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On perfect packings in dense graphs

József Balogh,* Alexandr V. Kostochka† and Andrew Treglown‡

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Abstract

We say that a graph $G$ has a perfect $H$-packing if there exists a set of vertex-disjoint copies of $H$ which cover all the vertices in $G$. We consider various problems concerning perfect $H$-packings: Given $n, r, D \in \mathbb{N}$, we characterise the edge density threshold that ensures a perfect $K_r$-packing in any graph $G$ on $n$ vertices and with minimum degree $\delta(G) \geq D$. We also give two conjectures concerning degree sequence conditions which force a graph to contain a perfect $H$-packing. Other related embedding problems are also considered. Indeed, we give a structural result concerning $K_r$-free graphs that satisfy a certain degree sequence condition.

1 Introduction

Given two graphs $H$ and $G$, a perfect $H$-packing in $G$ is a collection of vertex-disjoint copies of $H$ which cover all the vertices in $G$. Perfect $H$-packings are also referred to as $H$-factors or perfect $H$-tilings. Hell and Kirkpatrick [10] showed that the decision problem whether a graph $G$ has a perfect $H$-packing is NP-complete precisely when $H$ has a component consisting of at least 3 vertices. So for such graphs $H$, it is unlikely that there is a complete characterisation of those graphs containing a perfect $H$-packing. Thus, there has been significant attention on obtaining sufficient conditions that ensure a graph $G$ contains a perfect $H$-packing.

A seminal result in the area is the Hajnal-Szemerédi theorem [9] which states that a graph $G$ whose order $n$ is divisible by $r$ has a perfect $K_r$-packing provided that $\delta(G) \geq (r-1)n/r$. Kühn and Osthus [15, 16] characterised, up to an additive constant, the minimum degree which ensures a graph $G$ contains a perfect $H$-packing for an arbitrary graph $H$.

It is easy to see that the minimum degree condition in the Hajnal-Szemerédi theorem cannot be lowered. Of course, this does not mean that one cannot strengthen this result. Ore-type degree conditions consider the sum of the degrees of non-adjacent vertices in a graph. The following Ore-type result of Kierstead and Kostochka [12] implies the Hajnal-Szemerédi theorem.

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Theorem 1 (Kierstead and Kostochka [12]) Let \( n, r \in \mathbb{N} \) such that \( r \) divides \( n \). Suppose that \( G \) is a graph on \( n \) vertices such that for all non-adjacent \( x \neq y \in V(G) \),
\[
d(x) + d(y) \geq 2(1 - 1/r)n - 1.
\]

Then \( G \) contains a perfect \( K_r \)-packing.

Kühn, Osthus and Treglown [17] characterised, asymptotically, the Ore-type degree condition which ensures a graph \( G \) contains a perfect \( H \)-packing for an arbitrary graph \( H \).

1.1 Perfect packings in dense graphs of low minimum degree

It is easy to characterise the edge density that forces a graph \( G \) to contain a perfect \( K_r \)-packing when there are no other restrictions. Indeed, given \( n, r \in \mathbb{N} \) such that \( r \geq 2 \) divides \( n \), if \( G \) is a graph on \( n \) vertices and \( e(G) \geq \binom{n}{2} - n + r \) then \( G \) contains a perfect \( K_r \)-packing. Moreover, if \( G \) is a copy \( K \) of \( K_{n-1} \) together with a vertex which sends precisely \( r - 2 \) edges to \( K \), then \( e(G) = \binom{n}{2} - n + r - 1 \) and \( G \) does not contain a perfect \( K_r \)-packing. The following result of Akiyama and Frankl [1] refines this observation.

Theorem 2 (Akiyama and Frankl [1]) Let \( n, r \in \mathbb{N} \) such that \( r \) divides \( n \). Suppose \( G \) is a graph on \( n \) vertices and \( e(G) \leq \min\{\binom{n}{r+1}, n - r + 1\} \). Then \( G \) has a perfect \( K_r \)-packing unless \( G \) is isomorphic to one of the following graphs:

(i) A copy of \( K_{n/r+1} \) together with \( (1 - 1/r)n - 1 \) isolated vertices;

(ii) The disjoint union of \( K_{1,n-r-j+1}, j \) edges and \( r-j-2 \) isolated vertices, for some \( 1 \leq j \leq r-2 \).

When (for example) \( n \geq r^3 \), \( \binom{n}{r+1} > n - r + 1 \). Hence, in this case Theorem 2 is equivalent to the following: If \( G \) is a graph on \( n \) vertices and \( e(G) \geq \binom{n}{2} - n + r - 1 \) then either \( G \) contains a perfect \( K_r \)-packing or \( \overline{G} \) is isomorphic to a graph as in (ii).

In Sections 2 and 3 we consider the following natural problem: Let \( n, r \in \mathbb{N} \) such that \( r \) divides \( n \). Given some \( D \in \mathbb{N} \), what edge density condition ensures that any graph \( G \) on \( n \) vertices and of minimum degree \( \delta(G) \geq D \) contains a perfect \( K_r \)-packing?

We fully resolve the problem, and our answers for \( r = 2 \) and \( r \geq 3 \) differ.

Theorem 3 For an even positive \( n \) and integer \( 1 \leq d < n/2 \), let \( h(n, d) := \binom{n-d-1}{2} + d(d+1) \) and let \( f(2, n, d) \) denote the maximum integer \( c \) such that some \( n \)-vertex graph with minimum degree at least \( d \) and at least \( c \) edges has no perfect matching. Then
\[
f(2, n, d) = \max\{h(n, d), h(n, 0.5n - 1)\}.
\]

Theorem 4 Let \( n, r \in \mathbb{N} \) such that \( r \geq 3 \) and \( r \) divides \( n \). Given any \( D \in \mathbb{N} \) such that \( r - 1 \leq D \leq (r-1)n/r - 1 \) define
\[
g(n, r, D) := \max\left\{\binom{n}{2} - \binom{n/r + 1}{2}, D(n - D) + \binom{n - 1 - D}{2} + e(T(D, r-2))\right\}.
\]

Suppose that \( G \) is a graph on \( n \) vertices with \( \delta(G) \geq D \) and \( e(G) > g(n, r, D) \). Then \( G \) contains a perfect \( K_r \)-packing. Moreover, there exists a graph \( G' \) on \( n \) vertices with \( \delta(G') \geq D \) and \( e(G') = g(n, r, D) \) but such that \( G' \) does not contain a perfect \( K_r \)-packing.
Clearly a graph $G$ of minimum degree $\delta(G) < r - 1$ cannot contain a perfect $K_r$-packing. Further, regardless of edge density, every graph $G$ whose order $n$ is divisible by $r$ and with $\delta(G) \geq (r - 1)n/r$ contains a perfect $K_r$-packing. Thus, Theorem 4 covers all values of $D$ where our problem was not solved previously.

An equitable $k$-colouring of a graph $G$ is a proper $k$-colouring of $G$ such that any two colour classes differ in size by at most one. Let $n, r \in \mathbb{N}$ such that $r$ divides $n$. Notice that a graph $G$ on $n$ vertices has a perfect $K_r$-packing if and only if the complement $\overline{G}$ of $G$ has an equitable $n/r$-colouring. So, for example, the Hajnal-Szemerédi theorem can be stated in terms of equitable colourings: Let $G$ be a graph on $n$ vertices such that $\Delta(G) \leq n/r - 1$ then $G$ has an equitable $n/r$-colouring.

It is often easier to work in the language of equitable colourings compared to perfect packings. Indeed, rather than prove Theorem 1 directly, Kierstead and Kostochka proved the equivalent statement for equitable colourings. Here we also find it more convenient to work with equitable colourings. Thus, instead of proving Theorem 4 directly we prove the following equivalent result.

**Theorem 5** Let $n, r \in \mathbb{N}$ such that $r \geq 3$ and $r$ divides $n$. Recall that $T(n, r)$ denotes the Turán graph. Given any $D \in \mathbb{N}$ such that $n/r \leq D \leq n - r$ define

$$f(n, r, D) := \min \left\{ \left( \frac{n/r + 1}{2} \right), D + e(T(n - D - 1, r - 2)) \right\}.$$

Suppose that $G$ is a graph on $n$ vertices with $\Delta(G) \leq D$ and $e(G) < f(n, r, D)$. Then $G$ has an equitable $n/r$-colouring. Moreover, there exists a graph $G'$ on $n$ vertices with $\Delta(G') \leq D$ and $e(G') = f(n, r, D)$ but such that $G'$ does not have an equitable $n/r$-colouring.

We prove Theorem 3 and describe extremal constructions for Theorems 4 and 5 in Section 2. That is, we show that the edge density condition in Theorem 4 is best possible for all values of $D$ such that $r - 1 \leq D \leq (r - 1)n/r - 1$. Section 3 contains a proof of Theorem 5.

### 1.2 Degree sequence conditions forcing a perfect packing

Chvátal [4] gave a condition on the degree sequence of a graph which ensures Hamiltonicity: Suppose that $G$ is a graph on $n$ vertices and that the degrees of the graph are $d_1 \leq \cdots \leq d_n$. If $n \geq 3$ and $d_i \geq i + 1$ or $d_{n-i} \geq n - i$ for all $i < n/2$ then $G$ is Hamiltonian. The following is a simple consequence of Chvátal’s theorem.

**Theorem 6 (Chvátal [4])** Suppose that $G$ is a graph on $n \geq 2$ vertices and the degrees of the graph are $d_1 \leq \cdots \leq d_n$. If

$$d_i \geq i \text{ or } d_{n-i+1} \geq n - i \text{ for all } 1 \leq i \leq n/2$$

then $G$ contains a Hamilton path.

We propose the following conjecture on the degree sequence of a graph which forces a perfect $K_r$-packing.

**Conjecture 7** Let $n, r \in \mathbb{N}$ such that $r$ divides $n$. Suppose that $G$ is a graph on $n$ vertices with degree sequence $d_1 \leq \cdots \leq d_n$ such that:
(α) $d_i \geq (r - 2)n/r + i$ for all $i < n/r$;

(β) $d_{n/r+1} \geq (r - 1)n/r$.

Then $G$ contains a perfect $K_r$-packing.

Note that Conjecture 7, if true, is much stronger than the Hajnal-Szemerédi theorem since the degree condition allows for $n/r$ vertices to have degree less than $(r - 1)n/r$. Further, Proposition 17 in Section 4 shows that the condition on the degree sequence in Conjecture 7 is essentially “best possible”. It is easy to see that Theorem 6 implies Conjecture 7 in the case when $r = 2$. We prove the conjecture in the case when $G$ is additionally $K_{r+1}$-free (see Section 5).

If one can prove Conjecture 7, it seems likely it can be used to prove the next conjecture.

**Conjecture 8** Suppose $\gamma > 0$ and $H$ is a graph with $\chi(H) = r$. Then there exists an integer $n_0 = n_0(\gamma, H)$ such that the following holds. If $G$ is a graph whose order $n \geq n_0$ is divisible by $|H|$, and whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies

- $d_i \geq (r - 2)n/r + i + \gamma n$ for all $i < n/r$,

then $G$ contains a perfect $H$-packing.

Since first submitting this paper, the third author and Knox [13] have proven Conjecture 8 in the case when $r = 2$. (In fact, they have proven a much more general result concerning embedding spanning bipartite graphs of small bandwidth.)

The following result of Erdős [8] characterises those degree sequences which force a copy of $K_r$ in a graph $G$.

**Theorem 9 (Erdős [8])** Let $G$ be a graph on $n$ vertices with degree sequence $d_1 \leq \cdots \leq d_n$. If $G$ is $K_{r+1}$-free then there is an $r$-partite graph $G'$ on $n$ vertices whose degree sequence $d'_1 \leq \cdots \leq d'_n$ satisfies $d_i \leq d'_i$ for all $i \leq n$.

In Section 6 we prove the following related structural theorem.

**Theorem 10** Suppose that $n, r \in \mathbb{N}$ such that $n \geq r$ and so that $r$ divides $n$. Let $G$ be a $K_{r+1}$-free graph on $n$ vertices whose degree sequence $d_1 \leq \cdots \leq d_n$ is such that $d_{n/r} \geq (r - 1)n/r$. Then $G \subseteq T(n, r)$, where $T(n, r)$ is the complete $r$-partite Turán graph on $n$ vertices; so each vertex class has size $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$.

## 2 The case $r = 2$ and extremal examples for $r \geq 3$

### 2.1 Perfect matchings in dense graphs

In this section we establish the density threshold that ensures every graph $G$ on an even number $n$ of vertices and of minimum degree $\delta(G) \geq d$ contains a perfect matching. Note that we only consider values of $d$ such that $1 \leq d < n/2$, since if $\delta(G) \geq n/2$ then $G$ has a perfect matching, regardless of the edge density.

Recall that $h(n, d) := \left(\frac{n-d-1}{2}\right) + d(d+1)$. Note that for a fixed even $n$, $h(n, d)$ decreases with $d$ in the interval $[0, n/3 - 5/6]$ and increases with $d$ in $[n/3 - 5/6, 0.5n - 1]$. 

For a positive even $n$ and an integer $0 \leq d < n/2$, let $A, B$ and $C$ be disjoint sets with $|A| = d+1$, $|B| = d$, $|C| = n - 2d - 1$. Let $H = H(n, d)$ be the graph with the vertex set $A \cup B \cup C$ such that $H[A \cup C] = K_{n-d-1}$, and each vertex in $A$ is adjacent to each vertex in $B$ and to no vertex in $C$. So $H$ does not contain a perfect matching and has exactly $h(n, d)$ edges.

The examples of $H(n, d)$ show that $f(2, n, d) \geq \max\{h(n, d), h(n, 0.5n - 1)\}$. Thus to derive Theorem 3, it suffices to prove that an $n$-vertex graph $G$ with $\delta(G) \geq d$ and $e(G) > \max\{h(n, d), h(n, 0.5n - 1)\}$ has a perfect matching.

Consider such a graph $G$. Let $d_1 \leq \cdots \leq d_n$ denote the degree sequence of $G$. If $d_i \geq i$ for all $1 \leq i \leq n/2$ then Theorem 6 implies that $G$ contains a perfect matching. Suppose for a contradiction that $d_{i'} \leq i' - 1$ for some $1 \leq i' \leq n/2$. Note that $i' > d$ as $\delta(G) \geq d$.

Let $A$ denote the set of $i'$ vertices in $G$ that correspond to the first $i'$ terms $d_1, \ldots, d_{i'}$ of the degree sequence. Set $B := V(G) \setminus A$. Then

$$e(G[B]) \geq e(G) - e(i') = \max\{h(n, d), h(n, 0.5n - 1)\} - e(i') > e(G[A]) = e\left(\frac{n - i'}{2}\right),$$

since $d(x) \leq i' - 1$ for all $x \in A$. Note that $\max\{h(n, d), h(n, 0.5n - 1)\} \geq h(n, i' - 1)$ since $d < i' \leq n/2$. Therefore,

$$e(G[B]) > \max\{h(n, d), h(n, 0.5n - 1)\} - e(i') > h(n, i' - 1) - e(i') = \left(\frac{n - i'}{2}\right),$$

a contradiction as $|B| = n - i'$. Thus, $d_i \geq i$ for all $1 \leq i \leq n/2$, as desired.

### 2.2 Examples for $r \geq 3$

We will give the extremal examples for Theorem 5. Since Theorems 4 and 5 are equivalent, the complements of the extremal graphs for Theorem 5 are the extremal graphs for Theorem 4.

**Proposition 11** Suppose that $n, r \in \mathbb{N}$ such that $r \geq 3$ and $r$ divides $n$. Then there exists a graph $G_1$ on $n$ vertices such that $\Delta(G_1) = n/r$,

$$e(G_1) = \left(\frac{n/r + 1}{2}\right),$$

but such that $G_1$ does not have an equitable $n/r$-colouring.

**Proof.** Let $G_1$ denote the disjoint union of a clique $V$ on $n/r + 1$ vertices and an independent set $W$ of $(1 - 1/r)n - 1$ vertices. So every independent set in $G_1$ contains at most one vertex from $V$. But since $|V| = n/r + 1$, $G_1$ does not have an equitable $n/r$-colouring. Further, $\Delta(G_1) = n/r$ and $e(G_1) = \left(\frac{n/r + 1}{2}\right)$.

**Proposition 12** Suppose that $n, r \in \mathbb{N}$ such that $r \geq 3$ and $n = kr$ for some $k \geq 2$. Further, let $D \in \mathbb{N}$ such that $n/(r - 1) \leq D \leq n - r$. Then there exists a graph $G_2$ on $n$ vertices such that $\Delta(G_2) = D$,

$$e(G_2) = D + e(T(n - D - 1, k - 2)),$$

but such that $G_2$ does not have an equitable $n/r$-colouring.
The shows that Theorem 5 is best possible in the case when 

\[ \frac{n}{r} | \text{So} \]

Proposition 13

Let 

\[ n/r \leq D \]

The following simple consequence of Turán’s theorem will be used in the proof of Theorem 5.

\[ e(G_2) = D + e(T(n - D - 1, r - 2)). \]

Since \( n/(r - 1) \leq D \) we have that \( n - 1 \leq (r - 1)D \). Thus, every vertex in the copy of \( T(n - D - 1, r - 2) \) has degree at most

\[ \left\lfloor \frac{n - D - 1}{r - 2} \right\rfloor - 1 \leq \frac{n - D - 1}{r - 2} \leq D. \]

This implies that \( \Delta(G_2) = D. \)

Clearly Propositions 11 and 12 show that one cannot lower the edge density condition in Theorem 5 in the case when \( n/(r - 1) \leq D \leq n - r \). The following result, together with Proposition 11, shows that Theorem 5 is best possible in the case when \( n/r \leq D \leq n/(r - 1) \).

**Proposition 13** Let \( n, r \in \mathbb{N} \) such that \( r \geq 3 \) and \( r \) divides \( n \geq 2r \). Suppose that \( D \in \mathbb{N} \) such that \( n/r \leq D \leq n/(r - 1) \). Then

\[ f(n, r, D) = \left( \frac{n/r + 1}{2} \right). \]

The following simple consequence of Turán’s theorem will be used in the proof of Theorem 5.

**Fact 14** Let \( n, r \in \mathbb{N} \) such that \( r \leq n \). Then

\[ e(T(n, r)) \leq \left( 1 - \frac{1}{r} \right) \frac{n^2}{2} \quad \text{and thus} \quad e(T(n, r)) \geq \frac{n^2}{2} - \frac{n}{2}. \]

We will also require the following easy result.

**Lemma 15** Let \( n, r \in \mathbb{N} \) such that \( r \geq 4 \) and \( r \) divides \( n \geq 3r \). Suppose that \( D \in \mathbb{N} \) such that \( n/r \leq D < (n + r)/(r - 1) \). Then

\[ f(n, r, D) = \left( \frac{n/r + 1}{2} \right). \]

3 Proof of Theorem 5

3.1 Preliminaries

Suppose for a contradiction that the result is false. Let \( G \) be a counterexample with the fewest vertices. That is, \( n = |V(G)| = rk \) for some \( k \in \mathbb{N}, \Delta(G) \leq D \) for some \( D \in \mathbb{N} \) such that \( n/r \leq D \leq n - r \), \( e(G) < f(n, r, D) \) and \( G \) has no equitable \( n/r \)-colouring. By the Hajnal-Szemerédi theorem, \( \Delta(G) \geq n/r \). Notice that given fixed \( n \) and \( r \), \( f(n, r, D) \) is non-increasing with respect to \( D \). Thus, we may assume that \( \Delta(G) = D. \)

We first show that \( k \geq 4 \). Indeed, if \( n = 2r \) then \( f(n, r, D) \leq \left( \frac{3}{2} \right) = 3 \). But it is easy to see that every graph \( G_1 \) on \( 2r \) vertices and with \( e(G_1) \leq 2 \) has an equitable 2-colouring. If \( n = 3r \) then \( f(n, r, D) \leq \left( \frac{4}{3} \right) = 6 \). Consider any graph \( G_1 \) on \( 3r \) vertices with \( e(G_1) \leq 5 \) and \( 3 \leq \Delta(G_1) \leq 5 \). Let \( x \) denote the vertex in \( G_1 \) where \( d_{G_1}(x) = \Delta(G_1) \). Since \( 3 \leq d_{G_1}(x) \leq 5 \), \( x \) lies in an independent
Thus, \( G_1 - I \) contains \( 2r \) vertices and at most \( 2 \) edges. So \( G_1 - I \) has an equitable 2-colouring and hence \( G_1 \) has an equitable 3-colouring.

Let \( v \in V(G) \) such that \( d_G(v) = D \). Set \( G^* := G - (N_G(v) \cup \{v\}) \). Since \( f(n, r, D) \leq D + e(T(n - D - 1, r - 2)) \) we have that \( e(G^*) < e(T(n - D - 1, r - 2)) \). Thus, by Turán’s theorem, \( G^* \) contains an independent set of size \( r - 1 \). Hence, \( v \) lies in an independent set in \( G \) of size \( r \).

Amongst all such independent sets of size \( r \) that contain \( v \), choose a set \( I = \{v, x_1, \ldots, x_{r-1}\} \) such that \( d_G(x_1) + \cdots + d_G(x_{r-1}) \) is maximised.

Set \( G' := G - I \), \( n' := |V(G')| = n - r \) and \( D' := \Delta(G') \leq D \). Notice that \( D' \geq n'/r \). (Indeed, if not, then by the Hajnal-Szemerédi theorem \( G' \) contains an equitable \( n'/r \)-colouring. Thus, as \( I \) is an independent set in \( G \) this gives us an equitable \( n/r \)-colouring of \( G \), a contradiction.) Furthermore, \( D' \leq n' - r \). If not then

\[
e(G) \geq D + D' \geq 2D' \geq 2(n' - r + 1) = 2n - 4r + 2
\]

and further,

\[
e(G) < f(n, r, D) \leq f(n, r, n - 2r + 1) \leq (n - 2r + 1) + e(T(2r - 2, r - 2)) \\
\leq (n - 2r + 1) + (r + 3) = n - r + 4.
\]

Therefore, \( 2n - 4r + 2 < n - r + 4 \) and so \( n < 3r + 2 \) a contradiction since \( n = kr \geq 4r \).

Since \( n'/r \leq D' \leq n' - r \), if \( e(G') < f(n', r, D') \) then the minimality of \( G \) implies that \( G' \) has an equitable \( n'/r \)-colouring. This then implies that \( G \) has an equitable \( n/r \)-colouring, a contradiction. Thus,

\[
e(G') \geq f(n', r, D'). \tag{1}
\]

We now split our argument into three cases.

3.2 Case 1: \( f(n', r, D') = \binom{n'/r+1}{2} \).

By (1), \( e(G') \geq \binom{n'/r+1}{2} = \binom{n/r}{2} \). Since \( d_G(v) = D \geq n/r \),

\[
e(G) \geq \frac{n}{r} + \binom{n/r}{2} = \binom{n/r+1}{2} \geq f(n, r, D),
\]

a contradiction, as desired.

3.3 Case 2: \( D' \leq D - 1 \) and \( f(n', r, D') = D' + e(T(n' - D' - 1, r - 2)) \).

The following claim will be useful.

Claim 16 \( D' < \frac{r-1}{2r-3}n - \frac{(r^2-r+1)}{2r-3} \).

**Proof.** Note that

\[
D + D' + e(T(n' - D' - 1, r - 2)) \leq e(G) < f(n, r, D) \leq D + e(T(n - D - 1, r - 2)). \tag{2}
\]
Since $D' \leq D - 1$, clearly $e(\overline{T}(n' - D, r - 2)) \leq e(\overline{T}(n' - D' - 1, r - 2))$. Thus, (2) implies that
\[ D' + e(\overline{T}(n' - D, r - 2)) < e(\overline{T}(n - D - 1, r - 2)). \] (3)
One can obtain $\overline{T}(n - D - 1, r - 2)$ from $\overline{T}(n' - D, r - 2)$ by adding $r - 1$ vertices and at most
\[ (n' - D) + \frac{n - D - 2}{r - 2} \] edges. (4)
Hence (3) and (4) give
\[ D' < n' - D + \frac{n - D - 2}{r - 2}. \]
Rearranging, and using that $D' \leq D - 1$ and $n' = n - r$ we get that
\[ \left(2 + \frac{1}{r - 2}\right)D' < \left(1 + \frac{1}{r - 2}\right)n - \frac{(r^2 - r + 1)}{r - 2}. \]
Thus,
\[ D' < \frac{r - 1}{2r - 3}n - \frac{(r^2 - r + 1)}{2r - 3}, \]
as desired. □

Since we are in Case 2 we have that
\[ D' + e(\overline{T}(n - r - D' - 1, r - 2)) \leq \binom{n'/r + 1}{2} = \binom{n/r}{2}. \] (5)
Notice that for fixed $n$ and $r$, $D' + e(\overline{T}(n - r - D' - 1, r - 2))$ is non-increasing as $D'$ increases. Hence, (5) and Claim 16 imply that
\[ D'' + e(\overline{T}(n - r - D'' - 1, r - 2)) \leq \frac{n^2}{2r^2} - \frac{n}{2r} \] (6)
where $D'' := [(r - 1)n/(2r - 3) - (r^2 - r + 1)/(2r - 3)].$ Note that
\[ n - r - \frac{r - 1}{2r - 3}n + \frac{(r^2 - r + 1)}{2r - 3} - 1 = \frac{r - 2}{2r - 3}n + \frac{4 - r^2}{2r - 3}. \]
So Fact 14 and (6) imply that
\begin{align*}
&\left(\frac{r - 1}{2r - 3}n - \frac{(r^2 - r + 1)}{2r - 3} - \frac{(2r - 4)}{2r - 3}\right) + \frac{1}{2(r - 2)} \left(\frac{r - 2}{2r - 3}n + \frac{4 - r^2}{2r - 3}\right)^2 \\
&\quad - \frac{1}{2} \left(\frac{r - 2}{2r - 3}n + \frac{4 - r^2}{2r - 3}\right) \leq \frac{n^2}{2r^2} - \frac{n}{2r}.
\end{align*}
Next we will move all terms from the previous equation to the left hand side and simplify. The coefficient of $n^2$ is
\[ \frac{r - 2}{2(2r - 3)^2} - \frac{1}{2r^2} = \frac{r^3 - 6r^2 + 12r - 9}{2r^2(2r - 3)^2}. \] (7)
The coefficient of $n$ is
\[
\frac{r - 1}{2r - 3} - \frac{(r - 2)}{2(2r - 3)} + \frac{1}{2r} + \frac{(4 - r^2)}{2(2r - 3)^2} = \frac{r^2 - 4r + 9}{2r(2r - 3)^2}. \tag{8}
\]
The constant term is
\[
-\frac{(r^2 + r - 3)}{2r - 3} + \frac{(r^2 - 4)^2}{2(r - 2)(2r - 3)^2} + \frac{(r^2 - 4)}{2(2r - 3)} = \frac{-r^4 + 3r^3 + 4r^2 - 26r + 28}{2(r - 2)(2r - 3)^2}. \tag{9}
\]
Since $n \geq 4r$, (7)–(9) imply that
\[
\frac{8(r^3 - 6r^2 + 12r - 9)}{(2r - 3)^2} + \frac{2(r^2 - 4r + 9)}{(2r - 3)^2} + \frac{-r^4 + 3r^3 + 4r^2 - 26r + 28}{2(r - 2)(2r - 3)^2} \leq 0. \tag{10}
\]
Multiplying (10) by $2(r - 2)(2r - 3)^2$ we get
\[15r^4 - 121r^3 + 364r^2 - 486r + 244 \leq 0\]
This yields a contradiction, since it is easy to check that
\[15r^4 - 121r^3 + 364r^2 - 486r + 244 > 0\]
for all $r \in \mathbb{N}$ such that $r \geq 3$.

### 3.4 Case 3: $D' = D$ and $f(n', r, D') = D' + e(T(n' - D' - 1, r - 2))$.

By (1) we have that
\[e(G') \geq f(n', r, D') = D' + e(T(n' - D' - 1, r - 2)). \tag{11}\]
Consider any vertex $x \in V(G')$ such that $d_{G'}(x) = D' = D$. Since $\Delta(G) = D$, $x$ is not adjacent to any vertex in $I = \{v, x_1, \ldots, x_r\}$. Further, $I$ was chosen such that $d_G(x_1) + \cdots + d_G(x_r)$ is maximised. Thus, $d_G(x_1) = \cdots = d_G(x_{r-1}) = D$. Together with (11) this implies that
\[e(G) \geq (r + 1)D + e(T(n' - D - 1, r - 2)). \tag{12}\]
Since $e(G) < f(n, r, D) \leq D + e(T(n - D - 1, r - 2))$, (12) implies that
\[rD + e(T(n' - D - 1, r - 2)) < e(T(n - D - 1, r - 2)). \tag{13}\]
One can obtain $T(n - D - 1, r - 2)$ from $T(n' - D - 1, r - 2)$ by adding $r$ vertices and at most
\[(n' - D - 1) + \frac{2(n - D - 3)}{r - 2} + 1 \text{ edges.} \tag{14}\]
Thus, (13) and (14) imply that
\[rD < n - r - D + \frac{2(n - D - 3)}{r - 2} \]
and so
\[
\left(r + 1 + \frac{2}{r - 2}\right) D < \left(1 + \frac{2}{r - 2}\right) n + \frac{(-r^2 + 2r - 6)}{r - 2} < \left(1 + \frac{2}{r - 2}\right) n.
\]
(15)
If \( r = 3 \) then (15) implies that
\[
D < \frac{n}{2}.
\]
Since \( f(n', 3, D) = \min\left(\binom{n'+1}{2}, D + \binom{n'-D-1}{2}\right) \) it is easy to see that if \( f(n', 3, D) = D + \binom{n'-D-1}{2} \) then \( D \geq 2n'/3 + 1 = 2n/3 - 1 \). Thus, \( 2n/3 - 1 \leq D < n/2 \), a contradiction since \( n \geq 4r = 12 \).

If \( r \geq 4 \) then (15) implies that
\[
D < \frac{n}{r - 1} = \frac{n'}{r - 1} + \frac{r}{r - 1}.
\]
Since \( n' \geq 3r \), Lemma 15 implies that \( f(n', r, D') = \binom{n'/r+1}{2} \) and so we are in Case 1, which we have already dealt with.

4 The extremal examples for Conjecture 7

**Proposition 17** Suppose that \( n, r, k \in \mathbb{N} \) such that \( r \geq 2 \) divides \( n \) and \( 1 \leq k < n/r \). Then there exists a graph \( G \) on \( n \) vertices whose degree sequence \( d_1 \leq \cdots \leq d_n \) satisfies

- \( d_i = (r - 2)n/r + k - 1 \) for all \( 1 \leq i \leq k \);
- \( d_i = (r - 1)n/r \) for all \( k + 1 \leq i \leq (r - 2)n/r + k \);
- \( d_i = n - k - 1 \) for all \( (r - 2)n/r + k + 1 \leq i \leq n - k + 1 \);
- \( d_i = n - 1 \) for all \( n - k + 2 \leq i \leq n \),

but such that \( G \) does not contain a perfect \( K_r \)-packing.

**Proof.** Let \( G' \) denote the complete \( (r - 2) \)-partite graph whose vertex classes \( V_1, \ldots, V_{r-2} \) each have size \( n/r \). Obtain \( G \) from \( G' \) by adding the following vertices and edges: Add a set \( V_{r-1} \) of \( 2n/r - 2k + 1 \) vertices to \( G' \), a set \( V_r \) of \( k - 1 \) vertices and a set \( V_0 \) of \( k \) vertices. Add all edges from \( V_0 \cup V_{r-1} \cup V_r \) to \( V_1 \cup \cdots \cup V_{r-2} \). Further, add all edges with both endpoints in \( V_{r-1} \cup V_r \). Add all possible edges between \( V_0 \) and \( V_r \).

So \( V_0 \) is an independent set, and there are no edges between \( V_0 \) and \( V_{r-1} \). This implies that any copy of \( K_r \) in \( G \) containing a vertex from \( V_0 \) must also contain at least one vertex from \( V_r \). But since \( |V_0| > |V_r| \) this implies that \( G \) does not contain a perfect \( K_r \)-packing. Furthermore, \( G \) has our desired degree sequence. \( \square \)

Notice that the graphs \( G \) considered in Proposition 17 satisfy \((\beta)\) from Conjecture 7 and only fail to satisfy \((\alpha)\) in the case when \( i = k \) (and in this case \( d_k = (r - 2)n/r + k - 1 \)).

Let \( n, r \in \mathbb{N} \) such that \( r \) divides \( n \). Denote by \( T^*(n, r) \) the complete \( r \)-partite graph on \( n \) vertices with \( r - 2 \) vertex classes of size \( n/r \), one vertex class of size \( n/r - 1 \) and one vertex class of size \( n/r + 1 \). Then \( T^*(n, r) \) does not contain a perfect \( K_r \)-packing. Furthermore, \( T^*(n, r) \) satisfies \((\alpha)\) but condition \((\beta)\) fails; we have that \( d_{n/r+1} = (r - 1)n/r - 1 \) here. Thus, together \( T^*(n, r) \) and Proposition 17 show that, if true, Conjecture 7 is essentially best possible.
5 A special case of Conjecture 7

We now give a simple proof of Conjecture 7 in the case when \( G \) is \( K_{r+1} \)-free.

**Theorem 18** Let \( n, r \in \mathbb{N} \) such that \( r \geq 2 \) divides \( n \). Suppose that \( G \) is a graph on \( n \) vertices with degree sequence \( d_1 \leq \cdots \leq d_n \) such that:

- \( d_i \geq (r - 2)n/r + i \) for all \( i < n/r \);
- \( d_{n/r+1} \geq (r - 1)n/r \).

Further suppose that no vertex \( x \in V(G) \) of degree less than \((r - 1)n/r \) lies in a copy of \( K_{r+1} \). Then \( G \) contains a perfect \( K_r \)-packing.

**Proof.** We prove the theorem by induction on \( n \). In the case when \( n = r \) then \( d_{n/r+1} = d_2 \geq (r - 1)r/r = r - 1 \). This implies that every vertex in \( G \) has degree \( r - 1 \). Hence \( G = K_r \) as desired. So suppose that \( n > r \) and the result holds for smaller values of \( n \). Let \( x_1 \in V(G) \) such that \( d_G(x_1) = d_1 \geq (r - 2)n/r + 1 \). If \( d_G(x_1) \geq (r - 1)n/r \) then \( \delta(G) \geq (r - 1)n/r \). Thus \( G \) contains a perfect \( K_r \)-packing by the Hajnal-Szemerédi theorem. So we may assume that \((r - 2)n/r + 1 \leq d_G(x_1) < (r - 1)n/r \). In particular, \( x_1 \) does not lie in a copy of \( K_{r+1} \). We first find a copy of \( K_r \) containing \( x_1 \). If \( r = 2 \), \( x_1 \) has a neighbour and so we have our desired copy of \( K_2 \). So assume that \( r \geq 3 \). Certainly \( N_G(x_1) \) contains a vertex \( x_2 \) such that \( d_G(x_2) \geq (r - 1)n/r \). Thus, \( |N_G(x_1) \cap N_G(x_2)| \geq (r - 3)n/r + 1 > 0 \). So if \( r = 3 \) we obtain our desired copy of \( K_r \). Otherwise, we can find a vertex \( x_3 \in N_G(x_1) \cap N_G(x_2) \) such that \( d_G(x_3) \geq (r - 1)n/r \). We can repeat this argument until we have obtained vertices \( x_1, \ldots, x_r \) that together form a copy \( K'_r \) of \( K_r \).

Let \( G' := G - V(K'_r) \) and set \( n' := n - r = |V(G')| \). Since \( G \) does not contain a copy of \( K_{r+1} \) containing \( x_1 \), every vertex \( x \in V(G') \) sends at most \( r - 1 \) edges to \( K'_r \) in \( G \). Thus, \( d_G'(x) \geq d_G(x) - (r - 1) \) for all \( x \in V(G') \). So if \( d_G(x) \geq (r - 1)n/r \) then \( d_G'(x) \geq (r - 1)n/r - (r - 1) = (r - 1)n'/r \) for all \( x \in V(G') \). If a vertex \( y \in V(G') \) does not lie in a copy of \( K_{r+1} \) in \( G \) then clearly \( y \) does not lie in a copy of \( K_{r+1} \) in \( G' \). This means that no vertex \( y \in V(G') \) of degree less than \((r - 1)n'/r \) lies in a copy of \( K_{r+1} \).

Let \( d'_1 \leq \cdots \leq d'_{n'} \) denote the degree sequence of \( G' \). It is easy to check that \( d'_i \geq (r - 2)n'/r + i \) for all \( i < n'/r \) and that \( d'_{n'/r+1} \geq (r - 1)n'/r \). Indeed, since \( x_1 \in V(K'_r) \) where \( d_G(x_1) = d_1 \), we have that \( d'_i \geq d_{i+1} - (r - 1) \) for all \( 1 \leq i \leq n' \). Thus, for all \( 1 \leq i < n'/r = n/r - 1 \), \( d'_i \geq d_{i+1} - (r - 1) \geq (r - 2)n/r + (i + 1) - (r - 1) = (r - 2)n'/r + i \). Similarly, \( d'_{n'/r+1} = d'_{n/r} \geq d_{n/r+1} - (r - 1) \geq (r - 1)n/r - (r - 1) = (r - 1)n'/r \). Hence, by induction \( G' \) contains a perfect \( K_r \)-packing. Together with \( K'_r \) this gives us our desired perfect \( K_r \)-packing in \( G \). \( \square \)

6 Proof of Theorem 10

Consider any \( x_1 \in V(G) \) such that \( d_G(x_1) \geq (r - 1)n/r \). Since \( d_{n/r} \geq (r - 1)n/r \) we can greedily select vertices \( x_2, \ldots, x_{r-1} \) such that

- \( x_1, \ldots, x_{r-1} \) induce a copy of \( K_{r-1} \) in \( G \);
- \( d_G(x_i) \geq (r - 1)n/r \) for all \( 1 \leq i \leq r - 1 \).
Note that since $G$ is $K_{r+1}$-free, $\cap_{i=1}^{r-1} N_G(x_i)$ is an independent set. The choice of $x_1, \ldots, x_{r-1}$ implies that $|\cap_{i=1}^{r-1} N_G(x_i)| \geq n/r$. Let $V_1$ denote a subset of $\cap_{i=1}^{r-1} N_G(x_i)$ of size $n/r$. Thus $V_1$ contains a vertex $x_1^1$ of degree at least $(r-1)n/r$.

As before we can find vertices $x_2^1, \ldots, x_{r-1}^1$ such that

- $x_1^1, \ldots, x_{r-1}^1$ induce a copy of $K_{r-1}$ in $G$;
- $d_G(x_i^1) \geq (r-1)n/r$ for all $1 \leq i \leq r-1$.

So $\cap_{i=1}^{r-1} N_G(x_i^1)$ is an independent set of size at least $n/r$. Let $V_2$ denote a subset of $\cap_{i=1}^{r-1} N_G(x_i^1)$ of size $n/r$. Note that $N_G(x_1^1) \cap V_1 = \emptyset$ since $x_1^1 \notin V_1$ and $V_1$ is an independent set. Thus as $V_2 \subseteq N_G(x_1^1)$, $V_1 \cap V_2 = \emptyset$.

Our aim is to find disjoint sets $V_1, \ldots, V_r \subseteq V(G)$ of size $n/r$ and vertices $x_1^1, \ldots, x_{r-1}^1, x_2^1, \ldots, x_{r-1}^1, \ldots, x_1^r, \ldots, x_{r-1}^r$ with the following properties:

- $G[V_j]$ is an independent set for all $1 \leq j \leq r$;
- Given any $1 \leq j \leq r-1$, $x_k^j \in V_k$ for each $1 \leq k \leq j$;
- $d_G(x_k^j) \geq (r-1)n/r$ for all $1 \leq j \leq r-1$ and $1 \leq k \leq r-1$;
- $x_1^j, \ldots, x_{r-1}^j$ induce a copy of $K_{r-1}$ in $G$ for all $1 \leq j \leq r-1$.

Clearly finding such a partition $V_1, \ldots, V_r$ of $V(G)$ implies that $G \subseteq T(n,r)$.

Suppose that for some $1 \leq j < r$ we have defined sets $V_1, \ldots, V_j$ and vertices $x_1^1, \ldots, x_{r-1}^1, x_1^{j-1}, \ldots, x_{r-1}^{j-1}$ with our desired properties. Since $d_{n/r} \geq (r-1)n/r$ and $V_1, \ldots, V_j$ are independent sets of size $n/r$ we can choose vertices $x_1^j, \ldots, x_j^j$ such that for all $1 \leq k \leq j$,

- $x_k^j \in V_k$ and $d_G(x_k^j) \geq (r-1)n/r$.

This degree condition, together with the fact that $x_1^j, \ldots, x_j^j$ lie in different vertex classes, implies that these vertices form a copy of $K_j$ in $G$. We now greedily select further vertices $x_{j+1}^j, \ldots, x_{r-1}^j$ such that

- $x_1^j, \ldots, x_{r-1}^j$ induce a copy of $K_{r-1}$ in $G$;
- $d_G(x_k^j) \geq (r-1)n/r$ for all $j+1 \leq k \leq r-1$.

So $\cap_{i=1}^{r-1} N_G(x_i^j)$ is an independent set of size at least $n/r$. Let $V_{j+1}$ denote a subset of $\cap_{i=1}^{r-1} N_G(x_i^j)$ of size $n/r$. Note that, for each $1 \leq k \leq j$, $N_G(x_k^j) \cap V_k = \emptyset$ since $x_k^j \notin V_k$ and $V_k$ is an independent set. Thus as $V_{j+1} \subseteq N_G(x_k^j)$ for each $1 \leq k \leq j$, $V_{j+1}$ is disjoint from $V_1 \cup \cdots \cup V_j$.

Repeating this argument we obtain our desired sets $V_1, \ldots, V_r \subseteq V(G)$ and vertices $x_1^1, \ldots, x_{r-1}^1, x_2^1, \ldots, x_{r-1}^1, \ldots, x_1^r, \ldots, x_{r-1}^r$.

7 Possible extensions of Conjecture 7

We suspect that the following ‘Chvátal-type’ degree sequence condition forces a graph to contain a perfect $K_r$-packing.
Question 19 Let $n, r \in \mathbb{N}$ such that $r \geq 2$ divides $n$. Suppose that $G$ is a graph on $n$ vertices with degree sequence $d_1 \leq \cdots \leq d_n$ such that for all $i \leq n/r$:

- $d_i \geq (r - 2)n/r + i$ or $d_{n-i(r-1)+1} \geq n-i$.

Does this condition imply that $G$ contains a perfect $K_r$-packing?

Note that Theorem 6 answers this question in the affirmative when $r = 2$. The following example shows that we cannot have a lower value in the second part of the condition in Question 19.

Proposition 20 Suppose that $n, r, k \in \mathbb{N}$ such that $r \geq 2$ divides $n$ and $1 \leq k \leq n/r$. Then there exists a graph $G$ on $n$ vertices whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies

- $d_{n-i(r-1)+1} \geq n-i$ for all $i \in \lfloor n/r \rfloor \setminus \{k\}$;
- $d_{n-k(r-1)+1} = n-k-1$,

but such that $G$ does not contain a perfect $K_r$-packing.

Proof. Let $G$ be the graph on $n$ vertices with vertex classes $V_1, V_2$ and $V_3$ of sizes $k$, $(r-1)k - 1$ and $n - rk + 1$ respectively and with the following edges: There are all possible edges between $V_1$ and $V_2$ and between $V_2$ and $V_3$. Further add all possible edges in $V_2$ and all edges in $V_3$. Thus, $V_1$ is an independent set and there are no edges between $V_1$ and $V_3$.

The degree sequence of $G$ is

$$\underbrace{(r-1)k-1, \ldots, (r-1)k-1}_{k \text{ times}}, \underbrace{n-k-1, \ldots, n-k-1}_{n-rk+1 \text{ times}}, \underbrace{n-1, \ldots, n-1}_{(r-1)k-1 \text{ times}}.$$

Hence $G$ satisfies our desired degree sequence condition. Every copy $K'_r$ of $K_r$ in $G$ that contains a vertex from $V_1$ must contain $r-1$ vertices from $V_2$. But since $|V_1|(r-1) > |V_2|$ this implies that $G$ does not contain a perfect $K_r$-packing. \hfill \Box

The $r$th power of a Hamilton cycle $C$ is obtained from $C$ by adding an edge between every pair of vertices of distance at most $r$ on $C$. Seymour [18] conjectured the following strengthening of Dirac’s theorem.

Conjecture 21 (Pósa-Seymour, see [18]) Let $G$ be a graph on $n$ vertices. If $\delta(G) \geq \frac{r}{r+1} n$ then $G$ contains the $r$th power of a Hamilton cycle.

Pósa (see [7]) had earlier proposed the conjecture in the case of the square of a Hamilton cycle (that is, when $r = 2$). Komlós, Sárközy and Szemerédi [14] proved Conjecture 21 for graphs whose order is sufficiently larger than $r$. More recently, Cháu, DeBiasio and Kierstead [3] proved Pósa’s conjecture for graphs of order at least $2 \times 10^8$.

In the case when $r + 1$ divides $|G|$, a necessary condition for a graph $G$ to contain the $r$th power of a Hamilton cycle is that $G$ contains a perfect $K_{r+1}$-packing. Further, notice that the minimum degree condition in Conjecture 21 is the same as the condition in the Hajnal-Szemerédi theorem with respect to perfect $K_{r+1}$-packings. Thus an obvious question is whether the condition in Conjecture 7 forces a graph to contain the $(r-1)$th power of a Hamilton cycle. Interestingly though, when $r = 3$, this is not the case.
Proposition 22 Suppose that $C, n \in \mathbb{N}$ such that $C \ll n$ and 3 divides $n$. Then there exists a graph $G$ whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies

$$d_i \geq \frac{n}{3} + C + i \text{ for all } 1 \leq i \leq \frac{n}{3}$$

but such that $G$ does not contain the square of a Hamilton cycle.

Proof. Choose $C, K, n \in \mathbb{N}$ so that $C \ll K \ll n$. Let $G$ denote the graph on $n$ vertices consisting of three vertex classes $V_1 = \{v\}$, $V_2$ and $V_3$ where $|V_2| = n/3 + C + 1$ and $|V_3| = 2n/3 - C - 2$ which contains the following edges:

- All edges from $v$ to $V_2$;
- All edges between $V_2$ and $V_3$ and all possible edges in $V_3$;
- There are $K$ vertex-disjoint stars in $V_2$, each of size $\lceil |V_2|/K \rceil$, $\lfloor |V_2|/K \rfloor$, which cover all of $V_2$ (see Figure 1).

Let $d_1 \leq \cdots \leq d_n$ denote the degree sequence of $G$. There are $n/3 + C - K + 1 \leq n/3 - 2C - 1$ vertices in $V_2$ of degree $2n/3 - C$. Since $C \ll K \ll n$, the remaining $K$ vertices in $V_2$ have degree at least $2n/3 - C - 2 + \lceil |V_2|/K \rceil \geq 2n/3 + C + 1$. Since $d_G(v) = n/3 + C + 1$ and $d_G(x) = n - 2$ for all $x \in V_3$, we have that $d_i \geq \frac{n}{3} + C + i$ for all $1 \leq i \leq \frac{n}{3}$.

A necessary condition for a graph $G$ to contain the square of a Hamilton cycle is that, for every $x \in V(G)$, $G[N(x)]$ contains a path of length 3. Note that $N(v) = V_2$ and $G[V_2]$ does not contain a path of length 3. So $G$ does not contain the square of a Hamilton cycle. \hfill \square

Notice that we can set $C = o(\sqrt{n})$ in Proposition 22. We finish by raising the following question.

Question 23 What can be said about degree sequence conditions which force a graph to contain the $r$th power of a Hamilton cycle? In particular, can one establish a degree sequence condition that ensures a graph $G$ on $n$ vertices contains the $r$th power of a Hamilton cycle and which allows for “many” vertices of $G$ to have degree “much less” than $rn/(r+1)$?
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References


Appendix

Here we give proofs of Proposition 13 and Lemma 15. The following fact will be used in both of these proofs.

**Fact 24** Fix $n, r \in \mathbb{N}$ such that $r \geq 3$ and $r$ divides $n \geq 2r$. Define

$$ h(x) := x + \frac{(n-x-1)^2}{2(r-2)} - \frac{1}{2}(n-x-1). $$

Then $h(x)$ is a decreasing function for $x \in [0, n/(r-1)]$. Moreover, if $n \geq 3r$ then $h(x)$ is a decreasing function for $x \in [0, (n+r)/(r-1)]$.

**Proof.** Notice that

$$ h'(x) = \frac{3}{2} = \frac{n}{(r-1)(r-2)} + \frac{1-n}{r-2} + \frac{3}{2} = -\frac{n}{r-1} + \frac{1}{r-2} + \frac{3}{2}. $$

So for $x \leq n/(r-1),$

$$ h'(x) \leq -\frac{n}{r-1} + \frac{1}{r-2} + \frac{3}{2} < 0. $$

Note that $3(r-1)/2 + (r-1)/(r-2) < n$ since $n \geq 2r$ and $r \geq 3$. Thus,

$$ h'(x) \leq -\frac{n}{r-1} + \frac{1}{r-2} + \frac{3}{2} < 0. $$

If $x \leq (n+r)/(r-1)$ then

$$ h'(x) \leq \frac{n+r}{(r-1)(r-2)} + \frac{1-n}{r-2} + \frac{3}{2} = -\frac{n}{r-1} + \frac{1}{r-2} + \frac{r}{(r-1)(r-2)} + \frac{3}{2}. $$

If $n \geq 3r$ then $n > 3r/2 + 4$. So $n > 3(r-1)/2 + (2r-1)/(r-2)$. Thus,

$$ h'(x) \leq -\frac{n}{r-1} + \frac{1}{r-2} + \frac{r}{(r-1)(r-2)} + \frac{3}{2} < 0, $$

as desired. \qed
Proof of Proposition 13. We need to show that, for all \( D \in \mathbb{N} \) such that \( n/r \leq D \leq n/(r-1) \),

\[
\frac{n^2}{2r^2} + \frac{n}{2r} = \left( \frac{n/r + 1}{2} \right) \leq D + e(T(n-D-1, r-2)).
\]

Since \( D \leq n/(r-1) \), Facts 14 and 24 imply that

\[
D + e(T(n-D-1, r-2)) \geq D + \frac{(n-D-1)^2}{2(r-2)} - \frac{(n-D-1)}{2} \\
\geq \frac{n}{r-1} + \frac{1}{2(r-2)} \left[ \frac{(r-2)}{r-1} n - 1 \right]^2 - \frac{1}{2} \left[ \frac{(r-2)}{r-1} n - 1 \right] \\
\geq \frac{(r-2)}{2(r-1)^2} n^2 - \frac{(r-2)}{2(r-1)} n.
\]

Thus, it suffices to show that

\[
\frac{(r-2)}{2(r-1)^2} n^2 - \frac{r-2}{2(r-1)} n \geq \frac{n}{2r^2} + \frac{1}{2r}.
\]

Notice that

\[
\frac{r-2}{2(r-1)^2} - \frac{1}{2r^2} = \frac{(r-2)^2 - (r-1)^2}{2r^2(r-1)^2} = \frac{r^3 - 3r^2 + 2r - 1}{2r^2(r-1)^2}
\]

and

\[
\frac{r-2}{2(r-1)} + \frac{1}{2r} = \frac{r^2 - r - 1}{2r(r-1)}.
\]

Since \( n \geq 2r \), (16) implies that it suffices to show that

\[
\frac{r^3 - 3r^2 + 2r - 1}{r(r-1)^2} - \frac{r^2 - r - 1}{2r(r-1)} \geq 0.
\]

Note that \( r^3 \geq 4r^2 - 4r + 3 \) as \( r \geq 3 \). Thus, \( 2(r^3 - 3r^2 + 2r - 1) \geq 2(r^2 - r - 1)(r-1) \). So indeed (18) is satisfied, as desired.

\[\Box\]

Proof of Lemma 15. We need to show that, for all \( D \in \mathbb{N} \) such that \( n/r \leq D < (n+r)/(r-1) \),

\[
\frac{n^2}{2r^2} + \frac{n}{2r} = \left( \frac{n/r + 1}{2} \right) \leq D + e(T(n-D-1, r-2)).
\]

Since \( D < (n+r)/(r-1) \) we have that \( D \leq n/(r-1) + 1 \). So Facts 14 and 24 imply that

\[
D + e(T(n-D-1, r-2)) \geq D + \frac{(n-D-1)^2}{2(r-2)} - \frac{(n-D-1)}{2} \\
\geq \frac{n}{r-1} + 1 + \frac{1}{2(r-2)} \left[ \frac{(r-2)}{r-1} n - 2 \right]^2 - \frac{1}{2} \left[ \frac{(r-2)}{r-1} n - 2 \right] \\
\geq \frac{(r-2)}{2(r-1)^2} n^2 - \frac{(r-2)}{2(r-1)} n - \frac{n}{r-1}.
\]
Thus, it suffices to show that
\[
\frac{(r - 2)}{2(r - 1)^2} - \frac{1}{2(r - 1)} + \frac{1}{2r} \geq \frac{n}{2r^2} + \frac{1}{2r}.
\] (19)

Notice that
\[
\frac{r - 2}{2(r - 1)} + \frac{1}{r - 1} + \frac{1}{2r} = \frac{r^2 + r - 1}{2r(r - 1)}.
\]

Since \(n \geq 3r\), (17) and (19) imply that it suffices to show that
\[
\frac{3(r^3 - 3r^2 + 2r - 1)}{2r(r - 1)^2} - \frac{r^2 + r - 1}{2r(r - 1)} \geq 0.
\] (20)

Note that \(2r^3 - 9r^2 + 8r - 4 \geq 0\) as \(r \geq 4\). Thus, \(3(r^3 - 3r^2 + 2r - 1) \geq (r^2 + r - 1)(r - 1)\). So indeed (20) is satisfied, as desired. □