt{n}}{\log n} \right) = \Omega \left( \frac{n^{3/4}}{r^{1/2}\sqrt{\log n}} \right) > \frac{\sqrt{n}(\log n)^{3/2}}{r},$$

which contradicts our assumption on $d$.

Our final goal is to show that for any such balanced separator, there are many points inside. In this particular case, we will show that with high probability there are already many points in a subset of $S$. This will be particularly convenient since there are fewer candidates for subsets than for $S$, and a simple union bound will suffice to finish the proof. In particular, to form such a subset we will only count the boundary contribution of some connected components in $H \setminus S$.

We now proceed with step (2). We will get rid of all components in $H \setminus S$ of size at most $\sqrt{\frac{n \log n}{\nu^r}}$, while at the same time still keeping a large number of cells occupied by the separator. Recall that by our deterministic hypothesis we have $d \geq \eta \sqrt{n}/r$. Denote by $H_0, \ldots, H_p$ the connected components of the graph $H \setminus S$.

For every $0 \leq i \leq p$, let $D_{H_i}$ be the set of cells of size $d_{H_i}$ provided by Lemma 2.8. (Note that the union of them is the set of cells provided by
Corollary 2.10.) Some of the points in \( D_S \) may appear in more than one set \( D_{H_i} \), but recall that by Lemma 2.9 part (2) each cell can be counted for at most 24 components. Thus, \( Y_{D_S} \geq \frac{1}{24} \sum_{i=0}^{p} Y_{D_{H_i}} \), and hence

\[
\Pr(Y_{D_S} \leq \gamma r \sqrt{n}) \leq \Pr \left( \frac{1}{24} \sum_{i=0}^{p} Y_{D_{H_i}} \leq \gamma r \sqrt{n} \right),
\]

and we can focus on the individual random variables \( Y_{D_{H_i}} \).

Since \( S \) has size at most \( 2\gamma n \), we have that \( \sum_{i=0}^{p} |V(H_i)| \geq n/2 \). Thus, by Lemma 4.4, there exists some \( \delta > 0 \) such that \( \sum_{i=0}^{p} \text{vol}(H_i) \geq \delta n \). Without loss of generality, let \( H_0, \ldots, H_q \) be the connected components of \( H \setminus S \) whose cell graphs contain at least \( \sqrt{\frac{n}{\log n}} \) cells. Since almost all cells of \( C_H \setminus D_S \) are in such cell components, \( \sum_{i=0}^{q} \text{vol}(H_i) \geq \delta n/2 \); that is, large components of \( H \setminus S \) occupy a constant proportion of \( S_n \).

We continue with step (3). Recall that \( \partial S_n \) denotes the boundary of the square \( S_n \). Without loss of generality, let \( H_0 \) be the connected component that maximizes \( \text{vol}(H_i \cap \partial S_n) \). Since the curves \( H_i \) are disjoint for different values of \( i \), for every \( 1 \leq i \leq q \), we have

\[
\text{vol}(H_i \cap \partial S_n) \leq \frac{1}{2} \text{vol}(\partial S_n).
\]

Since \( S \) is balanced, we also have that \( \sum_{i=1}^{q} |V(H_i)| \geq n/4 \) (otherwise \( V(H_0) \) is too large, and the cut is not balanced), which by Lemma 4.4 implies \( \sum_{i=1}^{q} \text{vol}(H_i) \geq \delta'(n) \) for some \( \delta' > 0 \). The same argument also implies that, for every \( 1 \leq i \leq q \), we have \( \text{vol}(H_i) < (1 - \delta')n \) (provided that \( \delta' \) is small enough).

Similarly as before, Corollary 2.10 implies that

\[
\sum_{i=1}^{q} d_{H_i} = \Omega \left( \frac{\sqrt{n}}{r} \right).
\]

Let us now show that \( q \) cannot be too large. Since we use a tessellation with side length \( \ell = r/4 \), there are at most \( 16n/r^2 \) cells. Moreover, each cell induces a clique which implies that each cell belongs to at most one connected component. Since each large cell component contains at least \( \sqrt{\frac{n}{\log n}} \) cells, we have that

\[
q \leq \frac{16n \sqrt{n}}{\log n}.
\]

Although the number of large components is small, each separator set \( D_{H_i} \) may induce a lot of different connected components. This can cause problems when bounding (24). In order to fix this issue, we consider a tessellation of \( \mathbb{R}^2 \) and extend the random geometric graph from \( S_n \) to \( \mathbb{R}^2 \). Recall that \( C_{H_i} \) denotes the set of cells that contain at least one point of \( H_i \). For every \( 1 \leq i \leq q \), we consider the set \( D_{H_i}' \subseteq \mathbb{R}^2 \) of size \( d_{H_i}' \) defined as follows: a cell \( D \) belongs to \( D_{H_i}' \) if either \( D \in D_{H_i} \), or \( D \in \mathbb{R}^2 \setminus S_n \) and \( D \) intersects \( C_{H_i} \). In other words, \( D_{H_i}' \) contains in addition to the cells already present in \( D_{H_i} \) also all cells touching the boundary of \( S_n \) containing at least one point
Fig. 6. Examples of sets $D_{H_i}$, $D'_{H_i}$, $D''_{H_i}$, and $D'''_{H_i}$.

One can imagine $D'_{H_i}$ to be the extension of $D_{H_i}$ to the tessellation of $\mathbb{R}^2$. Observe that the number of connected components in $D'_{H_i}$ is at most the number of those in $D_{H_i}$, but this number can still be large (for instance, if the component contains some holes). Clearly, $d'_{H_i} \geq d_{H_i}$.

Let $1 \leq i_1 < \cdots < i_s \leq q$ be the indices of large cell components that are not contained in the fill-up of any other cell component. Since the components are now considered as subsets of $\mathbb{R}^2$, by the standard isoperimetric inequality and its extension to nonconnected sets as in (11), we obtain

$$d'' := \sum_{j=1}^{s} d''_{H_{i_j}} = \Omega \left( \frac{1}{r} \sum_{i=1}^{q} \text{vol}(\mathcal{H}_i) \right) = \Omega \left( \frac{\sqrt{n}}{r} \right),$$

where for the last equality we have used that $\sum_{i=1}^{q} \text{vol}(\mathcal{H}_i) \geq \delta'n$.

Now, consider $D'''_{H_i} := D''_{H_i} \cap S_n$ to be the restriction of the previous sets to the square $S_n$ and let $d'''_{H_i}$ be their size. Observe that $D'''_{H_i} \subseteq D_H \subseteq D_S$. 

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and thus, it suffices to prove that \( \sum_{j=1}^{s} Y_{D_{H_{ij}}}'' \leq \gamma r \sqrt{n} \) with sufficiently small probability. Recall that by (25), for every \( 1 \leq i \leq q \), the component \( H_i \) satisfies \( \text{vol}(H_i \cap \partial S_n) \leq \frac{1}{2} \text{vol}(\partial S_n) \). Using Lemma 2.7, this implies

\[
\text{vol} \left( \partial F_{H_i} \cap \mathcal{S}_n \right) = \Omega(\text{vol}(\partial F_{H_i})) = \Omega(d''_{H_i}),
\]

and therefore

\[
d''' := \sum_{j=1}^{s} d'''_{H_{ij}} = \sum_{j=1}^{s} \Omega \left( \text{vol} \left( \partial F_{H_{ij}} \cap \mathcal{S}_n \right) \right) = \Omega(d'').
\]

Finally, we arrive at step (5): to bound the number of lattice animals that are candidates for \( D'''_{H_{ij}}, \ldots, D'''_{H_{ij}} \), we will look at \( D'''_{H_{ij}}, \ldots, D'''_{H_{ij}} \), which are connected and whose sizes add up to \( d''' \). To show that the probability of having few points is small, we will look at the \( d''' = \Omega(d') \) cells that are inside some set \( D'''_{H_{ij}} \). Any point there is also in \( D_{S} \), and thus it suffices to restrict the counting to them. Therefore, each summand of (24) that satisfies the hypothesis of Case 4 is bounded by \( n^{2+e^{(d/e)\gamma r}} e^{-r \sqrt{d''}/128} \). Since \( s \leq q \leq \frac{16 \sigma r \sqrt{n}}{\log n} \), \( d''' = \Omega(d') \) and \( d'' = \Omega(\sqrt{n}/r) \), the previous term can be bounded by \( e^{-\Omega(r \sqrt{n})} \), provided that \( c \) is large enough.

We showed that each term of (24) can be bounded from above by \( e^{-\Omega(r \sqrt{n})} \) if \( d \geq \eta \sqrt{n} / r \). Since all properties which we have assumed deterministically throughout the proof hold with probability at least \( 1 - e^{-\Omega(n)} \), we have together with (20),

\[
\Pr \left( \exists S \text{ balanced, } |S| \leq \frac{\eta \sqrt{n}}{32} \right) \leq e^{-\Omega(n)} + \sum_{d \geq \eta \sqrt{n}/r} \sum_{\gamma \leq \eta \sqrt{n}} e^{-\Omega(r \sqrt{n})} = e^{-\Omega(r \sqrt{n})}.
\]

Having chosen \( \gamma \) sufficiently small such that \( \gamma \leq \eta / 32 \), the theorem follows.

**Proof of Theorem 1.2.** Theorem 1.2 follows directly by recalling that \( \text{tw}(G) \leq \text{td}(G) \) and combining Proposition 4.2 with Theorem 4.5.

**5. Conclusion.** Given a random geometric graph \( G \in \mathcal{G}(n, r) \) we showed that if \( 0 < r < r_c \), \( \text{tw}(G) = \Theta \left( \frac{\log n}{\log \log n} \right) \) and that if \( r \geq c \), for some sufficiently large \( c \), \( \text{tw}(G) = \Theta(\sqrt{r \log n}) \). We conjecture that the latter can be extended to the whole supercritical regime, that is, we conjecture that for every \( r > r_c \), \( \text{tw}(G) = \Theta(\sqrt{r \log n}) \). This is a natural thing to expect since \( r_c \) is already the threshold radius for the existence of a giant component. The conjecture is equivalent to the existence of a sharp threshold width of order \( o(1) \) at \( r = r_c \). We remark that the general result on sharp thresholds of monotone properties of [9] implies only a sharp threshold width of order \( \log^{3/4} n \). Our methods, however, require the knowledge of the exact threshold value \( r_c \) of the appearance of the giant component in a random geometric graph, which at the moment is not known.

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