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On perfect packings in dense graphs

József Balogh,∗ Alexandr V. Kostochka† and Andrew Treglown‡

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Abstract

We say that a graph $G$ has a perfect $H$-packing if there exists a set of vertex-disjoint copies of $H$ which cover all the vertices in $G$. We consider various problems concerning perfect $H$-packings: Given $n, r, D ∈ \mathbb{N}$, we characterise the edge density threshold that ensures a perfect $K_r$-packing in any graph $G$ on $n$ vertices and with minimum degree $\delta(G) ≥ D$. We also give two conjectures concerning degree sequence conditions which force a graph to contain a perfect $H$-packing. Other related embedding problems are also considered. Indeed, we give a structural result concerning $K_r$-free graphs that satisfy a certain degree sequence condition.

1 Introduction

Given two graphs $H$ and $G$, a perfect $H$-packing in $G$ is a collection of vertex-disjoint copies of $H$ which cover all the vertices in $G$. Perfect $H$-packings are also referred to as $H$-factors or perfect $H$-tilings. Hell and Kirkpatrick [10] showed that the decision problem whether a graph $G$ has a perfect $H$-packing is NP-complete precisely when $H$ has a component consisting of at least 3 vertices. So for such graphs $H$, it is unlikely that there is a complete characterisation of those graphs containing a perfect $H$-packing. Thus, there has been significant attention on obtaining sufficient conditions that ensure a graph $G$ contains a perfect $H$-packing.

A seminal result in the area is the Hajnal-Szemerédi theorem [9] which states that a graph $G$ whose order $n$ is divisible by $r$ has a perfect $K_r$-packing provided that $\delta(G) ≥ (r − 1)n/r$. Kühn and Osthus [15, 16] characterised, up to an additive constant, the minimum degree which ensures a graph $G$ contains a perfect $H$-packing for an arbitrary graph $H$.

It is easy to see that the minimum degree condition in the Hajnal-Szemerédi theorem cannot be lowered. Of course, this does not mean that one cannot strengthen this result. Ore-type degree conditions consider the sum of the degrees of non-adjacent vertices in a graph. The following Ore-type result of Kierstead and Kostochka [12] implies the Hajnal-Szemerédi theorem.

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Theorem 1 (Kierstead and Kostochka [12]) Let \( n, r \in \mathbb{N} \) such that \( r \) divides \( n \). Suppose that \( G \) is a graph on \( n \) vertices such that for all non-adjacent \( x \neq y \in V(G) \),

\[ d(x) + d(y) \geq 2(1 - 1/r)n - 1. \]

Then \( G \) contains a perfect \( K_r \)-packing.

Kühn, Osthus and Treglown [17] characterised, asymptotically, the Ore-type degree condition which ensures a graph \( G \) contains a perfect \( H \)-packing for an arbitrary graph \( H \).

1.1 Perfect packings in dense graphs of low minimum degree

It is easy to characterise the edge density that forces a graph \( G \) to contain a perfect \( K_r \)-packing when there are no other restrictions. Indeed, given \( n, r \in \mathbb{N} \) such that \( r \geq 2 \) divides \( n \), if \( G \) is a graph on \( n \) vertices and \( e(G) \geq \binom{n}{2} - n + r \) then \( G \) contains a perfect \( K_r \)-packing. Moreover, if \( G \) is a copy \( K \) of \( K_{n-1} \) together with a vertex which sends precisely \( r - 2 \) edges to \( K \), then \( e(G) = \binom{n}{2} - n + r - 1 \) and \( G \) does not contain a perfect \( K_r \)-packing. The following result of Akiyama and Frankl [1] refines this observation.

Theorem 2 (Akiyama and Frankl [1]) Let \( n, r \in \mathbb{N} \) such that \( r \) divides \( n \). Suppose \( G \) is a graph on \( n \) vertices and \( e(G) \leq \min\{\binom{n}{2} - n + 1, \binom{n}{2} - n + r + 1\} \). Then \( G \) has a perfect \( K_r \)-packing unless \( \overline{G} \) is isomorphic to one of the following graphs:

(i) A copy of \( K_{n/r + 1} \) together with \((1 - 1/r)n - 1 \) isolated vertices;

(ii) The disjoint union of \( K_{1, n-r-j+1}, j \) edges and \( r - j - 2 \) isolated vertices, for some \( 1 \leq j \leq r - 2 \).

When (for example) \( n \geq r^3 \), \( \binom{n}{2} - n + r + 1 \). Hence, in this case Theorem 2 is equivalent to the following: If \( G \) is a graph on \( n \) vertices and \( e(G) \geq \binom{n}{2} - n + r - 1 \) then either \( G \) contains a perfect \( K_r \)-packing or \( \overline{G} \) is isomorphic to a graph as in (ii).

In Sections 2 and 3 we consider the following natural problem: Let \( n, r \in \mathbb{N} \) such that \( r \) divides \( n \). Given some \( D \in \mathbb{N} \), what edge density condition ensures that any graph \( G \) on \( n \) vertices and of minimum degree \( \delta(G) \geq D \) contains a perfect \( K_r \)-packing?

We fully resolve the problem, and our answers for \( r = 2 \) and \( r \geq 3 \) differ.

Theorem 3 For an even positive \( n \) and integer \( 1 \leq d < n/2 \), let \( h(n, d) := \binom{n-d-1}{2} + d(d+1) \) and let \( f(2, n, d) \) denote the maximum integer \( c \) such that some \( n \)-vertex graph with minimum degree at least \( d \) and at least \( c \) edges has no perfect matching. Then

\[ f(2, n, d) = \max\{h(n, d), h(n, 0.5n - 1)\}. \]

Theorem 4 Let \( n, r \in \mathbb{N} \) such that \( r \geq 3 \) and \( r \) divides \( n \). Given any \( D \in \mathbb{N} \) such that \( r - 1 \leq D \leq (r - 1)n/r - 1 \) define

\[ g(n, r, D) := \max\left\{ \binom{n}{2} - \binom{n/r + 1}{2}, D(n-D) + \binom{n-1-D}{2} + e(T(D, r-2)) \right\}. \]

Suppose that \( G \) is a graph on \( n \) vertices with \( \delta(G) \geq D \) and \( e(G) > g(n, r, D) \). Then \( G \) contains a perfect \( K_r \)-packing. Moreover, there exists a graph \( G' \) on \( n \) vertices with \( \delta(G') \geq D \) and \( e(G') = g(n, r, D) \) but such that \( G' \) does not contain a perfect \( K_r \)-packing.
Clearly a graph $G$ of minimum degree $\delta(G) < r - 1$ cannot contain a perfect $K_r$-packing. Further, regardless of edge density, every graph $G$ whose order $n$ is divisible by $r$ and with $\delta(G) \geq (r-1)n/r$ contains a perfect $K_r$-packing. Thus, Theorem 4 covers all values of $D$ where our problem was not solved previously.

An equitable $k$-colouring of a graph $G$ is a proper $k$-colouring of $G$ such that any two colour classes differ in size by at most one. Let $n, r \in \mathbb{N}$ such that $r$ divides $n$. Notice that a graph $G$ on $n$ vertices has a perfect $K_r$-packing if and only if the complement $\overline{G}$ of $G$ has an equitable $n/r$-colouring. So, for example, the Hajnal-Szemerédi theorem can be stated in terms of equitable colourings: Let $G$ be a graph on $n$ vertices such that $\Delta(G) \leq n/r - 1$ then $G$ has an equitable $n/r$-colouring.

It is often easier to work in the language of equitable colourings compared to perfect packings. Indeed, rather than prove Theorem 1 directly, Kierstead and Kostochka proved the equivalent statement for equitable colourings. Here we also find it more convenient to work with equitable colourings. Thus, instead of proving Theorem 4 directly we prove the following equivalent result.

**Theorem 5** Let $n, r \in \mathbb{N}$ such that $r \geq 3$ and $r$ divides $n$. Recall that $T(n,r)$ denotes the Turán graph. Given any $D \in \mathbb{N}$ such that $n/r \leq D \leq n - r$

\[ f(n, r, D) := \min \left\{ \left( \frac{n/r + 1}{2} \right), D + e(T(n - D - 1, r - 2)) \right\}. \]

Suppose that $G$ is a graph on $n$ vertices with $\Delta(G) \leq D$ and $e(G) < f(n, r, D)$. Then $G$ has an equitable $n/r$-colouring. Moreover, there exists a graph $G'$ on $n$ vertices with $\Delta(G') \leq D$ and $e(G') = f(n, r, D)$ but such that $G'$ does not have an equitable $n/r$-colouring.

We prove Theorem 3 and describe extremal constructions for Theorems 4 and 5 in Section 2. That is, we show that the edge density condition in Theorem 4 is best possible for all values of $D$ such that $r - 1 \leq D \leq (r - 1)n/r - 1$. Section 3 contains a proof of Theorem 5.

### 1.2 Degree sequence conditions forcing a perfect packing

Chvátal [4] gave a condition on the degree sequence of a graph which ensures Hamiltonicity: Suppose that $G$ is a graph on $n$ vertices and that the degrees of the graph are $d_1 \leq \cdots \leq d_n$. If $n \geq 3$ and $d_i \geq i + 1$ or $d_{n-i} \geq n - i$ for all $1 < i < n/2$ then $G$ is Hamiltonian. The following is a simple consequence of Chvátal’s theorem.

**Theorem 6 (Chvátal [4])** Suppose that $G$ is a graph on $n \geq 2$ vertices and the degrees of the graph are $d_1 \leq \cdots \leq d_n$. If

\[ d_i \geq i \text{ or } d_{n-i+1} \geq n - i \text{ for all } 1 \leq i \leq n/2 \]

then $G$ contains a Hamilton path.

We propose the following conjecture on the degree sequence of a graph which forces a perfect $K_r$-packing.

**Conjecture 7** Let $n, r \in \mathbb{N}$ such that $r$ divides $n$. Suppose that $G$ is a graph on $n$ vertices with degree sequence $d_1 \leq \cdots \leq d_n$ such that:

3
\[(\alpha)\ d_i \geq (r - 2)n/r + i \text{ for all } i < n/r;\]

\[(\beta)\ d_{n/(r+1)} \geq (r - 1)n/r.\]

Then \(G\) contains a perfect \(K_r\)-packing.

Note that Conjecture 7, if true, is much stronger than the Hajnal-Szemerédi theorem since the degree condition allows for \(n/r\) vertices to have degree less than \((r - 1)n/r\). Further, Proposition 17 in Section 4 shows that the condition on the degree sequence in Conjecture 7 is essentially “best possible”. It is easy to see that Theorem 6 implies Conjecture 7 in the case when \(r = 2\). We prove the conjecture in the case when \(G\) is additionally \(K_{r+1}\)-free (see Section 5).

If one can prove Conjecture 7, it seems likely it can be used to prove the next conjecture.

**Conjecture 8** Suppose \(\gamma > 0\) and \(H\) is a graph with \(\chi(H) = r\). Then there exists an integer \(n_0(n_0, H)\) such that the following holds. If \(G\) is a graph whose order \(n \geq n_0\) is divisible by \(|H|\), and whose degree sequence \(d_1 \leq \cdots \leq d_n\) satisfies

- \(d_i \geq (r - 2)n/r + i + \gamma n\) for all \(i < n/r\),

then \(G\) contains a perfect \(H\)-packing.

Since first submitting this paper, the third author and Knox [13] have proven Conjecture 8 in the case when \(r = 2\). (In fact, they have proven a much more general result concerning embedding spanning bipartite graphs of small bandwidth.)

The following result of Erdős [8] characterises those degree sequences which force a copy of \(K_r\) in a graph \(G\).

**Theorem 9 (Erdős [8])** Let \(G\) be a graph on \(n\) vertices with degree sequence \(d_1 \leq \cdots \leq d_n\). If \(G\) is \(K_{r+1}\)-free then there is an \(r\)-partite graph \(G'\) on \(n\) vertices whose degree sequence \(d_1' \leq \cdots \leq d_n'\) satisfies

\(d_i \leq d_i'\) for all \(i \leq n\).

In Section 6 we prove the following related structural theorem.

**Theorem 10** Suppose that \(n, r \in \mathbb{N}\) such that \(n \geq r\) and so that \(r\) divides \(n\). Let \(G\) be an \(K_{r+1}\)-free graph on \(n\) vertices whose degree sequence \(d_1 \leq \cdots \leq d_n\) is such that \(d_{n/r} \geq (r - 1)n/r\). Then \(G \subseteq T(n, r)\), where \(T(n, r)\) is the complete \(r\)-partite Turán graph on \(n\) vertices; so each vertex class has size \([n/r]\) or \([n/r]\).

2 The case \(r = 2\) and extremal examples for \(r \geq 3\)

2.1 Perfect matchings in dense graphs

In this section we establish the density threshold that ensures every graph \(G\) on an even number \(n\) of vertices and of minimum degree \(d(G) \geq d\) contains a perfect matching. Note that we only consider values of \(d\) such that \(1 \leq d < n/2\), since if \(d(G) \geq n/2\) then \(G\) has a perfect matching, regardless of the edge density.

Recall that \(h(n, d) := \left(\frac{n - d - 1}{2}\right) + d(d + 1)\). Note that for a fixed even \(n\), \(h(n, d)\) decreases with \(d\) in the interval \([0, n/3 - 5/6]\) and increases with \(d\) in \([n/3 - 5/6, 0.5n - 1]\).
For a positive even $n$ and an integer $0 \leq d < n/2$, let $A, B$ and $C$ be disjoint sets with $|A| = d+1$, $|B| = d$, $|C| = n - 2d - 1$. Let $H = H(n, d)$ be the graph with the vertex set $A \cup B \cup C$ such that $H[B \cup C] = K_{n-d-1}$, and each vertex in $A$ is adjacent to each vertex in $B$ and to no vertex in $C$. So $H$ does not contain a perfect matching and has exactly $h(n, d)$ edges.

The examples of $H(n, d)$ show that $f(2, n, d) \geq \max\{h(n, d), h(n, 0.5n - 1)\}$. Thus to derive Theorem 3, it suffices to prove that an $n$-vertex graph $G$ with $\delta(G) \geq d$ and $e(G) > \max\{h(n, d), h(n, 0.5n - 1)\}$ has a perfect matching.

Consider such a graph $G$. Let $d_1 \leq \cdots \leq d_n$ denote the degree sequence of $G$. If $d_i \geq i$ for all $1 \leq i \leq n/2$ then Theorem 6 implies that $G$ contains a perfect matching. Suppose for a contradiction that $d_{i'} \leq i' - 1$ for some $1 \leq i' \leq n/2$. Note that $i' > d$ as $\delta(G) \geq d$.

Let $A$ denote the set of $i'$ vertices in $G$ that correspond to the first $i'$ terms $d_1, \ldots, d_{i'}$ of the degree sequence. Set $B := V(G) \setminus A$. Then

$$e(G[B]) \geq e(G) - i' (i' - 1) > \max\{h(n, d), h(n, 0.5n - 1)\} - i'(i' - 1)$$

since $d(x) \leq i' - 1$ for all $x \in A$. Note that $\max\{h(n, d), h(n, 0.5n - 1)\} \geq h(n, i' - 1)$ since $d < i' \leq n/2$. Therefore,

$$e(G[B]) > \max\{h(n, d), h(n, 0.5n - 1)\} - i'(i' - 1) \geq h(n, i' - 1) - i'(i' - 1) = \binom{n-i'}{2},$$

a contradiction as $|B| = n - i'$. Thus, $d_i \geq i$ for all $1 \leq i \leq n/2$, as desired.

### 2.2 Examples for $r \geq 3$

We will give the extremal examples for Theorem 5. Since Theorems 4 and 5 are equivalent, the complements of the extremal graphs for Theorem 5 are the extremal graphs for Theorem 4.

**Proposition 11** Suppose that $n, r \in \mathbb{N}$ such that $r \geq 3$ and $r$ divides $n$. Then there exists a graph $G_1$ on $n$ vertices such that $\Delta(G_1) = n/r$,

$$e(G_1) = \binom{n/r + 1}{2},$$

but such that $G_1$ does not have an equitable $n/r$-colouring.

**Proof.** Let $G_1$ denote the disjoint union of a clique $V$ on $n/r + 1$ vertices and an independent set $W$ of $(1 - 1/r)n - 1$ vertices. So every independent set in $G_1$ contains at most one vertex from $V$. But since $|V| = n/r + 1$, $G_1$ does not have an equitable $n/r$-colouring. Further, $\Delta(G_1) = n/r$ and $e(G_1) = \binom{n/r + 1}{2}$. \hfill \square

**Proposition 12** Suppose that $n, r \in \mathbb{N}$ such that $r \geq 3$ and $n = kr$ for some $k \geq 2$. Further, let $D \in \mathbb{N}$ such that $n/(r-1) \leq D \leq n-r$. Then there exists a graph $G_2$ on $n$ vertices such that $\Delta(G_2) = D$,

$$e(G_2) = D + e(T(n-D-1, r-2)),$$

but such that $G_2$ does not have an equitable $n/r$-colouring.
Proof. Let $G_2$ denote the disjoint union of a copy $K$ of $K_{1,D}$ and a copy of $T(n - D - 1, r - 2)$. So $|G| = n$. Let $v$ denote the vertex of degree $D$ in $K$. The largest independent set in $G_2$ that contains $v$ is of size $r - 1$. Thus, $G_2$ does not have an equitable $n/r$-colouring. Further, $e(G_2) = D + e(T(n - D - 1, r - 2))$.

Since $n/(r - 1) \leq D$ we have that $n - 1 \leq (r - 1)D$. Thus, every vertex in the copy of $T(n - D - 1, r - 2)$ has degree at most

$$\left\lceil \frac{n - D - 1}{r - 2} \right\rceil - 1 \leq \frac{n - D - 1}{r - 2} \leq D.$$ 

This implies that $\Delta(G_2) = D$. \qed

Clearly Propositions 11 and 12 show that one cannot lower the edge density condition in Theorem 5 in the case when $n/(r - 1) \leq D \leq n - r$. The following result, together with Proposition 11, shows that Theorem 5 is best possible in the case when $n/r \leq D \leq n/(r - 1)$.

**Proposition 13** Let $n, r \in \mathbb{N}$ such that $r \geq 3$ and $r$ divides $n \geq 2r$. Suppose that $D \in \mathbb{N}$ such that $n/r \leq D \leq n/(r - 1)$. Then

$$f(n, r, D) = \frac{n/r + 1}{2}.$$ 

The following simple consequence of Turán’s theorem will be used in the proof of Theorem 5.

**Fact 14** Let $n, r \in \mathbb{N}$ such that $r \leq n$. Then

$$e(T(n, r)) \leq \left(1 - \frac{1}{r}\right) \frac{n^2}{2} \text{ and thus } e(T(n, r)) \geq \frac{n^2}{2r} - \frac{n}{2}.$$ 

We will also require the following easy result.

**Lemma 15** Let $n, r \in \mathbb{N}$ such that $r \geq 4$ and $r$ divides $n \geq 3r$. Suppose that $D \in \mathbb{N}$ such that $n/r \leq D < (n + r)/(r - 1)$. Then

$$f(n, r, D) = \frac{n/r + 1}{2}.$$ 

3 Proof of Theorem 5

3.1 Preliminaries

Suppose for a contradiction that the result is false. Let $G$ be a counterexample with the fewest vertices. That is, $n = |V(G)| = rk$ for some $k \in \mathbb{N}$, $\Delta(G) \leq D$ for some $D \in \mathbb{N}$ such that $n/r \leq D \leq n - r$, $e(G) < f(n, r, D)$ and $G$ has no equitable $n/r$-colouring. By the Hajnal-Szemerédi theorem, $\Delta(G) \geq n/r$. Notice that given fixed $n$ and $r$, $f(n, r, D)$ is non-increasing with respect to $D$. Thus, we may assume that $\Delta(G) = D$.

We first show that $k \geq 4$. Indeed, if $n = 2r$ then $f(n, r, D) \leq \left(\frac{3}{2}\right) = 3$. But it is easy to see that every graph $G_1$ on $2r$ vertices and with $e(G_1) \leq 2$ has an equitable 2-colouring. If $n = 3r$ then $f(n, r, D) \leq \left(\frac{4}{3}\right) = 6$. Consider any graph $G_1$ on $3r$ vertices with $e(G_1) \leq 5$ and $3 \leq \Delta(G_1) \leq 5$. Let $x$ denote the vertex in $G_1$ where $d_{G_1}(x) = \Delta(G_1)$. Since $3 \leq d_{G_1}(x) \leq 5$, $x$ lies in an independent
set $I$ in $G_1$ of size $r$. But then $G_1 - I$ contains $2r$ vertices and at most 2 edges. So $G_1 - I$ has an equitable 2-colouring and hence $G_1$ has an equitable 3-colouring.

Let $v \in V(G)$ such that $d_G(v) = D$. Set $G^* := G - (N_G(v) \cup \{v\})$. Since $f(n, r, D) \leq D + e(T(n - D - 1, r - 2))$ we have that $e(G^*) < e(T(n - D - 1, r - 2))$. Thus, by Turán’s theorem, $G^*$ contains an independent set of size $r - 1$. Hence, $v$ lies in an independent set in $G$ of size $r$. Amongst all such independent sets of size $r$ that contain $v$, choose a set $I = \{v, x_1, \ldots, x_{r-1}\}$ such that $d_G(x_1) + \cdots + d_G(x_{r-1})$ is maximised.

Set $G' := G - I$, $n' := |V(G')| = n - r$ and $D' := \Delta(G') \leq D$. Notice that $D' \geq n'/r$. (Indeed, if not, then by the Hajnal-Szemerédi theorem $G'$ contains an equitable $n'/r$-colouring. Thus, as $I$ is an independent set in $G$ this gives us an equitable $n/r$-colouring of $G$, a contradiction.) Furthermore, $D' \leq n' - r$. If not then

$$e(G) \geq D + D' \geq 2D' \geq 2(n' - r + 1) = 2n - 4r + 2$$

and further,

$$e(G) < f(n, r, D) \leq f(n, r, n - 2r + 1) \leq (n - 2r + 1) + e(T(2r - 2, r - 2)) \leq (n - 2r + 1) + (r + 3) = n - r + 4.$$

Therefore, $2n - 4r + 2 < n - r + 4$ and so $n < 3r + 2$ a contradiction since $n = kr \geq 4r$.

Since $n'/r \leq D' \leq n' - r$, if $e(G') < f(n', r, D')$ then the minimality of $G$ implies that $G'$ has an equitable $n'/r$-colouring. This then implies that $G$ has an equitable $n/r$-colouring, a contradiction. Thus,

$$e(G') \geq f(n', r, D').$$

(1)

We now split our argument into three cases.

### 3.2 Case 1: $f(n', r, D') = \binom{n'/r + 1}{2}$

By (1), $e(G') \geq \binom{n'/r + 1}{2} = \binom{n/r}{2}$. Since $d_G(v) = D \geq n/r$,

$$e(G) \geq \frac{n}{r} + \binom{n/r}{2} = \binom{n/r + 1}{2} \geq f(n, r, D),$$

a contradiction, as desired.

### 3.3 Case 2: $D' \leq D - 1$ and $f(n', r, D') = D' + e(T(n' - D' - 1, r - 2))$.

The following claim will be useful.

Claim 16 $D' < \frac{r - 1}{2}n - \frac{(r - r + 1)}{2r - 3}$.

**Proof.** Note that

$$D + D' + e(T(n' - D' - 1, r - 2)) \leq e(G) < f(n, r, D) \leq D + e(T(n - D - 1, r - 2)).$$

(2)
Since \( D' \leq D - 1 \), clearly 
\[
e(T(n' - D, r - 2)) \leq e(T(n' - D' - 1, r - 2)).
\]
Thus, (2) implies that
\[
D' + e(T(n' - D, r - 2)) < e(T(n - D - 1, r - 2)).
\]
(3)

One can obtain 
\( T(n - D - 1, r - 2) \) from 
\( T(n' - D, r - 2) \) by adding \( r - 1 \) vertices and at most
\[
(n' - D) + \frac{n - D - 2}{r - 2}
\]
edges. (4)

Hence (3) and (4) give
\[
D' < n' - D + \frac{n - D - 2}{r - 2}.
\]

Rearranging, and using that 
\( D' \leq D - 1 \) and 
\( n' = n - r \) we get that
\[
\left(2 + \frac{1}{r - 2}\right) D' < \left(1 + \frac{1}{r - 2}\right) n - \frac{(r^2 - r + 1)}{r - 2}.
\]

Thus,
\[
D' < \frac{r - 1}{2r - 3} n - \frac{(r^2 - r + 1)}{2r - 3},
\]
as desired. □

Since we are in Case 2 we have that
\[
D' + e(T(n - r - D' - 1, r - 2)) \leq \binom{n'/r + 1}{2} = \binom{n/r}{2}.
\]
(5)

Notice that for fixed \( n \) and \( r \), 
\( D' + e(T(n - r - D' - 1, r - 2)) \) is non-increasing as \( D' \) increases. Hence, (5) and Claim 16 imply that
\[
D'' + e(T(n - r - D'' - 1, r - 2)) \leq \frac{n^2}{2r^2} - \frac{n}{2r}
\]
where 
\( D'' := [(r - 1)n/(2r - 3) - (r^2 - r + 1)/(2r - 3)] \). Note that
\[
n - r - \frac{r - 1}{2r - 3} n + \frac{(r^2 - r + 1)}{2r - 3} - 1 = \frac{r - 2}{2r - 3} n + \frac{4 - r^2}{2r - 3}.
\]
So Fact 14 and (6) imply that
\[
\left(\frac{r - 1}{2r - 3} n - \frac{(r^2 - r + 1)}{2r - 3} - \frac{2r - 4}{2r - 3}\right) + \frac{1}{2(r - 2)} \left(\frac{r - 2}{2r - 3} n + \frac{4 - r^2}{2r - 3}\right)^2
\]
\[
- \frac{1}{2} \left(\frac{r - 2}{2r - 3} n + \frac{4 - r^2}{2r - 3}\right) \leq \frac{n^2}{2r^2} - \frac{n}{2r}.
\]

Next we will move all terms from the previous equation to the left hand side and simplify. The coefficient of \( n^2 \) is
\[
\frac{r - 2}{2(2r - 3)^2} - \frac{1}{2r^2} = \frac{r^3 - 6r^2 + 12r - 9}{2r^2(2r - 3)^2}.
\]
(7)
The coefficient of \( n \) is

\[
\frac{r - 1}{2r - 3} - \frac{(r - 2)}{2(2r - 3)} + \frac{1}{2r} + \frac{(4 - r^2)}{(2r - 3)^2} = \frac{r^2 - 4r + 9}{2r(2r - 3)^2}.
\]

(8)

The constant term is

\[
-\frac{(r^2 + r - 3)}{2r - 3} + \frac{(r^2 - 4)^2}{2(r - 2)(2r - 3)^2} + \frac{(r^2 - 4)}{2(2r - 3)} = \frac{-r^4 + 3r^3 + 4r^2 - 26r + 28}{2(r - 2)(2r - 3)^2}.
\]

(9)

Since \( n \geq 4r \), (7)–(9) imply that

\[
\frac{8(r^3 - 6r^2 + 12r - 9)}{(2r - 3)^2} + \frac{2(r^2 - 4r + 9)}{(2r - 3)^2} + \frac{-r^4 + 3r^3 + 4r^2 - 26r + 28}{2(r - 2)(2r - 3)^2} \leq 0.
\]

(10)

Multiplying (10) by \( 2(r - 2)(2r - 3)^2 \) we get

\[
15r^4 - 121r^3 + 364r^2 - 486r + 244 \leq 0
\]

This yields a contradiction, since it is easy to check that

\[
15r^4 - 121r^3 + 364r^2 - 486r + 244 > 0
\]

for all \( r \in \mathbb{N} \) such that \( r \geq 3 \).

### 3.4 Case 3: \( D' = D \) and \( f(n', r, D') = D' + e(\overline{T}(n' - D' - 1, r - 2)) \).

By (1) we have that

\[
e(G') \geq f(n', r, D') = D' + e(\overline{T}(n' - D' - 1, r - 2)).
\]

(11)

Consider any vertex \( x \in V(G') \) such that \( d_{G'}(x) = D' = D \). Since \( \Delta(G) = D \), \( x \) is not adjacent to any vertex in \( I = \{v, x_1, \ldots, x_{r-1}\} \). Further, \( I \) was chosen such that \( d_G(x_1) + \cdots + d_G(x_{r-1}) \) is maximised. Thus, \( d_G(x_1) = \cdots = d_G(x_{r-1}) = D \). Together with (11) this implies that

\[
e(G) \geq (r + 1)D + e(\overline{T}(n' - D - 1, r - 2)).
\]

(12)

Since \( e(G) < f(n, r, D) \leq D + e(\overline{T}(n - D - 1, r - 2)) \), (12) implies that

\[
rD + e(\overline{T}(n' - D - 1, r - 2)) < e(\overline{T}(n - D - 1, r - 2)).
\]

(13)

One can obtain \( \overline{T}(n - D - 1, r - 2) \) from \( \overline{T}(n' - D - 1, r - 2) \) by adding \( r \) vertices and at most

\[
\frac{2(n - D - 3)}{r - 2} + 1 \text{ edges.}
\]

(14)

Thus, (13) and (14) imply that

\[
rD < n - r - D + \frac{2(n - D - 3)}{r - 2}
\]
and so
\[
\left( r + 1 + \frac{2}{r-2} \right) D < \left( 1 + \frac{2}{r-2} \right) n + \frac{(-r^2 + 2r - 6)}{r-2} < \left( 1 + \frac{2}{r-2} \right) n.
\]
(15)

If \( r = 3 \) then (15) implies that
\[
D < \frac{n}{2}.
\]

Since \( f(n', 3, D) = \min\{ (\binom{n'+3}{2}, D + (\binom{n'-D}{2}) \} \) it is easy to see that if \( f(n', 3, D) = D + (\binom{n'-D}{2}) \) then \( D \geq 2n'/3 + 1 = 2n/3 - 1 \). Thus, \( 2n/3 - 1 \leq D < n/2 \), a contradiction since \( n \geq 4r = 12 \).

If \( r \geq 4 \) then (15) implies that
\[
D < \frac{n}{r-1} = \frac{n'+r}{r-1} + \frac{r}{r-1}.
\]

Since \( n' \geq 3r \), Lemma 15 implies that \( f(n', r, D') = (\binom{n'}{2}) \) and so we are in Case 1, which we have already dealt with.

4 The extremal examples for Conjecture 7

Proposition 17 Suppose that \( n, r, k \in \mathbb{N} \) such that \( r \geq 2 \) divides \( n \) and \( 1 \leq k < n/r \). Then there exists a graph \( G \) on \( n \) vertices whose degree sequence \( d_1 \leq \cdots \leq d_n \) satisfies

- \( d_i = (r-2)n/r + k - 1 \) for all \( 1 \leq i \leq k \);
- \( d_i = (r-1)n/r \) for all \( k + 1 \leq i \leq (r-2)n/r + k \);
- \( d_i = n - k - 1 \) for all \( (r-2)n/r + k + 1 \leq i \leq n - k + 1 \);
- \( d_i = n - 1 \) for all \( n - k + 2 \leq i \leq n \),

but such that \( G \) does not contain a perfect \( K_r \)-packing.

Proof. Let \( G' \) denote the complete \( (r-2) \)-partite graph whose vertex classes \( V_1, \ldots, V_{r-2} \) each have size \( n/r \). Obtain \( G \) from \( G' \) by adding the following vertices and edges: Add a set \( V_{r-1} \) of \( 2n/r - 2k + 1 \) vertices to \( G' \), a set \( V_r \) of \( k - 1 \) vertices and a set \( V_0 \) of \( k \) vertices. Add all edges from \( V_0 \cup V_{r-1} \cup V_r \) to \( V_1 \cup \cdots \cup V_{r-2} \). Further, add all edges with both endpoints in \( V_{r-1} \cup V_r \). Add all possible edges between \( V_0 \) and \( V_r \).

So \( V_0 \) is an independent set, and there are no edges between \( V_0 \) and \( V_{r-1} \). This implies that any copy of \( K_r \) in \( G \) containing a vertex from \( V_0 \) must also contain at least one vertex from \( V_r \). But since \( |V_0| > |V_r| \) this implies that \( G \) does not contain a perfect \( K_r \)-packing. Furthermore, \( G \) has our desired degree sequence.

Notice that the graphs \( G \) considered in Proposition 17 satisfy (\( \beta \)) from Conjecture 7 and only fail to satisfy (\( \alpha \)) in the case when \( i = k \) (and in this case \( d_k = (r-2)n/r + k - 1 \)).

Let \( n, r \in \mathbb{N} \) such that \( r \) divides \( n \). Denote by \( T^*(n, r) \) the complete \( r \)-partite graph on \( n \) vertices with \( r-2 \) vertex classes of size \( n/r \), one vertex class of size \( n/r - 1 \) and one vertex class of size \( n/r + 1 \). Then \( T^*(n, r) \) does not contain a perfect \( K_r \)-packing. Furthermore, \( T^*(n, r) \) satisfies (\( \alpha \)) but condition (\( \beta \)) fails; we have that \( d_{n/r+1} = (r-1)n/r - 1 \) here. Thus, together \( T^*(n, r) \) and Proposition 17 show that, if true, Conjecture 7 is essentially best possible.
5 A special case of Conjecture 7

We now give a simple proof of Conjecture 7 in the case when $G$ is $K_{r+1}$-free.

**Theorem 18** Let $n, r \in \mathbb{N}$ such that $r \geq 2$ divides $n$. Suppose that $G$ is a graph on $n$ vertices with degree sequence $d_1 \leq \cdots \leq d_n$ such that:

- $d_i \geq (r-2)n/r + i$ for all $i < n/r$;
- $d_{n/r+1} \geq (r-1)n/r$.

Further suppose that no vertex $x \in V(G)$ of degree less than $(r-1)n/r$ lies in a copy of $K_{r+1}$. Then $G$ contains a perfect $K_r$-packing.

**Proof.** We prove the theorem by induction on $n$. In the case when $n = r$ then $d_{n/r+1} = d_2 \geq (r-1)r/r = r - 1$. This implies that every vertex in $G$ has degree $r - 1$. Hence $G = K_r$ as desired. So suppose that $n > r$ and the result holds for smaller values of $n$. Let $x_1 \in V(G)$ such that $d_G(x_1) = d_1 \geq (r-2)n/r + 1$. If $d_G(x_1) \geq (r-1)n/r$ then $\delta(G) \geq (r-1)n/r$. Thus $G$ contains a perfect $K_r$-packing by the Hajnal-Szemerédi theorem. So we may assume that $(r-2)n/r + 1 \leq d_G(x_1) < (r-1)n/r$. In particular, $x_1$ does not lie in a copy of $K_{r+1}$. We first find a copy of $K_r$ containing $x_1$. If $r = 2$, $x_1$ has a neighbour and so we have our desired copy of $K_2$. So assume that $r \geq 3$. Certainly $N_G(x_1)$ contains a vertex $x_2$ such that $d_G(x_2) \geq (r-1)n/r$. Thus, $|N_G(x_1) \cap N_G(x_2)| \geq (r-3)n/r + 1 > 0$. So if $r = 3$ we obtain our desired copy of $K_r$. Otherwise, we can find a vertex $x_3 \in N_G(x_1) \cap N_G(x_2)$ such that $d_G(x_3) \geq (r-1)n/r$. We can repeat this argument until we have obtained vertices $x_1, \ldots, x_r$ that together form a copy $K'_r$ of $K_r$.

Let $G' := G - V(K'_r)$ and set $n' := n - r = |V(G')|$. Since $G$ does not contain a copy of $K_{r+1}$ containing $x_1$, every vertex $x \in V(G) \setminus V(K'_r)$ sends at most $r-1$ edges to $K'_r$ in $G$. Thus, $d_{G'}(x) \geq d_G(x) - (r-1)$ for all $x \in V(G')$. So if $d_G(x) \geq (r-1)n/r$ then $d_{G'}(x) \geq (r-1)n/r - (r-1) = (r-1)n'/r$ for all $x \in V(G')$. If a vertex $y \in V(G')$ does not lie in a copy of $K_{r+1}$ in $G$ then clearly $y$ does not lie in a copy of $K_{r+1}$ in $G'$. This means that no vertex $y \in V(G')$ of degree less than $(r-1)n'/r$ lies in a copy of $K_{r+1}$.

Let $d'_1 \leq \cdots \leq d'_{n'}$ denote the degree sequence of $G'$. It is easy to check that $d'_i \geq (r-2)n'/r + i$ for all $i < n'/r$ and that $d'_{n'/r+1} \geq (r-1)n'/r$. Indeed, since $x_1 \in V(K'_r)$ where $d_G(x_1) = d_1$, we have that $d'_i \geq d_{i+1} - (r-1)$ for all $1 \leq i \leq n'$. Thus, for all $1 \leq i < n'/r = n/r - 1$, $d'_i \geq d_{i+1} - (r-1) \geq (r-2)n/r + (i+1) - (r-1) = (r-2)n'/r + i$. Similarly, $d'_{n'/r+1} = d'_{n/r} \geq d_{n/r+1} - (r-1) \geq (r-1)n/r - (r-1) = (r-1)n'/r$. Hence, by induction $G'$ contains a perfect $K_r$-packing. Together with $K'_r$ this gives us our desired perfect $K_r$-packing in $G$. \(\square\)

6 Proof of Theorem 10

Consider any $x_1 \in V(G)$ such that $d_G(x_1) \geq (r-1)n/r$. Since $d_{n/r} \geq (r-1)n/r$ we can greedily select vertices $x_2, \ldots, x_{r-1}$ such that

- $x_1, \ldots, x_{r-1}$ induce a copy of $K_{r-1}$ in $G$;
- $d_G(x_i) \geq (r-1)n/r$ for all $1 \leq i \leq r-1$. 

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Note that since $G$ is $K_r$-free, $\cap_{i=1}^{r-1} N_G(x_i)$ is an independent set. The choice of $x_1, \ldots, x_{r-1}$ implies that $|\cap_{i=1}^{r-1} N_G(x_i)| \geq n/r$. Let $V_1$ denote a subset of $\cap_{i=1}^{r-1} N_G(x_i)$ of size $n/r$. Thus $V_1$ contains a vertex $x_1$ of degree at least $(r-1)n/r$. As before we can find vertices $x_2^1, \ldots, x_{r-1}^1$ such that

- $x_1^1, \ldots, x_{r-1}^1$ induce a copy of $K_{r-1}$ in $G$;
- $d_G(x_i^1) \geq (r-1)n/r$ for all $1 \leq i \leq r-1$.

So $\cap_{i=1}^{r-1} N_G(x_i)$ is an independent set of size at least $n/r$. Let $V_2$ denote a subset of $\cap_{i=1}^{r-1} N_G(x_i)$ of size $n/r$. Note that $N_G(x_i^1) \cap V_1 = \emptyset$ since $x_i^1 \in V_1$ and $V_1$ is an independent set. Thus as $V_2 \subseteq N_G(x_i^1)$, $V_1 \cap V_2 = \emptyset$.

Our aim is to find disjoint sets $V_1, \ldots, V_r \subseteq V(G)$ of size $n/r$ and vertices $x_1^1, \ldots, x_{r-1}^1$, $x_1^{r-1}, x_2^{r-1}, \ldots, x_1^{r-1}, x_2^{r-1}, \ldots$ with the following properties:

- $G[V_j]$ is an independent set for all $1 \leq j \leq r$;
- Given any $1 \leq j \leq r-1$, $x_k^j \in V_k$ for each $1 \leq k \leq j$;
- $d_G(x_k^j) \geq (r-1)n/r$ for all $1 \leq j \leq r-1$ and $1 \leq k \leq r-1$;
- $x_1^j, \ldots, x_{r-1}^j$ induce a copy of $K_{r-1}$ in $G$ for all $1 \leq j \leq r$.

Clearly finding such a partition $V_1, \ldots, V_r$ of $V(G)$ implies that $G \subseteq T(n,r)$.

Suppose that for some $1 < j < r$ we have defined sets $V_1, \ldots, V_j$ and vertices $x_1^1, \ldots, x_1^{j-1}, x_1^j, \ldots, x_1^{j-1}$ with our desired properties. Since $d_{n/r} \geq (r-1)n/r$ and $V_1, \ldots, V_j$ are independent sets of size $n/r$ we can choose vertices $x_1^j, \ldots, x_j^j$ such that for all $1 \leq k \leq j$,

- $x_k^j \in V_k$ and $d_G(x_k^j) \geq (r-1)n/r$.

This degree condition, together with the fact that $x_1^j, \ldots, x_j^j$ lie in different vertex classes, implies that these vertices form a copy of $K_j$ in $G$. We now greedily select further vertices $x_{j+1}^j, \ldots, x_{r-1}^j$ such that

- $x_1^j, \ldots, x_{r-1}^j$ induce a copy of $K_{r-1}$ in $G$;
- $d_G(x_k^j) \geq (r-1)n/r$ for all $j+1 \leq k \leq r-1$.

So $\cap_{i=1}^{r-1} N_G(x_i^j)$ is an independent set of size at least $n/r$. Let $V_{j+1}$ denote a subset of $\cap_{i=1}^{r-1} N_G(x_i^j)$ of size $n/r$. Note that, for each $1 \leq k \leq j$, $N_G(x_k^j) \cap V_k = \emptyset$ since $x_k^j \in V_k$ and $V_k$ is an independent set. Thus as $V_{j+1} \subseteq N_G(x_k^j)$ for each $1 \leq k \leq j$, $V_{j+1}$ is disjoint from $V_1 \cup \cdots \cup V_j$.

Repeating this argument we obtain our desired sets $V_1, \ldots, V_r \subseteq V(G)$ and vertices $x_1^1, \ldots, x_{r-1}^1$, $x_1^{r-1}, x_2^{r-1}, \ldots, x_1^{r-1}, x_2^{r-1}, \ldots$.

7 Possible extensions of Conjecture 7

We suspect that the following ‘Chvátal-type’ degree sequence condition forces a graph to contain a perfect $K_r$-packing.
Question 19 Let \( n, r \in \mathbb{N} \) such that \( r \geq 2 \) divides \( n \). Suppose that \( G \) is a graph on \( n \) vertices with degree sequence \( d_1 \leq \cdots \leq d_n \) such that for all \( i \leq n/r \):

- \( d_i \geq (r - 2)n/r + i \) or \( d_{n-i(r-1)+1} \geq n - i \).

Does this condition imply that \( G \) contains a perfect \( K_r \)-packing?

Note that Theorem 6 answers this question in the affirmative when \( r = 2 \). The following example shows that we cannot have a lower value in the second part of the condition in Question 19.

Proposition 20 Suppose that \( n, r, k \in \mathbb{N} \) such that \( r \geq 2 \) divides \( n \) and \( 1 \leq k \leq n/r \). Then there exists a graph \( G \) on \( n \) vertices whose degree sequence \( d_1 \leq \cdots \leq d_n \) satisfies

- \( d_{n-i(r-1)+1} \geq n - i \) for all \( i \in [n/r] \setminus \{k\} \);
- \( d_{n-k(r-1)+1} = n - k - 1 \),

but such that \( G \) does not contain a perfect \( K_r \)-packing.

Proof. Let \( G \) be the graph on \( n \) vertices with vertex classes \( V_1, V_2 \) and \( V_3 \) of sizes \( k, (r - 1)k - 1 \) and \( n - rk + 1 \) respectively and with the following edges: There are all possible edges between \( V_1 \) and \( V_2 \) and between \( V_2 \) and \( V_3 \). Further add all possible edges in \( V_2 \) and all edges in \( V_3 \). Thus, \( V_1 \) is an independent set and there are no edges between \( V_1 \) and \( V_3 \).

The degree sequence of \( G \) is

\[
(r-1)k-1, \ldots, (r-1)k-1, n-k-1, \ldots, n-k-1, n-1, \ldots, n-1.
\]

Hence \( G \) satisfies our desired degree sequence condition. Every copy \( K'_r \) of \( K_r \) in \( G \) that contains a vertex from \( V_1 \) must contain \( r - 1 \) vertices from \( V_2 \). But since \( |V_1|(r - 1) > |V_2| \) this implies that \( G \) does not contain a perfect \( K_r \)-packing. \( \square \)

The \( r \)th power of a Hamilton cycle \( C \) is obtained from \( C \) by adding an edge between every pair of vertices of distance at most \( r \) on \( C \). Seymour [18] conjectured the following strengthening of Dirac’s theorem.

Conjecture 21 (Pósa-Seymour, see [18]) Let \( G \) be a graph on \( n \) vertices. If \( \delta(G) \geq \frac{r}{r+1}n \) then \( G \) contains the \( r \)th power of a Hamilton cycle.

Pósa (see [7]) had earlier proposed the conjecture in the case of the square of a Hamilton cycle (that is, when \( r = 2 \)). Komlós, Sárközy and Szemerédi [14] proved Conjecture 21 for graphs whose order is sufficiently larger than \( r \). More recently, Cháu, DeBiasio and Kierstead [3] proved Pósa’s conjecture for graphs of order at least \( 2 \times 10^8 \).

In the case when \( r + 1 \) divides \( |G| \), a necessary condition for a graph \( G \) to contain the \( r \)th power of a Hamilton cycle is that \( G \) contains a perfect \( K_{r+1} \)-packing. Further, notice that the minimum degree condition in Conjecture 21 is the same as the condition in the Hajnal-Szemerédi theorem with respect to perfect \( K_{r+1} \)-packings. Thus an obvious question is whether the condition in Conjecture 7 forces a graph to contain the \((r - 1)\)th power of a Hamilton cycle. Interestingly though, when \( r = 3 \), this is not the case.

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Proposition 22 Suppose that $C, n \in \mathbb{N}$ such that $C \ll n$ and $3$ divides $n$. Then there exists a graph $G$ whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies

$$d_i \geq \frac{n}{3} + C + i \text{ for all } 1 \leq i \leq \frac{n}{3}$$

but such that $G$ does not contain the square of a Hamilton cycle.

Proof. Choose $C, K, n \in \mathbb{N}$ so that $C \ll K \ll n$. Let $G$ denote the graph on $n$ vertices consisting of three vertex classes $V_1 = \{v\}, V_2$ and $V_3$ where $|V_2| = n/3 + C + 1$ and $|V_3| = 2n/3 - C - 2$ which contains the following edges:

- All edges from $v$ to $V_2$;
- All edges between $V_2$ and $V_3$ and all possible edges in $V_3$;
- There are $K$ vertex-disjoint stars in $V_2$, each of size $\lfloor |V_2|/K \rfloor$, $\lceil |V_2|/K \rceil$, which cover all of $V_2$ (see Figure 1).

Let $d_1 \leq \cdots \leq d_n$ denote the degree sequence of $G$. There are $n/3 + C - K + 1 \leq n/3 - 2C - 1$ vertices in $V_2$ of degree $2n/3 - C$. Since $C \ll K \ll n$, the remaining $K$ vertices in $V_2$ have degree at least $2n/3 - C - 2 + \lfloor |V_2|/K \rfloor \geq 2n/3 + C + 1$. Since $d_G(v) = n/3 + C + 1$ and $d_G(x) = n - 2$ for all $x \in V_3$, we have that $d_i \geq \frac{n}{3} + C + i$ for all $1 \leq i \leq \frac{n}{3}$.

A necessary condition for a graph $G$ to contain the square of a Hamilton cycle is that, for every $x \in V(G)$, $G[N(x)]$ contains a path of length 3. Note that $N(v) = V_2$ and $G[V_2]$ does not contain a path of length 3. So $G$ does not contain the square of a Hamilton cycle. \qed

Notice that we can set $C = o(\sqrt{n})$ in Proposition 22. We finish by raising the following question.

Question 23 What can be said about degree sequence conditions which force a graph to contain the $r$th power of a Hamilton cycle? In particular, can one establish a degree sequence condition that ensures a graph $G$ on $n$ vertices contains the $r$th power of a Hamilton cycle and which allows for “many” vertices of $G$ to have degree “much less” than $rn/(r+1)$?
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References


Appendix

Here we give proofs of Proposition 13 and Lemma 15. The following fact will be used in both of these proofs.

Fact 24 Fix \( n, r \in \mathbb{N} \) such that \( r \geq 3 \) and \( r \) divides \( n \geq 2r \). Define

\[
h(x) := x + \frac{(n-x-1)^2}{2(r-2)} - \frac{1}{2}(n-x-1).
\]

Then \( h(x) \) is a decreasing function for \( x \in [0, n/(r-1)] \). Moreover, if \( n \geq 3r \) then \( h(x) \) is a decreasing function for \( x \in [0, (n+r)/(r-1)] \).

Proof. Notice that

\[
h'(x) = \frac{3}{2} - \frac{(n-x-1)}{r-2} = \frac{x}{r-2} + \frac{1-n}{r-2} + \frac{3}{2}.
\]

So for \( x \leq n/(r-1) \),

\[
h'(x) \leq \frac{n}{(r-1)(r-2)} + \frac{1-n}{r-2} + \frac{3}{2} = -\frac{n}{r-1} + \frac{1}{r-2} + \frac{3}{2}.
\]

Note that \( 3(r-1)/2 + (r-1)/(r-2) < n \) since \( n \geq 2r \) and \( r \geq 3 \). Thus,

\[
h'(x) \leq -\frac{n}{r-1} + \frac{1}{r-2} + \frac{3}{2} < 0.
\]

If \( x \leq (n+r)/(r-1) \) then

\[
h'(x) \leq \frac{n+r}{(r-1)(r-2)} + \frac{1-n}{r-2} + \frac{3}{2} = -\frac{n}{r-1} + \frac{1}{r-2} + \frac{r}{(r-1)(r-2)} + \frac{3}{2}.
\]

If \( n \geq 3r \) then \( n > 3r/2 + 4 \). So \( n > 3(r-1)/2 + (2r-1)/(r-2) \). Thus,

\[
h'(x) \leq -\frac{n}{r-1} + \frac{1}{r-2} + \frac{r}{(r-1)(r-2)} + \frac{3}{2} < 0,
\]

as desired. \( \square \)
Proof of Proposition 13. We need to show that, for all $D \in \mathbb{N}$ such that $n/r \leq D \leq n/(r-1)$,

$$\frac{n^2}{2r^2} + \frac{n}{2r} = \left(\frac{n/r + 1}{2}\right) \leq D + e(\overline{T}(n - D - 1, r - 2)).$$

Since $D \leq n/(r-1)$, Facts 14 and 24 imply that

$$D + e(\overline{T}(n - D - 1, r - 2)) \geq D + \frac{(n - D - 1)^2}{2(r - 2)} - \frac{(n - D - 1)}{2} \geq \frac{n}{r - 1} + 1 + \frac{1}{2} \left[\frac{(r - 2)}{r - 1} n - 1\right]^2 - \frac{1}{2} \left[\frac{(r - 2)}{r - 1} n - 1\right] \geq \frac{(r - 2)}{2(r - 1)^2} n^2 - \frac{(r - 2)}{2(r - 1)} n.$$  

Thus, it suffices to show that

$$\frac{(r - 2)}{2(r - 1)^2} n^2 - \frac{r - 2}{2(r - 1)} \geq \frac{n}{2r^2} + \frac{1}{2r}.$$  

(16)

Notice that

$$\frac{r - 2}{2(r - 1)^2} - \frac{1}{2r^2} = \frac{(r - 2)^2 - (r - 1)^2}{2r^2(r - 1)^2} = \frac{r^3 - 3r^2 + 2r - 1}{2r^2(r - 1)^2}$$  

(17)

and

$$\frac{r - 2}{2(r - 1)^2} + \frac{1}{2r} = \frac{r^2 - r - 1}{2r(r - 1)}.$$  

Since $n \geq 2r$, (16) implies that it suffices to show that

$$\frac{r^3 - 3r^2 + 2r - 1}{r(r - 1)^2} - \frac{r^2 - r - 1}{2r(r - 1)} \geq 0.$$  

(18)

Note that $r^3 \geq 4r^2 - 4r + 3$ as $r \geq 3$. Thus, $2(r^3 - 3r^2 + 2r - 1) \geq (r^2 - r - 1)(r - 1)$. So indeed (18) is satisfied, as desired. □

Proof of Lemma 15. We need to show that, for all $D \in \mathbb{N}$ such that $n/r \leq D < (n+r)/(r-1)$,

$$\frac{n^2}{2r^2} + \frac{n}{2r} = \left(\frac{n/r + 1}{2}\right) \leq D + e(\overline{T}(n - D - 1, r - 2)).$$

Since $D < (n+r)/(r-1)$ we have that $D \leq n/(r-1) + 1$. So Facts 14 and 24 imply that

$$D + e(\overline{T}(n - D - 1, r - 2)) \geq D + \frac{(n - D - 1)^2}{2(r - 2)} - \frac{(n - D - 1)}{2} \geq \frac{n}{r - 1} + 1 + \frac{1}{2} \left[\frac{(r - 2)}{r - 1} n - 2\right]^2 - \frac{1}{2} \left[\frac{(r - 2)}{r - 1} n - 2\right] \geq \frac{(r - 2)}{2(r - 1)^2} n^2 - \frac{(r - 2)}{2(r - 1)} n - \frac{n}{r - 1}.$$  

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Thus, it suffices to show that
\[
\frac{(r - 2)}{2(r - 1)^2} n - \frac{(r - 2)}{2(r - 1)} - \frac{1}{r - 1} \geq \frac{n}{2r^2} + \frac{1}{2r}. \tag{19}
\]

Notice that
\[
\frac{r - 2}{2(r - 1)} + \frac{1}{r - 1} + \frac{1}{2r} = \frac{r^2 + r - 1}{2r(r - 1)}. \tag{18}
\]

Since \( n \geq 3r \), (17) and (19) imply that it suffices to show that
\[
\frac{3(r^3 - 3r^2 + 2r - 1)}{2r(r - 1)^2} - \frac{r^2 + r - 1}{2r(r - 1)} \geq 0. \tag{20}
\]

Note that \( 2r^3 - 9r^2 + 8r - 4 \geq 0 \) as \( r \geq 4 \). Thus, \( 3(r^3 - 3r^2 + 2r - 1) \geq (r^2 + r - 1)(r - 1) \). So indeed (20) is satisfied, as desired. \( \square \)