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ON EXTREMIZERS FOR STRICHARTZ ESTIMATES FOR HIGHER ORDER SCHRÖDINGER EQUATIONS

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ABSTRACT. For an appropriate class of convex functions ϕ, we study the Fourier extension operator on the surface $\{(y, |y|^2 + \phi(y)) : y \in \mathbb{R}^2\}$ equipped with projection measure. For the corresponding extension inequality, we compute optimal constants and prove that extremizers do not exist. The main tool is a new comparison principle for convolutions of certain singular measures that holds in all dimensions. Using tools of concentration-compactness flavor, we further investigate the behavior of general extremizing sequences. Our work is directly related to the study of extremizers and optimal constants for Strichartz estimates of certain higher order Schrödinger equations. In particular, we resolve a dichotomy from the recent literature concerning the existence of extremizers for a family of fourth order Schrödinger equations, and compute the corresponding operator norms exactly where only lower bounds were previously known.

CONTENTS

1. Introduction

Recently there has been considerable interest in the study of extremizers, optimal constants, and sharp instances of various Fourier extension inequalities. The purpose of the present paper is three-fold. Firstly, we establish a sharp Fourier extension inequality on certain non-compact hypersurfaces in Euclidean space. Secondly, we use concentration-compactness tools to study the qualitative behavior of extremizing sequences for this sharp inequality. Thirdly, we explore the link between Fourier extension inequalities and Strichartz estimates for certain higher order Schrödinger equations, and resolve some dichotomies concerning the existence of extremizers that have appeared in the recent literature.

Throughout the paper, we normalize the Fourier transform as follows:

$$\widehat{f}(y) = \int_{\mathbb{R}^d} f(x) e^{-i\langle x, y\rangle} \, \mathrm{d}x,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^d$. Given a sufficiently nice function $\phi : \mathbb{R}^d \to \mathbb{R}$, consider the hypersurface in $\mathbb{R}^{d+1}$

$$\Sigma_\phi = \{(y, |y|^2 + \phi(y)) : y \in \mathbb{R}^d\} \tag{1.1}$$

endowed with projection measure

$$\sigma(y, t) = \boldsymbol{\delta}\big(t - |y|^2 - \phi(y)\big) \, \mathrm{d}y \, \mathrm{d}t, \tag{1.2}$$

which in turn is defined by requiring that the identity

$$\int_{\mathbb{R}^{d+1}} g(y, t) \, \mathrm{d}\sigma(y, t) = \int_{\mathbb{R}^d} g(y, |y|^2 + \phi(y)) \, \mathrm{d}y$$

holds for every Schwartz function g. The Fourier extension operator for the hypersurface Σ_ϕ is defined as

$$\widehat{f\sigma}(x, t) = \int_{\mathbb{R}^d} f(y) e^{-i\langle x, y\rangle} e^{-it(|y|^2 + \phi(y))} \, \mathrm{d}y, \quad (x, t) \in \mathbb{R}^{d+1}.$$

Estimates for this operator stem from the seminal works of Tomas [31], Stein [29] and Strichartz [30]. In particular, under certain fairly general convexity assumptions on ϕ, the inequality

$$\|\widehat{f\sigma}\|_{L^{2+\frac{4}{d}}(\mathbb{R}^{d+1})} \lesssim_{d,\phi} \|f\|_{L^2(\mathbb{R}^d)}$$

holds in dimensions $d \geqslant 1$, see e.g. [18, 19, 21]. To pursue this point further, let us specialize the discussion to the two-dimensional case $d = 2$. Using the fact that in this case the exponent $2 + \frac{4}{d} = 4$ is an even integer together with Plancherel's Theorem, the inequality

$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} \lesssim_\phi \|f\|_{L^2(\mathbb{R}^2)}$$

can be rewritten in bilinear convolution form as

$$\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} \lesssim_\phi \|f\|_{L^2(\mathbb{R}^2)}^2. \tag{1.3}$$

We emphasize that, since the surface Σ_ϕ is not compact, inequality (1.3) does not hold in general if the projection measure is replaced by the usual surface measure

on Σ_ϕ. Inequality (1.3) will be established under some mild assumptions on ϕ in Theorem 1.2 below. As we will see, it will follow from the fact that the convolution $\sigma * \sigma$ defines a measure which is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^3$, and whose Radon–Nikodym derivative is given by an essentially bounded function.

In the first part of the paper, we address the question of existence of extremizers for the sharp version of inequality (1.3), and compute the optimal constant. More precisely, consider the sharp inequality

$$\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} \leqslant \mathcal{R}_\phi^2 \|f\|_{L^2(\mathbb{R}^2)}^2, \tag{1.4}$$

where the optimal constant is given by

$$\mathcal{R}_\phi := \sup_{0 \neq f \in L^2(\mathbb{R}^2)} \frac{\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}}{\|f\|_{L^2(\mathbb{R}^2)}}. \tag{1.5}$$

Definition 1.1. An **extremizing sequence** for inequality (1.4) is a sequence $\{f_n\} \subset L^2(\mathbb{R}^2)$ satisfying $\|f_n\|_{L^2(\mathbb{R}^2)} \leqslant 1$, such that

$$\|f_n\sigma * f_n\sigma\|_{L^2(\mathbb{R}^3)} \to \mathcal{R}_\phi^2, \text{ as } n \to \infty.$$

An **extremizer** for inequality (1.4) is a nonzero function $f \in L^2(\mathbb{R}^2)$ which satisfies

$$\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} = \mathcal{R}_\phi^2 \|f\|_{L^2(\mathbb{R}^2)}^2.$$

The unperturbed case $\phi = 0$ was treated by Foschi [11], who proved that extremizers for the corresponding extension inequality on the paraboloid Σ_0 are given by Gaussians, and computed $\mathcal{R}_0 = (\frac{\pi}{2})^{\frac{1}{4}}$. A key step in Foschi's program was the elementary but crucial observation that the convolution of projection measure on the two-dimensional paraboloid defines a constant function in the interior of its support, see [11, Lemma 3.2], and Remark 2.2 below. Alternative approaches are available: Hundertmark–Zharnitski [17] base their analysis on a novel representation of the Strichartz integral, and Bennett et al. [3] identify a monotonicity property of such integrals under a certain quadratic heat-flow. These proofs rely on the large symmetry group enjoyed by the paraboloid. Perturbed paraboloids Σ_ϕ with $\phi \neq 0$ no longer enjoy this special feature, and understanding the associated Fourier extension operator in sharp form is an important step towards the understanding of general manifolds with positive Gaussian curvature. This motivates our first main result.

Theorem 1.2. *Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a nonnegative, twice continuously differentiable, strictly convex function, whose Hessian $H(\phi)$ satisfies one of the following conditions:*
  (i) *$H(\phi)(y_0) = 0$ for some $y_0 \in \mathbb{R}^2$, or*
  (ii) *There exists a sequence $\{y_n\} \subset \mathbb{R}^2$ with $|y_n| \to \infty$, such that $H(\phi)(y_n) \to 0$, as $n \to \infty$.*

Let σ denote the projection measure on the surface Σ_ϕ. Then the inequality

$$\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} \leqslant \mathcal{R}_\phi^2 \|f\|_{L^2(\mathbb{R}^2)}^2 \tag{1.6}$$

holds for every $f \in L^2(\mathbb{R}^2)$, and is sharp with optimal constant given by

$$\mathcal{R}_\phi = \left(\frac{\pi}{2}\right)^{\frac{1}{4}}. \tag{1.7}$$

The sequence $\{f_n/\|f_n\|_{L^2}\}$ defined via

$$f_n(y) := \begin{cases} e^{-n(\psi(y)-\psi(y_0)-\langle\nabla\psi(y_0),y-y_0\rangle)}, & \text{in case (i)}, \\ e^{-a_n(\psi(y)-\psi(y_n)-\langle\nabla\psi(y_n),y-y_n\rangle)}, & \text{in case (ii)}, \end{cases} \tag{1.8}$$

where $\psi := |\cdot|^2 + \phi$ and $\{a_n\}$ is an appropriately chosen sequence, is extremizing for inequality (1.6). Moreover, extremizers for inequality (1.6) do not exist.

Let us briefly comment on the proof of Theorem 1.2. In order to compute the optimal constant $\mathcal{R}_\phi$ and to show that extremizers do not exist, we employ methods from [24, 26] that are based on Foschi's ideas [11], with a novel ingredient which we highlight below. The main steps are the following:

- One shows that $\mathcal{R}_\phi^4 \leqslant \|\sigma * \sigma\|_{L^\infty} < \infty$.
- One exhibits an explicit sequence $\{f_n\} \subset L^2(\mathbb{R}^2)$ such that

$$\liminf_{n\to\infty} \frac{\|f_n\sigma * f_n\sigma\|_{L^2}^2}{\|f_n\|_{L^2}^4} \geqslant \|\sigma * \sigma\|_{L^\infty}.$$

- From the previous two steps, one concludes $\mathcal{R}_\phi^4 = \|\sigma * \sigma\|_{L^\infty}$.
- One proves that the set $\{(\xi,\tau) \in \mathbb{R}^{2+1} : (\sigma * \sigma)(\xi,\tau) = \|\sigma * \sigma\|_{L^\infty}\}$ has Lebesgue measure zero.
- A careful review of Foschi's method then implies

$$\|f\sigma * f\sigma\|_{L^2} < \mathcal{R}_\phi^2\|f\|_{L^2}^2,$$

for every nonzero $f \in L^2(\mathbb{R}^2)$. In particular, extremizers do not exist.

The first and fourth steps above are based on a new comparison principle that translates into a pointwise inequality between convolution of projection measures on the perturbed surface Σ_ϕ and the paraboloid Σ_0, respectively. It leads to the computation of the exact numerical value of the optimal constant $\mathcal{R}_\phi$. The comparison principle holds in all dimensions $d \geqslant 2$, and we state it precisely as follows.

Theorem 1.3. *For $d \geqslant 2$, let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a nonnegative, continuously differentiable, strictly convex function. Let $\varphi = |\cdot|^2$ and $\psi = |\cdot|^2 + \phi$. Let σ_0, σ denote the projection measures on the hypersurfaces Σ_0, Σ_ϕ, respectively. Then*

$$\big(\sigma * \sigma\big)(\xi, 2\psi(\xi/2) + \tau) \leqslant \big(\sigma_0 * \sigma_0\big)(\xi, 2\varphi(\xi/2) + \tau), \tag{1.9}$$

for every $\xi \in \mathbb{R}^d$ and $\tau > 0$. Moreover, this inequality is strict for almost every point in the support of the measure $\sigma * \sigma$.

Certain related but distinct comparison principles have already proved useful in understanding the effect of global smoothing. See [28] for an instance in which such a principle was used to derive new estimates for dispersive (and non-dispersive) equations from known ones, as well as an effective means to compare estimates for different equations. The link with optimal constants and extremizers for a broad class of

smoothing estimates is established in [4].

In the second part of the present paper, we adapt ideas from the concentration-compactness principle of Lions [22] to examine the behavior of general extremizing sequences for inequality (1.6). Generally speaking, the theory of concentration-compactness has proved a very efficient tool to exhibit the precise mechanisms which are responsible for the loss of compactness in a variety of settings. In our concrete problem, extremizers fail to exist because extremizing sequences concentrate. Concentration can only occur at points where the convolution $\sigma * \sigma$ attains its maximum value, or at spatial infinity. To make these concepts precise, we introduce the relevant definitions.

Definition 1.4. A sequence $\{f_n\} \subset L^2(\mathbb{R}^2)$ **concentrates at a point** $y_0 \in \mathbb{R}^2$ if, for every $\varepsilon, \rho > 0$, there exists $N \in \mathbb{N}$ such that, for every $n \geqslant N$,

$$\int_{\{|y-y_0| \geqslant \rho\}} |f_n(y)|^2 \, \mathrm{d}y \leqslant \varepsilon \|f_n\|^2_{L^2(\mathbb{R}^2)}.$$

A sequence $\{f_n\} \subset L^2(\mathbb{R}^2)$ **concentrates along a sequence** $\{y_n\} \subset \mathbb{R}^2$ if, for every $\varepsilon, \rho > 0$, there exists $N \in \mathbb{N}$ such that, for every $n \geqslant N$,

$$\int_{\{|y-y_n| \geqslant \rho\}} |f_n(y)|^2 \, \mathrm{d}y \leqslant \varepsilon \|f_n\|^2_{L^2(\mathbb{R}^2)}.$$

A sequence $\{f_n\} \subset L^2(\mathbb{R}^2)$ **concentrates at infinity** if, for every $\varepsilon, \rho > 0$, there exists $N \in \mathbb{N}$ such that, for every $n \geqslant N$,

$$\int_{\{|y| \leqslant \rho\}} |f_n(y)|^2 \, \mathrm{d}y \leqslant \varepsilon \|f_n\|^2_{L^2(\mathbb{R}^2)}.$$

The following result holds under the general hypotheses of Theorem 1.2.

Theorem 1.5. *Let* $\phi : \mathbb{R}^2 \to \mathbb{R}$ *be a nonnegative, twice continuously differentiable, strictly convex function, whose Hessian satisfies condition* (i) *or condition* (ii) *from Theorem 1.2. Then any extremizing sequence for inequality* (1.6) *has a further subsequence which either concentrates at some point* $y_0 \in \mathbb{R}^2$ *satisfying* $H(\phi)(y_0) = 0$, *or concentrates at infinity.*

It has long been understood that Tomas–Stein extension type inequalities are related to Strichartz estimates for linear partial differential equations of dispersive type. To illustrate this connection in the present situation, consider the multiplier operator

$$\widehat{M_\phi g} = \phi \, \widehat{g}$$

acting on Schwartz functions g, and the associated Schrödinger equation

$$\begin{cases} iu_t + M_\phi u - \mu \Delta u = 0, & \mu \geqslant 0, \\ u(\cdot, 0) = f \in L^2(\mathbb{R}^d), \end{cases} \tag{1.10}$$

whose solution can be written as

$$u(x, t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(y) e^{i\langle x, y \rangle} e^{it(\mu|y|^2 + \phi(y))} \, \mathrm{d}y, \quad (x, t) \in \mathbb{R}^{d+1}. \tag{1.11}$$

In the third part of the paper, we consider Strichartz inequalities for solutions of equation (1.10) in the two-dimensional case $d = 2$. In particular, we investigate the family of inequalities

$$\|(\mu + |\nabla|^2)^{\frac{1}{4}} e^{it(\phi(|\nabla|) - \mu\Delta)} f\|_{L^4(\mathbb{R}^3)} \lesssim_{\mu,\phi} \|f\|_{L^2(\mathbb{R}^2)}, \tag{1.12}$$

and mostly focus on the particular instance of a quartic perturbation, $\phi = |\cdot|^4$. In this case, inequality (1.12) can be proved via the method of stationary phase together with the main theorem of [20], see the remarks preceding [19, Proposition 2.4], and [2, 21, 25] for further details. In the spirit of what had been done in the one-dimensional setting in [18], this instance of inequality (1.12) was refined in [19], with the goal of establishing a linear profile decomposition for a family of fourth order Schrödinger equations. As a consequence, the authors of [19] obtained a dichotomy result for the existence of extremizers in the cases $\mu \in \{0, 1\}$, which by scaling extends to the general case $\mu \geqslant 0$, and can be summarized as follows: Either extremizers exist, or extremizing sequences exhibit a certain classical Schrödinger behavior. See [19, Theorems 4.1 and 4.2] for a precise formulation of these results. Along the way, the authors of [19] obtained lower bounds for the norms of the corresponding Fourier extension operators. The methods we use to study the sharp bilinear convolution inequality (1.6) are robust enough to resolve this dichotomy, and to determine which situation actually happens. In particular, we prove that extremizers exist if $\mu = 0$, but fail to exist if $\mu = 1$. In the latter case, we also compute the operator norm exactly.

To state our results precisely, let us start by considering the case of $\mu = 0$ and $\phi = |\cdot|^4$. Then inequality (1.12) can be restated with the help of the Fourier transform, here denoted by $\mathcal{F}$, as

$$\|\mathcal{F}(f|\cdot|^{\frac{1}{2}}\nu)\|_{L^4(\mathbb{R}^3)} \lesssim \|f\|_{L^2(\mathbb{R}^2)}, \tag{1.13}$$

where the measure ν is given by $\nu(y, t) = \boldsymbol{\delta}(t - |y|^4)\, dy\, dt$. By Plancherel's Theorem, inequality (1.13) can be rewritten in sharp form as

$$\|f|\cdot|^{\frac{1}{2}}\nu * f|\cdot|^{\frac{1}{2}}\nu\|_{L^2(\mathbb{R}^3)} \leqslant \mathcal{Q}^2 \|f\|_{L^2(\mathbb{R}^2)}^2, \tag{1.14}$$

with optimal constant $\mathcal{Q}$. The following result should be compared to [19, Theorem 4.1].

Theorem 1.6. *The optimal constant for inequality* (1.14) *satisfies the bounds*

$$\frac{\pi}{4\sqrt{3}} < \mathcal{Q}^4 < \frac{\pi}{4}. \tag{1.15}$$

Moreover, there exists an extremizer for inequality (1.14).

Still taking $\phi = |\cdot|^4$, let us now consider the case of $\mu = 1$. Then inequality (1.12) can be restated as

$$\|\mathcal{F}(f(1 + |\cdot|^2)^{\frac{1}{4}}\sigma)\|_{L^4(\mathbb{R}^3)} \lesssim \|f\|_{L^2(\mathbb{R}^2)}, \tag{1.16}$$

where the measure σ is given by $\sigma(y, t) = \boldsymbol{\delta}(t - |y|^2 - |y|^4)\, dy\, dt$. By Plancherel's Theorem, inequality (1.16) can be rewritten in sharp form as

$$\|f\sqrt{w}\sigma * f\sqrt{w}\sigma\|_{L^2(\mathbb{R}^3)} \leqslant \mathcal{S}^2 \|f\|_{L^2(\mathbb{R}^2)}^2, \tag{1.17}$$

with weight $w = (1 + |\cdot|^2)^{\frac{1}{2}}$ and optimal constant $\mathcal{S}$. The following result is a special case of Theorem 6.2 below.

Theorem 1.7. *The value of the optimal constant for inequality* (1.17) *is given by* $\mathcal{S}^4 = \frac{\pi}{2}$. *Moreover, extremizers for inequality* (1.17) *do not exist, and extremizing sequences concentrate at the origin.*

In particular, Theorems 1.6 and 1.7 imply that $\mathcal{Q}^4 < \mathcal{S}^4 = \frac{\pi}{2}$. The shape of (the Fourier transform of) a general extremizing sequence for inequality (1.17) is then given by [19, Theorem 4.2] and the remarks following it. Furthermore, as mentioned in [19], it is of interest to extend the analysis to more general perturbations of the Schrödinger equation. Our methods allow to make progress in a number of previously untreated cases, and we comment on this in Remark 6.3 and §6.4 below.

Our results complement the recent, vast and very interesting body of work concerning sharp Fourier extension and Strichartz estimates. In addition to the works previously cited in this introduction, see [6, 7, 8, 12, 14] for results in sharp Fourier extension theory on spheres, and [5, 9, 10, 15, 23, 27] for other instances.

Overview. The paper is organized as follows. In Chapter 2, we briefly comment on the model case of a pure power perturbation of the paraboloid, and derive a useful integral formula for the convolution of projection measure on a generic convex perturbation of the two-dimensional paraboloid. Chapter 3 is the technical heart of the first part of the paper, and is devoted to the aforementioned comparison principle. In particular, we prove Theorem 1.3, and briefly remark on possible extensions of this result to n-fold convolutions if $n \geqslant 3$. The proof of Theorem 1.2 is presented in Chapter 4. We discuss the behavior of general extremizing sequences in Chapter 5. In particular, we establish a precise form of the geometric principle that distant caps interact weakly, show some auxiliary results of concentration-compactness flavor, and prove Theorem 1.5. Finally, we deal with sharp Strichartz inequalities in Chapter 6. We treat the case of quartic perturbations in §6.1, establishing a generalization of Theorem 1.7. We study the convolution of projection measure associated to pure powers in §6.2, and use this knowledge to tackle the case of the pure quartic in §6.3, establishing Theorem 1.6, and of other pure powers in §6.4.

A word on forthcoming notation. The usual inner product between vectors $x, y \in \mathbb{R}^d$ will continue to be denoted by $\langle x, y \rangle$. This is to clarify the distinction from the $d \times d$ matrix obtained as the matrix product between the vector x and the transpose of the vector y, denoted $x \cdot y^T$. The usual matrix product between a $d \times d$ matrix A and a vector $x \in \mathbb{R}^d$ will likewise be indicated by $A \cdot x$. The $d \times d$ identity matrix will be denoted by I_d, or simply by I if no confusion arises. The open ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$ will be denoted by $B_r(x)$. If $x = 0$, then we will simply write B_r instead of $B_r(0)$. The corresponding closed balls will be denoted by $\bar{B}_r(x)$ and $\bar{B}_r = \bar{B}_r(0)$, respectively. The alternative notation for the Fourier transform $\mathcal{F}(f) = \hat{f}$ will occasionally be used. Finally, $\mathbb{1}_E$ will stand for the

indicator function of a given subset $E \subset \mathbb{R}^d$, and the complement of E will at times be denoted by $E^{\complement}$.

2. On scaling and convolutions

2.1. **An explicit example.** For $d \geqslant 1$, $a > 0$ and $p > 2$, consider the family of Fourier extension operators associated to certain polynomial perturbations of the paraboloid equipped with projection measure, given by

$$T_a(f)(x,t) = \int_{\mathbb{R}^d} f(y) e^{-i\langle x,y \rangle} e^{-it(|y|^2 + a|y|^p)} \, \mathrm{d}y, \quad (x,t) \in \mathbb{R}^{d+1}. \tag{2.1}$$

The family $\{T_a\}$ enjoys the following scaling property. Given $a, b > 0$, let $\lambda = (\frac{b}{a})^{\frac{1}{p-2}}$. Changing variables $y \rightsquigarrow \lambda y$ in the integral (2.1), we see that

$$T_a(f)(x,t) = \lambda^{\frac{d}{2}} \int_{\mathbb{R}^d} f_\lambda(y) e^{-i\langle \lambda x, y \rangle} e^{-i\lambda^2 t(|y|^2 + b|y|^p)} \, \mathrm{d}y = \lambda^{\frac{d}{2}} T_b(f_\lambda)(\lambda x, \lambda^2 t),$$

where the rescaled function $f_\lambda(y) = \lambda^{\frac{d}{2}} f(\lambda y)$ satisfies $\|f_\lambda\|_{L^2} = \|f\|_{L^2}$. It follows that

$$\|T_a(f)\|_{L^{2+\frac{4}{d}}(\mathbb{R}^{d+1})} = \|T_b(f_\lambda)\|_{L^{2+\frac{4}{d}}(\mathbb{R}^{d+1})},$$

and therefore

$$\sup_{0 \neq f \in L^2(\mathbb{R}^d)} \frac{\|T_a(f)\|_{L^{2+\frac{4}{d}}(\mathbb{R}^{d+1})}}{\|f\|_{L^2(\mathbb{R}^d)}} = \sup_{0 \neq f \in L^2(\mathbb{R}^d)} \frac{\|T_b(f)\|_{L^{2+\frac{4}{d}}(\mathbb{R}^{d+1})}}{\|f\|_{L^2(\mathbb{R}^d)}}.$$

From this identity, we conclude that optimal constants are independent of a, and that extremizers exist for some value of $a > 0$ if and only if they exist for every value of $a > 0$. If extremizers exist for one value of $a > 0$, then the simple dilation indicated above produces an extremizer for any other value of $a > 0$. Theorem 1.2 provides a refinement of this rudimentary analysis in the two-dimensional case. In particular, it states that the optimal constant is also independent of p, and that extremizers do not exist.

2.2. **Convolution of singular measures.** The goal of this section is to exhibit an explicit formula for the convolution of projection measure on perturbed paraboloids. For the sake of concreteness, we limit our discussion to the two dimensional case $d = 2$. See [1, Lemma 3.1] for a formula in the same spirit of identity (2.2) below.

Proposition 2.1. *Let $\psi := |\cdot|^2 + \phi$, where $\phi \geqslant 0$ is a convex $C^2(\mathbb{R}^2)$ function. Let σ denote projection measure on the surface Σ_ϕ. Then the following assertions hold for the convolution measure $\sigma * \sigma$:*

(a) *It is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^3$.*

(b) *Its support is given by*

$$\mathrm{supp}(\sigma * \sigma) = \{(\xi, \tau) \in \mathbb{R}^{2+1} : \tau \geqslant 2\psi(\xi/2)\}.$$

(c) *Its Radon–Nikodym derivative, also denoted by $\sigma * \sigma$, is given by the formula*

$$(\sigma * \sigma)(\xi, \tau) = \int_{\mathbb{S}^1} \left(\int_{-1}^{1} \langle \omega, H(\psi)(\xi/2 + t\alpha(\xi, \tau, \omega)\omega) \cdot \omega \rangle \, \mathrm{d}t \right)^{-1} \mathrm{d}\mu_\omega, \tag{2.2}$$

provided $\tau > 2\psi(\xi/2)$. Here, the measure μ denotes arc length measure on the unit circle $\mathbb{S}^1$, and the function α is given by

$$\alpha(\xi, \tau, \omega) = \sqrt{\tau/2 - \psi(\xi/2)}\lambda\left(\sqrt{\tau/2 - \psi(\xi/2)}\omega\right), \tag{2.3}$$

where the function λ is implicitly defined via identity (2.6) below.

(d) *The convolution $\sigma * \sigma$ defines a continuous function of the variables ξ, τ in the interior of its support. It extends continuously to the boundary of the support, with values given by*

$$(\sigma * \sigma)(\xi, 2\psi(\xi/2)) = \frac{\pi}{\sqrt{\det(H(\psi)(\xi/2))}}. \tag{2.4}$$

Remark 2.2. In the special case $\phi = 0$, the Hessian of ψ is a constant multiple of the identity matrix, and formula (2.2) recovers the result from [11, Lemma 3.2] for the convolution of projection measure σ_0 on the two-dimensional paraboloid $\Sigma_0 \subset \mathbb{R}^3$: For $\tau > |\xi|^2/2$,

$$(\sigma_0 * \sigma_0)(\xi, \tau) = \int_{\mathbb{S}^1}\left(\int_{-1}^1 \langle\omega, 2\omega\rangle \, \mathrm{d}t\right)^{-1} \mathrm{d}\mu_\omega = \frac{\pi}{2}.$$

Proof of Proposition 2.1. The absolute continuity of $\sigma * \sigma$ with respect to Lebesgue measure follows in the same way as in the proof of [1, Lemma 3.1 (b)], with minor modifications only. We provide the details for the convenience of the reader. In order to show that the pairing $\langle\sigma * \sigma, \mathbb{1}_E\rangle = 0$ for each set E of Lebesgue measure zero in $\mathbb{R}^3$, set $y = (y_1, y_2)$, $z = (z_1, z_2)$, and change variables $t_j = y_j + z_j$, $s_j = y_j - z_j$ ($j = 1, 2$) to get

$$\langle\sigma * \sigma, \mathbb{1}_E\rangle = \int_{(\mathbb{R}^2)^2} \mathbb{1}_E(y + z, \psi(y) + \psi(z)) \, \mathrm{d}y \, \mathrm{d}z$$

$$= \frac{1}{4}\int_{\mathbb{R}}\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}} \mathbb{1}_E(t_1, t_2, F_t(s_1, s_2)) \, \mathrm{d}s_1 \, \mathrm{d}t\right) \mathrm{d}s_2,$$

where the function F_t is defined as

$$F_t(s) = \psi\left(\frac{t + s}{2}\right) + \psi\left(\frac{t - s}{2}\right).$$

The key observation is that $F_t(s_1, s_2)$ is a strictly convex function of s_1 for each fixed s_2 and t. The change of variables $s_1 \mapsto u$ given by the (at most 2–to–1) map $u = F_t(s_1, s_2)$ shows that the triple integral in (s_1, t) is zero (for each s_2) since E is a Lebesgue null set. This establishes (a).

For (b), consider vectors $y, y' \in \mathbb{R}^2$, and note that

$$\xi := y + y' \text{ and } \tau := \psi(y) + \psi(y')$$

satisfy $\tau \geqslant 2\psi(\xi/2)$ because the function ψ is convex. For the reverse inclusion, let $(\xi, \tau) \in \mathbb{R}^{2+1}$ be given, such that $\tau \geqslant 2\psi(\xi/2)$. We want to find $y, y' \in \mathbb{R}^2$, such that

$$y + y' = \xi \text{ and } \psi(y) + \psi(y') = \tau.$$

It is enough to find y such that $\psi(y) + \psi(\xi - y) = \tau$, for then $y' = \xi - y$. Note that $\psi(y) + \psi(\xi - y) \geqslant 2\psi(\xi/2)$ by convexity of ψ, with equality if $y = \xi/2$. Moreover,

$$\psi(y) + \psi(\xi - y) \to \infty, \text{ as } |y| \to \infty.$$

The function $y \mapsto \psi(y) + \psi(\xi - y)$ is continuous because ψ is convex, and the result follows from applying the Intermediate Value Theorem in the appropriate direction. We now come to part (c). Let (ξ, τ) be such that $\tau > 2\psi(\xi/2)$. Fubini's Theorem and a simple change of variables yield[1]

$$(\sigma * \sigma)(\xi, \tau) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} \boldsymbol{\delta}\big(\tau - t - \psi(\xi - y)\big) \, \boldsymbol{\delta}\big(t - \psi(y)\big) \, \mathrm{d}y \right) \mathrm{d}t$$

$$= \int_{\mathbb{R}^2} \boldsymbol{\delta}\big(\tau - \psi(\xi/2 + y) - \psi(\xi/2 - y)\big) \, \mathrm{d}y. \tag{2.5}$$

We would like to perform another change of variables $y = T(w)$, where $T(w) = \lambda w$, and $\lambda = \lambda(w) > 0$ is an implicit real-valued function of w which takes only positive values, and is defined via

$$\psi(\xi/2 + \lambda w) + \psi(\xi/2 - \lambda w) = 2|w|^2 + 2\psi(\xi/2). \tag{2.6}$$

For fixed ξ, a unique positive solution $\lambda = \lambda(w)$ exists if $w \neq 0$. By the Implicit Function Theorem, equation (2.6) defines λ as a continuously differentiable function of w, as long as the derivative of the map

$$\lambda \mapsto \psi(\xi/2 + \lambda w) + \psi(\xi/2 - \lambda w)$$

is nonzero. In view of the strict convexity of the function ψ, this is indeed the case if $\lambda > 0$. Further details in a more general context will be provided in the course of the proof of Lemma 3.2 below. Since the function λ is continuously differentiable and $T(w) = \lambda(w)w$, we have that

$$T'(w) = \lambda I + \nabla\lambda \cdot w^T, \tag{2.7}$$

where I denotes the 2×2 identity matrix, the gradient is taken with respect to w, and the term $\nabla\lambda \cdot w^T$ stands for the 2×2 matrix obtained as the product of the gradient $\nabla\lambda$ (seen as a vector in $\mathbb{R}^2$) and the transpose of the vector w. To compute the gradient $\nabla\lambda$, note that implicit differentiation of (2.6) with respect to w yields

$$T'(w) \cdot u = 4w, \text{ where } u = u(w, \xi) := \nabla\psi(\xi/2 + \lambda w) - \nabla\psi(\xi/2 - \lambda w). \tag{2.8}$$

From (2.7) and (2.8), it follows that

$$\nabla\lambda = \frac{4w - \lambda u}{\langle w, u \rangle}. \tag{2.9}$$

One easily computes

$$\det T'(w) = \det(\lambda I + \nabla\lambda \cdot w^T) = (1 + \lambda^{-1}\langle w, \nabla\lambda \rangle) \det(\lambda I),$$

and identity (2.9) then implies

$$\det T'(w) = \frac{4|w|^2 \lambda(w)}{\langle w, u(w, \xi) \rangle}. \tag{2.10}$$

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[1]For a treatment of integration on manifolds using delta calculus, see [13, Appendix A].

Note that this is a nonnegative quantity because of the strict convexity of ψ. Going back to the integral expression for $\sigma * \sigma$, changing variables as announced, and switching to polar coordinates, yields

$$(\sigma * \sigma)(\xi, \tau) = \int_{\mathbb{R}^2} \boldsymbol{\delta}\big(\tau - 2\psi(\xi/2) - 2|w|^2\big) \det T'(w) \, \mathrm{d}w$$

$$= \int_0^\infty \boldsymbol{\delta}\big(\tau - 2\psi(\xi/2) - 2r^2\big) \Big(\int_{\mathbb{S}^1} \det T'(r\omega) \, \mathrm{d}\mu_\omega \Big) r \, \mathrm{d}r,$$

where μ denotes arc length measure on the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$. Using expression (2.10) for the Jacobian factor $\det T'$, changing variables $2r^2 = s$, and appealing to Fubini's theorem, we have that

$$(\sigma * \sigma)(\xi, \tau) = \int_{\mathbb{S}^1} \Big(\int_0^\infty \boldsymbol{\delta}\big(\tau - 2\psi(\xi/2) - s\big) \frac{\sqrt{s/2}\lambda(\sqrt{s/2}\omega)}{\langle \omega, u(\sqrt{s/2}\omega, \xi)\rangle} \, \mathrm{d}s \Big) \mathrm{d}\mu_\omega.$$

Evaluating the inner integral,

$$(\sigma * \sigma)(\xi, \tau) = \int_{\mathbb{S}^1} \frac{\sqrt{\tau/2 - \psi(\xi/2)}\lambda(\sqrt{\tau/2 - \psi(\xi/2)}\omega)}{\langle \omega, u(\sqrt{\tau/2 - \psi(\xi/2)}\omega, \xi)\rangle} \, \mathrm{d}\mu_\omega.$$

Defining the function $\alpha = \alpha(\xi, \tau, \omega)$ as in (2.3), and recalling the expression in (2.8) for the vector u,

$$(\sigma * \sigma)(\xi, \tau) = \int_{\mathbb{S}^1} \Big\langle \omega, \frac{\nabla\psi(\xi/2 + \alpha\omega) - \nabla\psi(\xi/2 - \alpha\omega)}{\alpha} \Big\rangle^{-1} \mathrm{d}\mu_\omega. \tag{2.11}$$

Formula (2.2) now follows from the Fundamental Theorem of Calculus.

As for part (d), the continuity in the interior of the support follows from an inspection of representation formula (2.2), after recalling the fact that the function λ is continuous. The boundary value is obtained by noting that, for each $\omega \in \mathbb{S}^1$, the function $\alpha(\xi, \tau, \omega)$ tends to 0 as (ξ, τ) approaches the boundary of the support from its interior, since the function λ satisfies $0 \leqslant \lambda \leqslant 1$. This yields

$$(\sigma * \sigma)(\xi, 2\psi(\xi/2)) = \frac{1}{2} \int_{\mathbb{S}^1} \frac{1}{\langle \omega, H(\psi)(\xi/2) \cdot \omega \rangle} \, \mathrm{d}\mu_\omega,$$

from which identity (2.4) follows by using an orthonormal basis of $\mathbb{R}^2$ consisting of eigenvectors of the Hessian matrix $H(\psi)(\xi/2)$. The proof is now complete. $\qquad\square$

Remark 2.3. Identity (2.11) already implies a weak form of the comparison principle (Theorem 1.3) in the two-dimensional case. Analogous reasoning leads to similar formulae for higher dimensional hypersurfaces. This is one of the motivations for the next chapter, which shares some features with the proof of Proposition 2.1. However, the analysis there seems more flexible, and may be adaptable to other situations as well.

3. A COMPARISON RESULT

This chapter is devoted to the proof of Theorem 1.3, which holds in dimensions $d \geqslant 2$. Before stating the technical lemmata that will be used in its proof, let us consider two convex functions $\psi, \varphi : \mathbb{R}^d \to \mathbb{R}$. Given $\xi, y \in \mathbb{R}^d$, define the following auxiliary functions of one real variable:

$$g(t) := \psi(\xi/2 - ty) + \psi(\xi/2 + ty) - 2\psi(\xi/2), \tag{3.1}$$

$$h(t) := \varphi(\xi/2 - ty) + \varphi(\xi/2 + ty) - 2\varphi(\xi/2). \tag{3.2}$$

Note that $g = h \equiv 0$ if $y = 0$. Some properties of the functions g, h in a useful special case are collected in the following lemma.

Lemma 3.1. *Let $\psi, \varphi : \mathbb{R}^d \to \mathbb{R}$ be differentiable, convex functions, such that their difference $\psi - \varphi$ is also convex. Given $\xi, y \in \mathbb{R}^d$, define the functions g, h as above. Then:*

(a) *$g(t) \geqslant h(t) \geqslant 0$, for every $t \in \mathbb{R}$.*
(b) *The functions g and h are convex.*
(c) *$g'(0) = h'(0) = 0$.*
(d) *If ψ is strictly convex and $y \neq 0$, then g attains its unique global minimum at $t = 0$.*
(e) *If ψ is strictly convex and $y \neq 0$, then there exists a unique nonnegative $\lambda = \lambda(y, \xi)$ such that*
$$h(1) = g(\lambda),$$
and moreover $\lambda \in [0, 1]$.
(f) *If $h(1) > 0$, then $\lambda > 0$. If $h(1) < g(1)$, then $\lambda < 1$.*

Proof. The inequality $h \geqslant 0$ follows from the (midpoint) convexity of the function φ. The inequality $g \geqslant h$ follows from the (midpoint) convexity of the function $\psi - \varphi$. This establishes (a). Statement (b) is a consequence of the following two general facts: sums of convex functions are convex, and restrictions of convex functions to lines are convex. Differentiability of the functions g, h follows from that of ψ, φ. In light of (a), both g and h attain a (local, and therefore global) minimum at $t = 0$ since $g(0) = h(0) = 0$, and (c) follows. Further notice that g is strictly convex if ψ is strictly convex, provided $y \neq 0$. Since a strictly convex function can have at most one global minimum, (d) follows from (c). We now consider statement (e). Since g is continuous and $g(0) \leqslant h(1) \leqslant g(1)$, the Intermediate Value Theorem ensures the existence of $\lambda \in [0, 1]$ such that $h(1) = g(\lambda)$. There exists no λ in the interval $(1, \infty)$ with the same property because g is strictly convex, and therefore $g(t) > g(1)$ if $t > 1$. The uniqueness of λ also follows from the strict convexity of g. Statement (f) is immediate, and this concludes the proof of the lemma. $\square$

Henceforth we restrict attention to continuously differentiable functions ψ, φ which are strictly convex, and introduce two sets which will play a role in the proof of Theorem 1.3. Given $\xi \in \mathbb{R}^d$ and $c \in \mathbb{R}$, define the *ψ-ellipsoid* as

$$\mathcal{E}_\psi(\xi, c) := \{y \in \mathbb{R}^d : \psi(\xi/2 - y) + \psi(\xi/2 + y) - 2\psi(\xi/2) = c\}, \tag{3.3}$$

and similarly for the φ-ellipsoid $\mathcal{E}_\varphi(\xi,c)$. We will abuse notation mildly by occasionally referring to these sets simply as "ellipsoids". The sets $\mathcal{E}_\psi(\xi,c)$ and $\mathcal{E}_\varphi(\xi,c)$ are non-empty provided $c \geqslant 0$, and codimension 1 hypersurfaces if $c > 0$. This claim requires a short justification which goes as follows. Since the function ψ is differentiable and strictly convex, its gradient $\nabla\psi$ is a strictly monotone mapping, in the sense that

$$\langle \nabla\psi(x) - \nabla\psi(x'), x - x' \rangle > 0, \text{ for every } x \neq x',$$

see, for instance, [16, p. 112]. As a consequence, any positive number $c > 0$ is a regular value of the function $F_\psi : \mathbb{R}^d \to \mathbb{R}$, defined via

$$y \mapsto F_\psi(y) = \psi(\xi/2 - y) + \psi(\xi/2 + y) - 2\psi(\xi/2),$$

and the claim follows for the ellipsoid $\mathcal{E}_\psi(\xi,c) = F_\psi^{-1}(c)$. The assertion for φ can be verified in an identical way. Further note that, for each fixed $\xi \in \mathbb{R}^d$, the disjoint union of the ellipsoids $\mathcal{E}_\psi(\xi,c)$ as the parameter $c \geqslant 0$ ranges over the nonnegative real numbers equals the whole of $\mathbb{R}^d$, and similarly for φ.

After these preliminary observations, define the transformation

$$T : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d, \quad T(y) = \lambda(y,\xi)y, \tag{3.4}$$

where $\lambda(y,\xi)$ is given by part (e) of Lemma 3.1. In other words, the real number $\lambda = \lambda(y,\xi)$ is the unique nonnegative solution of the equation

$$\varphi(\xi/2 - y) + \varphi(\xi/2 + y) - 2\varphi(\xi/2) = \psi(\xi/2 - \lambda y) + \psi(\xi/2 + \lambda y) - 2\psi(\xi/2). \tag{3.5}$$

Relevant properties of the transformation T are recorded in the next result.

Lemma 3.2. *Let $\psi, \varphi : \mathbb{R}^d \to \mathbb{R}$ be continuously differentiable, strictly convex functions with a convex difference $\psi - \varphi$. Let $\xi \in \mathbb{R}^d$ be given, and consider the transformation T given by (3.4). Then:*

(a) *T is injective.*
(b) *T is continuously differentiable.*
(c) *If $T'(y)$ denotes the Jacobian matrix of T at a point $y \neq 0$, then*

$$\det T'(y) = \lambda(y)^{d-1} \frac{\langle \nabla\varphi(\xi/2 + y) - \nabla\varphi(\xi/2 - y), y \rangle}{\langle \nabla\psi(\xi/2 + T(y)) - \nabla\psi(\xi/2 - T(y)), y \rangle}. \tag{3.6}$$

(d) *T defines a bijection from the ellipsoid $\mathcal{E}_\varphi(\xi,c)$ onto the ellipsoid $\mathcal{E}_\psi(\xi,c)$, for every $c > 0$.*

Proof. To prove (a), let us consider nonzero vectors $y, z \in \mathbb{R}^d$ such that $T(y) = T(z)$. Then

$$\psi(\xi/2 - \lambda(y)y) + \psi(\xi/2 + \lambda(y)y) = \psi(\xi/2 - \lambda(z)z) + \psi(\xi/2 + \lambda(z)z),$$

where, for notational convenience, we have dropped the dependence of λ on ξ. This implies

$$\varphi(\xi/2 - y) + \varphi(\xi/2 + y) = \varphi(\xi/2 - z) + \varphi(\xi/2 + z).$$

Since $y = rz$ for some $r > 0$, and the function $t \mapsto \varphi(\xi/2 - tz) + \varphi(\xi/2 + tz)$ is strictly increasing on $(0, \infty)$, we obtain $r = 1$. This means $y = z$, as desired.

Property (b) will follow from the Implicit Function Theorem, after showing that the derivative of the map $t \mapsto g(t) = \psi(\xi/2 - ty) + \psi(\xi/2 + ty) - 2\psi(\xi/2)$ is nonzero for each $\xi, y \in \mathbb{R}^d$ with $y \neq 0$, provided $t > 0$. This derivative equals

$$g'(t) = \langle \nabla\psi(\xi/2 + ty) - \nabla\psi(\xi/2 - ty), y \rangle,$$

which is nonzero because of the strict convexity of ψ. Indeed, in the proof of Lemma 3.1 we have already argued that g is a strictly convex C^1 function which attains its unique global minimum at $t = 0$, hence $g'(t) > 0$ for every $t > 0$. Alternatively, recall that the gradient $\nabla\psi$ is a strictly monotone mapping.

To verify (c), we compute the Jacobian matrix of T in an analogous way to what was done in the proof of Proposition 2.1. Implicit differentiation with respect to the variable y of identity (3.5) with $\lambda = \lambda(y)$ yields

$$(\lambda I + \nabla\lambda \cdot y^T) \cdot u = v,$$

where the vectors $u, v \in \mathbb{R}^d$ are defined by

$$u = \nabla\psi(\xi/2 + T(y)) - \nabla\psi(\xi/2 - T(y)),$$
$$v = \nabla\varphi(\xi/2 + y) - \nabla\varphi(\xi/2 - y).$$

For $y \neq 0$, it follows that

$$\nabla\lambda = \frac{v - \lambda u}{\langle u, y \rangle},$$

where the denominator $\langle u, y \rangle$ is strictly positive because the gradient $\nabla\psi$ is strictly monotone and the vector $T(y)$ is collinear with y. Using this together with the Matrix Determinant Lemma, we arrive at identity (3.6):

$$\det T'(y) = \det(\lambda I + \nabla\lambda \cdot y^T) = \det(\lambda I)(1 + \lambda^{-1}\langle y, \nabla\lambda \rangle) = \lambda^{d-1}\frac{\langle v, y \rangle}{\langle u, y \rangle}.$$

We finally turn to (d). That the transformation T has the desired mapping properties from $\mathcal{E}_\varphi$ into $\mathcal{E}_\psi$ follows from the defining identity (3.5). In view of (a), the restriction of T to the set $\mathcal{E}_\varphi$ is an injective map. So we are left with verifying surjectivity. The previous considerations show that, given $c > 0$ and $z \in \mathcal{E}_\psi(\xi, c)$, it suffices to find *any* vector $y \in \mathbb{R}^d$ for which $T(y) = z$ (for such y will then necessarily belong to $\mathcal{E}_\varphi(\xi, c)$). But T is a continuous map which preserves rays emanating from the origin, such that $|Ty| \leqslant |y|$ for every $y \neq 0$, and

$$\lim_{|y| \to \infty} |Ty| = \infty.$$

The result follows from the Intermediate Value Theorem. $\qquad\square$

Recall that $|T(y)| \leqslant |y|$, for every $y \neq 0$. We would like to argue that the transformation T is contractive in the sense that $|\det T'| < 1$. Unfortunately, an explicit computation involving the example $\varphi(x) = |x|^4$ and $\psi(x) = |x|^2 + |x|^4 + |x|^6$ reveals that, perhaps unintuitively, one should not expect that to be the case in general. We will be interested in convex perturbations of the paraboloid, and so the following result will suffice for our purposes.

Lemma 3.3. *Let $d \geqslant 2$. Let $\varphi = |\cdot|^2$ and $\psi = |\cdot|^2 + \phi$, where $\phi \geqslant 0$ is a strictly convex $C^1(\mathbb{R}^d)$ function. Let $\xi \in \mathbb{R}^d$ be given, and consider the transformation T given by (3.4). Then*

$$|\det T'(y)| < 1, \text{ for every } y \neq 0. \tag{3.7}$$

Proof. Fix $y \neq 0$. For the particular choices of ψ, φ as in the statement of the lemma, define real-valued functions g, h via identities (3.1) and (3.2). In this case, $h'(t) = 4|y|^2 t$, a homogenous function of degree 1. Identity (3.6) then implies

$$\det T'(y) = \lambda(y)^{d-1} \frac{h'(1)}{g'(\lambda(y))} = \lambda(y)^{d-2} \frac{h'(\lambda(y))}{g'(\lambda(y))}. \tag{3.8}$$

We have already argued that $g - h$ is a nonnegative, differentiable, strictly convex function satisfying $(g - h)(0) = 0$ and $(g - h)'(0) = 0$. It follows that $(g - h)'(t) > 0$ for every $t > 0$, which means that the fraction on the right-hand side of identity (3.8) is strictly less than 1 as long as $\lambda(y) > 0$. That this is indeed the case follows from part (f) of Lemma 3.1, since $h(1) = 2|y|^2 > 0$. The proof is finished by noting that $\lambda(y) \leqslant 1$ and $d \geqslant 2$ together imply $\lambda(y)^{d-2} \leqslant 1$. $\qquad\square$

We have now collected all the ingredients needed to prove Theorem 1.3.

Proof of Theorem 1.3. As in the proof of Proposition 2.1, the convolutions can be written as

$$(\sigma * \sigma)(\xi, \tau) = \int_{\mathbb{R}^d} \boldsymbol{\delta}\big(\tau - \psi(\xi/2 - y) - \psi(\xi/2 + y)\big) \, \mathrm{d}y, \tag{3.9}$$

$$(\sigma_0 * \sigma_0)(\xi, \tau) = \int_{\mathbb{R}^d} \boldsymbol{\delta}\big(\tau - \varphi(\xi/2 - y) - \varphi(\xi/2 + y)\big) \, \mathrm{d}y.$$

A straightforward adaptation of the arguments there shows that the convolution $\sigma * \sigma$ is supported on the region $\{(\xi, \tau) : \tau \geqslant 2\psi(\xi/2)\}$. Since $\phi \geqslant 0$, this region is contained in the support of the convolution $\sigma_0 * \sigma_0$, i.e., the set $\{(\xi, \tau) : \tau \geqslant 2\varphi(\xi/2)\}$. For each fixed $\xi \in \mathbb{R}^d$, consider the transformation T given by (3.4), which by Lemma 3.2 maps the ellipsoid $\mathcal{E}_\varphi(\xi, \tau)$ bijectively onto $\mathcal{E}_\psi(\xi, \tau)$, for every $\tau > 0$. Changing variables $y \rightsquigarrow Ty$ in the expression (3.9) for $\sigma * \sigma$, and appealing to the defining identity (3.5), yields

$$(\sigma * \sigma)(\xi, \tau) = \int_{\mathbb{R}^d} \boldsymbol{\delta}\big(\tau - \psi(\xi/2 - Ty) - \psi(\xi/2 + Ty)\big)|\det T'(y)| \, \mathrm{d}y$$

$$= \int_{\mathbb{R}^d} \boldsymbol{\delta}\big(\tau - 2\phi(\xi/2) - \varphi(\xi/2 - y) - \varphi(\xi/2 + y)\big)|\det T'(y)| \, \mathrm{d}y. \tag{3.10}$$

From Lemma 3.3, we know that $|\det T'| \leqslant 1$, and so Hölder's inequality implies

$$(\sigma * \sigma)(\xi, \tau) \leqslant (\sigma_0 * \sigma_0)(\xi, \tau - 2\phi(\xi/2)),$$

for every $\xi \in \mathbb{R}^d$ and $\tau > 0$. This is equivalent to inequality (1.9). We now use the full power of (3.7) to argue that this inequality must be strict at every point in the interior of the support of $\sigma * \sigma$. Let (ξ, τ) be one such point, for which $c := \tau - 2\psi(\xi/2) > 0$. It is straightforward to check that the singular measure that is being integrated in (3.10) is supported on the ellipsoid $\mathcal{E}_\varphi(\xi, c)$. Since $c > 0$, this ellipsoid does not

contain the origin, and by Lemma 3.3 the strict inequality $|\det T'(y)| < 1$ holds at every point $y \in \mathcal{E}_\varphi(\xi, c)$. This can be strengthened to $|\det T'(y)| \leqslant c_0$ for some fixed $c_0 < 1$ (which depends on ϕ, ξ, τ but not on y), since the set $\mathcal{E}_\varphi(\xi, c)$ is compact and the function $y \mapsto \det T'(y)$ is continuous. The result now follows from replacing the δ-function appearing in the integral (3.10) by an appropriate ε-neighborhood of the ellipsoid $\mathcal{E}_\varphi(\xi, c)$, and then analyzing the cases of equality in Hölder's inequality. To conclude the proof of the theorem, let $\varepsilon \to 0^+$. $\square$

Remark 3.4. The previous discussion can be partially generalized to the case of n-fold convolutions for $n \geqslant 3$. Defining the functions

$$g_n(t) = \sum_{j=1}^{n-1} \psi(\xi/n - ty_j) + \psi(\xi/n + t\sum_{j=1}^{n-1} y_j) - n\psi(\xi/n),$$

$$h_n(t) = \sum_{j=1}^{n-1} \varphi(\xi/n - ty_j) + \varphi(\xi/n + t\sum_{j=1}^{n-1} y_j) - n\varphi(\xi/n),$$

we have the following generalization of Lemma 3.1, whose straightforward proof (omitted) can be done by induction on n.

Lemma 3.5. *Let $n \geqslant 2$. Let $\psi, \varphi : \mathbb{R}^d \to \mathbb{R}$ be differentiable, convex functions, such that their difference $\psi - \varphi$ is also convex. Given $\xi, y_1, \ldots, y_{n-1} \in \mathbb{R}^d$, define the functions g_n, h_n as above. Then:*

(a) *$g_n(t) \geqslant h_n(t) \geqslant 0$, for every $t \in \mathbb{R}$.*
(b) *The functions g_n and h_n are convex.*
(c) *$g_n'(0) = h_n'(0) = 0$.*
(d) *If ψ is strictly convex and $(y_1, \ldots, y_{n-1}) \neq (0, \ldots, 0)$, then g_n attains its unique global minimum at $t = 0$.*
(e) *If ψ is strictly convex and $(y_1, \ldots, y_{n-1}) \neq (0, \ldots, 0)$, then there exists a unique nonnegative $\lambda = \lambda(y_1, \ldots, y_{n-1}, \xi)$ such that*

$$h_n(1) = g_n(\lambda),$$

moreover $\lambda \in [0, 1]$.
(f) *If $h_n(1) > 0$, then $\lambda > 0$. If $h_n(1) < g_n(1)$, then $\lambda < 1$.*

An n-linear version of Theorem 1.3 would follow from satisfactory substitutes for Lemmata 3.2 and 3.3. The latter is more intricate if $n \geqslant 3$, and the authors have not investigated the extent to which the argument would need to be changed.

4. Optimal constants and nonexistence of extremizers

This chapter is devoted to the proof of Theorem 1.2. In what follows, the function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is assumed to be nonnegative, twice continuously differentiable and strictly convex, σ denotes projection measure on the surface $\Sigma_\phi \subset \mathbb{R}^3$, and $\psi = |\cdot|^2 + \phi$. We start by stating two lemmata which explore the connection between pointwise values of the convolution measure $\sigma * \sigma$, and concentration at a point.

Lemma 4.1. *Let $y_0 \in \mathbb{R}^2$ be given, and let $\{f_n\} \subset L^2(\mathbb{R}^2)$ be a sequence concentrating at y_0. Then*

$$\limsup_{n\to\infty} \frac{\|f_n\sigma * f_n\sigma\|^2_{L^2(\mathbb{R}^3)}}{\|f_n\|^4_{L^2(\mathbb{R}^2)}} \leqslant (\sigma * \sigma)(2y_0, 2\psi(y_0)). \tag{4.1}$$

Lemma 4.2. *Let $y_0 \in \mathbb{R}^2$ be given, and let $f_n(y) = e^{-n(\psi(y)-\psi(y_0)-\langle\nabla\psi(y_0), y-y_0\rangle)}$. Then the sequence $\{f_n/\|f_n\|_{L^2}\}$ concentrates at y_0, and*

$$\lim_{n\to\infty} \frac{\|f_n\sigma * f_n\sigma\|^2_{L^2(\mathbb{R}^3)}}{\|f_n\|^4_{L^2(\mathbb{R}^2)}} = (\sigma * \sigma)(2y_0, 2\psi(y_0)). \tag{4.2}$$

Proof of Lemma 4.2. We first prove that the given sequence concentrates at y_0. With that purpose in mind, fix $\rho > 0$. The function

$$\gamma(y) := \psi(y) - \psi(y_0) - \langle\nabla\psi(y_0), y - y_0\rangle$$

satisfies $\gamma \geqslant 0$, $\gamma(y_0) = 0$, $\nabla\gamma(y_0) = 0$ and $H(\gamma)(y_0) = 2I + H(\phi)(y_0)$. It follows that, for any sufficiently small $\varepsilon > 0$, there exists $r = r_\varepsilon > 0$ such that the inequality

$$\gamma(y) \leqslant (1 + \varepsilon)\Big(|y - y_0|^2 + \frac{1}{2}\langle y - y_0, H(\phi)(y_0) \cdot (y - y_0)\rangle\Big)$$

holds, for every $y \in \mathbb{R}^2$ satisfying $|y - y_0| \leqslant r$. The L^2 norm of the function f_n can be bounded from below as follows:

$$\|f_n\|^2_{L^2} = \int_{\mathbb{R}^2} e^{-2n\gamma(y)} \, \mathrm{d}y \geqslant \int_{\{|y-y_0|\leqslant r\}} e^{-2n(1+\varepsilon)\big(|y-y_0|^2 + \frac{1}{2}\langle y-y_0, H(\phi)(y_0)\cdot(y-y_0)\rangle\big)} \, \mathrm{d}y$$

$$= \int_{\{|y|\leqslant r\}} e^{-2n(1+\varepsilon)\langle y, A\cdot y\rangle} \, \mathrm{d}y \geqslant \frac{1}{(\det A)^{\frac{1}{2}}} \int_{\{|y|\leqslant \alpha r\}} e^{-2n(1+\varepsilon)|y|^2} \, \mathrm{d}y$$

$$= \frac{2\pi}{(\det A)^{\frac{1}{2}}} \frac{1 - e^{-2n(1+\varepsilon)\alpha^2 r^2}}{4n(1+\varepsilon)},$$

where the (positive-definite) matrix A is given by $A = I + \frac{1}{2}H(\phi)(y_0)$, and $\alpha > 0$ denotes the square root of the smallest eigenvalue of A. Noting that

$$\gamma(y) = |y - y_0|^2 + \phi(y) - \phi(y_0) - \langle\nabla\phi(y_0), y - y_0\rangle \geqslant |y - y_0|^2,$$

we obtain

$$\int_{\{|y-y_0|\geqslant\rho\}} |f_n(y)|^2 \, \mathrm{d}y \leqslant \int_{\{|y-y_0|\geqslant\rho\}} e^{-2n|y-y_0|^2} \, \mathrm{d}y = 2\pi\frac{e^{-2n\rho^2}}{4n}.$$

Therefore

$$\|f_n\|^{-2}_{L^2} \int_{\{|y-y_0|\geqslant\rho\}} |f_n(y)|^2 \, \mathrm{d}y \leqslant (1+\varepsilon)(\det A)^{\frac{1}{2}} \frac{e^{-2n\rho^2}}{1 - e^{-2n(1+\varepsilon)\alpha^2 r^2}} \to 0,$$

as $n \to \infty$, as had to be shown. We now turn to the proof of identity (4.2). Start by noting that the function γ equals the restriction of the linear affine function

$$(\xi, \tau) \mapsto \tau - \psi(y_0) - \langle\nabla\psi(y_0), \xi - y_0\rangle$$

to the surface $\Sigma_\phi \subset \mathbb{R}^3$. It follows that

$$(f_n\sigma * f_n\sigma)(\xi,\tau) = e^{-n(\tau-\langle\nabla\psi(y_0),\xi\rangle)}e^{2n(\psi(y_0)-\langle\nabla\psi(y_0),y_0\rangle)}(\sigma * \sigma)(\xi,\tau),$$

which in turn implies the pointwise identity

$$(f_n\sigma * f_n\sigma)^2 = (f_n^2\sigma * f_n^2\sigma)(\sigma * \sigma). \tag{4.3}$$

Given $r > 0$, let

$$E_r := \{(y,\psi(y)) \in \mathbb{R}^{2+1} | \, y \in B_r(y_0)\} \subset \Sigma_\phi$$

denote the cap of radius r and center $(y_0,\psi(y_0))$ on the surface Σ_ϕ. From identity (4.3), it follows that

$$\|f_n\sigma * f_n\sigma\|_{L^2(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} \big(f_n^2\mathbb{1}_{E_r}\sigma * f_n^2\mathbb{1}_{E_r}\sigma\big)(\xi,\tau)\big(\sigma * \sigma\big)(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

$$+ \int_{\mathbb{R}^3} \big(f_n^2\mathbb{1}_{E_r^\complement}\sigma * f_n^2\mathbb{1}_{E_r^\complement}\sigma\big)(\xi,\tau)\big(\sigma * \sigma\big)(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

$$+ 2\int_{\mathbb{R}^3} \big(f_n^2\mathbb{1}_{E_r}\sigma * f_n^2\mathbb{1}_{E_r^\complement}\sigma\big)(\xi,\tau)\big(\sigma * \sigma\big)(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau,$$

where $E_r^\complement$ stands for the complement of the set E_r in Σ_ϕ. Dividing by $\|f_n\|_{L^2}^4$, we can bound the last summand by

$$\|f_n\|_{L^2}^{-4}\int_{\mathbb{R}^3} \big(f_n^2\mathbb{1}_{E_r}\sigma * f_n^2\mathbb{1}_{E_r^\complement}\sigma\big)(\xi,\tau)\big(\sigma * \sigma\big)(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

$$\leqslant \sup_{(\xi,\tau)\in\mathbb{R}^3}(\sigma * \sigma)(\xi,\tau)\frac{\|f_n\mathbb{1}_{E_r'}\|_{L^2}^2}{\|f_n\|_{L^2}^2}\frac{\|f_n\mathbb{1}_{E_r'^\complement}\|_{L^2}^2}{\|f_n\|_{L^2}^2},$$

where $E_r' := B_r(y_0) \subset \mathbb{R}^2$, and $E_r'^\complement$ stands for the complement of the set E_r' in $\mathbb{R}^2$. The right-hand side of this inequality tends to zero, as $n \to \infty$, because the sequence $\{f_n/\|f_n\|_{L^2}\}$ concentrates at the point y_0. The second summand can be treated in an analogous way. The first summand, after appropriate normalization, is bounded from above by

$$\|f_n\|_{L^2}^{-4}\int_{\mathbb{R}^3} \big(f_n^2\mathbb{1}_{E_r}\sigma * f_n^2\mathbb{1}_{E_r}\sigma\big)(\xi,\tau)\big(\sigma * \sigma\big)(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

$$\leqslant \frac{\|f_n\mathbb{1}_{E_r'}\|_{L^2}^4}{\|f_n\|_{L^2}^4}\sup_{(\xi,\tau)\in E_r+E_r}(\sigma * \sigma)(\xi,\tau),$$

and from below by

$$\|f_n\|_{L^2}^{-4}\int_{\mathbb{R}^3} \big(f_n^2\mathbb{1}_{E_r}\sigma * f_n^2\mathbb{1}_{E_r}\sigma\big)(\xi,\tau)\big(\sigma * \sigma\big)(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

$$\geqslant \frac{\|f_n\mathbb{1}_{E_r'}\|_{L^2}^4}{\|f_n\|_{L^2}^4}\inf_{(\xi,\tau)\in E_r+E_r}(\sigma * \sigma)(\xi,\tau).$$

Since $\|f_n \mathbb{1}_{E'_r}\|_{L^2}/\|f_n\|_{L^2} \to 1$, as $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \frac{\|f_n\sigma * f_n\sigma\|^2_{L^2(\mathbb{R}^3)}}{\|f_n\|^4_{L^2(\mathbb{R}^2)}} \leqslant \sup_{(\xi,\tau)\in E_r+E_r} (\sigma * \sigma)(\xi,\tau),$$

and

$$\liminf_{n\to\infty} \frac{\|f_n\sigma * f_n\sigma\|^2_{L^2(\mathbb{R}^3)}}{\|f_n\|^4_{L^2(\mathbb{R}^2)}} \geqslant \inf_{(\xi,\tau)\in E_r+E_r} (\sigma * \sigma)(\xi,\tau).$$

Identity (4.2) follows because the convolution $\sigma * \sigma$ defines a continuous function up to the boundary of its support, and $r > 0$ was arbitrary. Taking $r \to 0^+$ finishes the proof. $\qquad\square$

Sketch of proof of Lemma 4.1. Integrate the pointwise bound

$$|(f\sigma * f\sigma)(\xi,\tau)|^2 \leqslant \big(|f|^2\sigma * |f|^2\sigma\big)(\xi,\tau)\big(\sigma * \sigma\big)(\xi,\tau),$$

which was observed in [11, 24, 26] to hold almost everywhere, and proceed as in the proof of the corresponding inequality in Lemma 4.2. $\qquad\square$

Proof of Theorem 1.2. As in the proof of Lemma 4.1, the Cauchy–Schwarz inequality implies

$$\|f\sigma * f\sigma\|^2_{L^2(\mathbb{R}^3)} \leqslant \|\sigma * \sigma\|_{L^\infty(\mathbb{R}^3)}\|f\|^4_{L^2(\mathbb{R}^2)}. \tag{4.4}$$

It follows that (a possibly non-sharp version of) inequality (1.6) holds, as long as the L^∞ norm of the convolution $\sigma * \sigma$ is finite. This, in turn, can be seen using identity (2.2), since the Hessian of ψ satisfies $H(\psi) = 2I + H(\phi)$, and the matrix $H(\phi)(x)$ is positive semidefinite, for every $x \in \mathbb{R}^2$. Estimate (4.4) also shows that the optimal constant in inequality (1.6) satisfies

$$\mathcal{R}^4_\phi \leqslant \|\sigma * \sigma\|_{L^\infty}.$$

Now, let σ_0 denote the projection measure on the paraboloid Σ_0. From Theorem 1.3 and Remark 2.2, we know that $\|\sigma * \sigma\|_{L^\infty} \leqslant \|\sigma_0 * \sigma_0\|_{L^\infty} = \frac{\pi}{2}$. That these two quantities are actually the same follows from the fact that the convolution $\sigma * \sigma$ attains the value $\pi/2$ at the boundary point $(2y_0, 2\psi(y_0))$ in case (i), or at infinity in case (ii). Identity (1.7) will then follow from the inequality

$$\mathcal{R}^4_\phi \geqslant \frac{\pi}{2}, \tag{4.5}$$

which we establish using the sequences given by (1.8). We consider the two cases separately. In case (i), since $H(\phi)(y_0) = 0$, it follows from identity (2.4) that $(\sigma * \sigma)(2y_0, 2\psi(y_0)) = \frac{\pi}{2}$, and therefore the sequence $\{f_n/\|f_n\|_{L^2}\}$, where

$$f_n(y) = e^{-n(\psi(y)-\psi(y_0)-\langle\nabla\psi(y_0),y-y_0\rangle)},$$

is extremizing for inequality (1.6) in light of Lemma 4.2. In case (ii), we have that $(\sigma * \sigma)(2y_n, 2\psi(y_n)) \to \frac{\pi}{2}$, as $n \to \infty$. Choose a sequence $\{a_n\} \subset \mathbb{N}$ in such a way that, for every $n \in \mathbb{N}$, the function given by

$$f_n(y) = e^{-a_n(\psi(y)-\psi(y_n)-\langle\nabla\psi(y_n),y-y_n\rangle)}$$

satisfies

$$\left| \frac{\|f_n\sigma * f_n\sigma\|_{L^2}^2}{\|f_n\|_{L^2}^4} - (\sigma * \sigma)(2y_n, 2\psi(y_n)) \right| \leqslant \frac{1}{n},$$

and

$$\int_{\{|y-y_n| \geqslant \frac{1}{n}\}} |f_n(y)|^2 \, \mathrm{d}y \leqslant \frac{1}{n} \|f_n\|_{L^2}^2. \tag{4.6}$$

That this is possible follows again from Lemma 4.2. Since

$$\frac{\|f_n\sigma * f_n\sigma\|_{L^2}^2}{\|f_n\|_{L^2}^4} \to \frac{\pi}{2}, \text{ as } n \to \infty,$$

the sequence $\{f_n/\|f_n\|_{L^2}\}$ is again extremizing for inequality (1.6). This establishes (4.5) in both cases (i) and (ii), and therefore identity (1.7) is proved. Incidentally, note that condition (4.6) ensures that $\{f_n/\|f_n\|_{L^2}\}$ concentrates along the sequence $\{y_n\}$. Since $|y_n| \to \infty$, as $n \to \infty$, it concentrates at infinity.

We finish by showing that extremizers for inequality (1.6) do not exist. Aiming at a contradiction, let f be an extremizer. An application of Cauchy–Schwarz and Hölder's inequalities yields

$$\mathcal{R}_\phi^4 \|f\|_{L^2}^4 = \|f\sigma * f\sigma\|_{L^2}^2$$

$$\leqslant \int_{\mathbb{R}^3} |(f^2\sigma * f^2\sigma)(\xi, \tau)|(\sigma * \sigma)(\xi, \tau) \, \mathrm{d}\xi \, \mathrm{d}\tau$$

$$\leqslant \|\sigma * \sigma\|_{L^\infty} \int_{\mathbb{R}^3} |(f^2\sigma * f^2\sigma)(\xi, \tau)| \, \mathrm{d}\xi \, \mathrm{d}\tau$$

$$= \|\sigma * \sigma\|_{L^\infty} \|f\|_{L^2}^4.$$

Since $\mathcal{R}_\phi^4 = \|\sigma * \sigma\|_{L^\infty} = \frac{\pi}{2}$ and $f \neq 0$, all inequalities in this chain of inequalities must be equalities. In particular, the convolution $\sigma * \sigma$ must be constant equal to $\|\sigma*\sigma\|_{L^\infty}$ almost everywhere inside the support of $f^2\sigma * f^2\sigma$, which is a set of positive Lebesgue measure since $f \neq 0$. This contradicts the strict inequality

$$(\sigma * \sigma)(\xi, \tau) < \|\sigma * \sigma\|_{L^\infty}, \text{ for almost every } (\xi, \tau) \in \text{supp}(\sigma * \sigma),$$

which in turn is an immediate consequence of the second part of Theorem 1.3. This contradiction shows that extremizers do not exist. The proof of the theorem is now complete. □

5. On extremizing sequences

From the previous chapter, we know that extremizers for inequality (1.6) do not exist. As mentioned in the Introduction, this failure of compactness can be understood via the concentration-compactness principle, which is the subject of the present chapter. Heuristically, an extremizing sequence for inequality (1.6) should concentrate around the points where the function $\sigma * \sigma$ achieves its essential supremum. Lemma 4.1 and formula (2.4) imply that, if an extremizing sequence concentrates at a point y_0, then necessarily $H(\phi)(y_0) = 0$. Lemma 4.2 provides the construction of an explicit extremizing sequence concentrating at any point $y_0 \in \mathbb{R}^2$, provided

$H(\phi)(y_0) = 0$. Therefore, concentration occurs at a point if and only if the Hessian vanishes at that point. Further information concerning extremizing sequences concentrating at spatial infinity will be obtained below.

5.1. **Weak interaction between distant caps.** Reasoning in a similar way to the proof of (2.2) from Proposition 2.1, we find that the identity

$$(f\sigma * g\sigma)(\xi, \tau) = \int_{\mathbb{S}^1} \frac{f(\xi/2 + \alpha(\xi, \tau, \omega)\omega)g(\xi/2 - \alpha(\xi, \tau, \omega)\omega)}{\int_{-1}^1 \langle \omega, H(\psi)(\xi/2 + t\alpha(\xi, \tau, \omega)\omega) \cdot \omega \rangle \, dt} \, d\mu_\omega \tag{5.1}$$

holds, in particular, in the case when f, g are indicator functions of balls or their complements. Formula (5.1) allows for a quantification of the general principle that "distant caps interact weakly". This is a geometric feature that translates into useful bilinear estimates, and has been observed in a variety of related contexts; see, for instance, [7, 23]. The precise statement is as follows.

Lemma 5.1. *Let $r, \rho > 0$ satisfy $\rho > 3r$. Then, for any $y_0 \in \mathbb{R}^2$,*

$$\|\mathbb{1}_{B_r(y_0)}\sigma * \mathbb{1}_{B_\rho^\complement(y_0)}\sigma\|_{L^\infty(\mathbb{R}^3)} \leqslant \frac{1}{2}\arcsin\left(\frac{2r}{\rho - r}\right). \tag{5.2}$$

As a result, the following statements hold:
  (a) *For any $r > 0$ and $y_0 \in \mathbb{R}^2$,*

$$\lim_{\rho \to \infty} \|\mathbb{1}_{B_r(y_0)}\sigma * \mathbb{1}_{B_\rho^\complement(y_0)}\sigma\|_{L^\infty(\mathbb{R}^3)} = 0. \tag{5.3}$$

  (b) *For any $\rho > 0$ and $y_0 \in \mathbb{R}^2$,*

$$\lim_{r \to 0^+} \|\mathbb{1}_{B_r(y_0)}\sigma * \mathbb{1}_{B_\rho^\complement(y_0)}\sigma\|_{L^\infty(\mathbb{R}^3)} = 0. $$

  (c) *For any $r > 0$,*

$$\lim_{\rho \to \infty} \sup_{y \in \mathbb{R}^2} \|\mathbb{1}_{B_r(y)}\sigma * \mathbb{1}_{B_\rho^\complement(y)}\sigma\|_{L^\infty(\mathbb{R}^3)} = 0. \tag{5.4}$$

Moreover,
  (d) *If $B, B' \subseteq \mathbb{R}^2$ are disjoint balls, then*

$$\|\mathbb{1}_B\sigma * \mathbb{1}_{B'}\sigma\|_{L^\infty(\mathbb{R}^3)} \leqslant \frac{\pi}{4}. $$

Proof. We establish identity (5.2) for $y_0 = 0$ only, the case of general $y_0 \in \mathbb{R}^2$ being similar. Let $\rho > r > 0$. If $f = \mathbb{1}_{B_r}$ and $g = \mathbb{1}_{B_\rho^\complement}$, then the integrand in (5.1) is nonzero only if the point (ξ, τ) satisfies

$$\xi/2 + \alpha(\xi, \tau, \omega)\omega \in B_r, \quad \text{and} \quad \xi/2 - \alpha(\xi, \tau, \omega)\omega \notin B_\rho. $$

By the triangle inequality, this can only happen if

$$|\xi| \geqslant |\xi/2 - \alpha(\xi, \tau, \omega)\omega| - |\xi/2 + \alpha(\xi, \tau, \omega)\omega| \geqslant \rho - r. \tag{5.5}$$

In this case, if $\rho > 3r$, then $|\xi/2| > r$, and therefore the ray $\xi/2 + t\omega$, $t > 0$, intersects the ball B_r only if ω belongs to an arc of $\mathbb{S}^1$ of measure exactly $2\arcsin(\frac{2r}{|\xi|})$. Denoting arc length measure on the unit circle by μ as usual, we conclude that

$$\mu(\{\omega \in \mathbb{S}^1 : \xi/2 + \alpha(\xi, \tau, \omega)\omega \in B_r, \xi/2 - \alpha(\xi, \tau, \omega)\omega \notin B_\rho\})$$
$$\leqslant 2\arcsin\left(\frac{2r}{|\xi|}\right) \leqslant 2\arcsin\left(\frac{2r}{\rho - r}\right).$$

It follows that, for every $(\xi, \tau) \in \mathbb{R}^3$,

$$(\mathbb{1}_{B_r}\sigma * \mathbb{1}_{B_\rho^\complement}\sigma)(\xi, \tau) \leqslant \frac{1}{2}\arcsin\left(\frac{2r}{\rho - r}\right),$$

where we bounded the denominator in (5.1) from below by 4. Parts (a) and (b) follow at once, and a similar reasoning for $y_0 \neq 0$ establishes (c). For part (d), note that the definition (2.6) of the function λ implies $\lambda(-w, \xi) = \lambda(w, \xi)$ for every w, ξ, and therefore the function α satisfies $\alpha(\xi, \tau, -\omega) = \alpha(\xi, \tau, \omega)$, for every $\omega \in \mathbb{S}^1$. It then follows that, if $\xi/2 + \alpha(\xi, \tau, \omega)\omega \in B$ and $\xi/2 - \alpha(\xi, \tau, \omega)\omega \in B'$, then $\xi/2 + \alpha(\xi, \tau, -\omega)(-\omega) \notin B$ and $\xi/2 - \alpha(\xi, \tau, -\omega)(-\omega) \notin B'$. As a consequence, the subset of $\mathbb{S}^1$ where the integrand in (5.1) is nonzero has measure bounded from above by π, and the result follows as before. $\qquad\square$

5.2. **Concentration-compactness.** The three lemmata in this section hold under the general hypotheses of Theorem 1.2, which for brevity will not be included in the corresponding statements.

Lemma 5.2. *Under the hypotheses of Theorem 1.2, suppose that there exist a subset $X \subset (\mathbb{R}^2)^2$ and $\delta > 0$ such that, for every $(y, z) \in X$,*

$$\frac{(\sigma * \sigma)(y + z, \psi(y) + \psi(z))}{\|\sigma * \sigma\|_{L^\infty(\mathbb{R}^3)}} \leqslant 1 - \delta. \tag{5.6}$$

Let $\{f_n\} \subset L^2(\mathbb{R}^2)$ be any extremizing sequence for inequality (1.6). Then

$$\int_X |f_n(y)|^2|f_n(z)|^2 \, \mathrm{d}y\,\mathrm{d}z \to 0, \ \ as \ n \to \infty.$$

In particular, if X contains a subset of the form $A \times B$, for some $A, B \subset \mathbb{R}^2$, then

$$\int_A |f_n(y)|^2 \, \mathrm{d}y \int_B |f_n(z)|^2 \, \mathrm{d}z \to 0, \ \ as \ n \to \infty.$$

Proof. Let $\{f_n\} \subset L^2(\mathbb{R}^2)$ be an extremizing sequence for inequality (1.6). The first step is to verify that

$$\liminf_{n\to\infty} \int_{(\mathbb{R}^2)^2} |f_n(y)|^2|f_n(z)|^2 \frac{(\sigma * \sigma)(y + z, \psi(y) + \psi(z))}{\|\sigma * \sigma\|_{L^\infty}} \, \mathrm{d}y\,\mathrm{d}z = 1. \tag{5.7}$$

With this goal in mind, estimate

$$\int_{\mathbb{R}^3} |(f_n\sigma * f_n\sigma)(\xi,\tau)|^2 \, d\xi \, d\tau \leqslant \int_{\mathbb{R}^3} \big(|f_n|^2\sigma * |f_n|^2\sigma\big)(\xi,\tau)\big(\sigma * \sigma\big)(\xi,\tau) \, d\xi \, d\tau$$

$$= \int_{(\mathbb{R}^2)^2} |f_n(y)|^2 |f_n(z)|^2 (\sigma * \sigma)(y+z, \psi(y)+\psi(z)) \, dy \, dz$$

$$\leqslant \|\sigma * \sigma\|_{L^\infty} \|f_n\|_{L^2}^4.$$

The first and the last terms in this chain of inequalities converge to $\|\sigma * \sigma\|_{L^\infty}$, as $n \to \infty$, and therefore so does the third term, and (5.7) follows. We next observe

$$\liminf_{n\to\infty} \int_{(\mathbb{R}^2)^2} |f_n(y)|^2 |f_n(z)|^2 \, dy \, dz = \lim_{n\to\infty} \|f_n\|_{L^2}^4 = 1.$$

Writing $X^\complement$ for the complement of the set X in $(\mathbb{R}^2)^2$, we have an inequality

$$\int_{(\mathbb{R}^2)^2} |f_n(y)|^2 |f_n(z)|^2 \frac{(\sigma * \sigma)(y+z, \psi(y)+\psi(z))}{\|\sigma * \sigma\|_{L^\infty}} \, dy \, dz$$

$$\leqslant (1-\delta) \int_X |f_n(y)|^2 |f_n(z)|^2 \, dy \, dz + \int_{X^\complement} |f_n(y)|^2 |f_n(z)|^2 \, dy \, dz.$$

Since $\|f_n\|_{L^2} \to 1$ as $n \to \infty$, we conclude from (5.7) that

$$1 \leqslant \liminf_{n\to\infty} \Big(\Big(\int_{\mathbb{R}^2} |f_n(y)|^2 \, dy \Big)^2 - \delta \int_X |f_n(y)|^2 |f_n(z)|^2 \, dy \, dz \Big)$$

$$= 1 - \delta \limsup_{n\to\infty} \int_X |f_n(y)|^2 |f_n(z)|^2 \, dy \, dz.$$

It follows that

$$\limsup_{n\to\infty} \int_X |f_n(y)|^2 |f_n(z)|^2 \, dy \, dz = 0,$$

which establishes the first statement. The second statement follows at once, and the proof is complete. $\qquad\square$

The preceding lemma implies the following modest amount of control over extremizing sequences that split their mass in a nontrivial way.

Lemma 5.3. *Under the hypotheses of Theorem 1.2, let $\{f_n\} \subset L^2(\mathbb{R}^2)$ be any extremizing sequence for inequality (1.6). Let $0 < r_1 < r_2 < r_3 < \infty$ be arbitrary. Then*

$$\int_{B_{r_1}} |f_n(y)|^2 \, dy \int_{B_{r_3} \setminus B_{r_2}} |f_n(z)|^2 \, dz \to 0, \ \text{as } n \to \infty.$$

Proof. Let $X = B_{r_1} \times (B_{r_3} \setminus B_{r_2})$. Appealing to the continuity of the convolution $\sigma * \sigma$ on its support, to the fact that the essential supremum is only achieved on the boundary of the support (as observed in the course of the proof of Theorem 1.3), together with the compactness of the set $\overline{X}$ and the fact that $r_1 < r_2$, we can ensure the existence of $\delta = \delta_{r_1,r_2,r_3} > 0$ such that

$$\frac{(\sigma * \sigma)(y+z, \psi(y)+\psi(z))}{\|\sigma * \sigma\|_{L^\infty}} \leqslant 1-\delta,$$

for every $(y, z) \in X$. The conclusion now follows from Lemma 5.2. $\qquad\square$

Lemma 5.3 can be upgraded in a way that reveals that an extremizing sequence can only split its mass in a nontrivial way if neither of the corresponding supports remains in a bounded region. We formulate one version of this principle which will be useful for our purposes.

Lemma 5.4. *Under the hypotheses of Theorem 1.2, let $\{f_n\} \subset L^2(\mathbb{R}^2)$ be any extremizing sequence for inequality (1.6). Let $0 < r_1 < r_2 < \infty$ be arbitrary. Then*

$$\int_{B_{r_1}} |f_n(y)|^2 \, dy \int_{\mathbb{R}^2 \setminus B_{r_2}} |f_n(z)|^2 \, dz \to 0, \ as \ n \to \infty. \tag{5.8}$$

Remark 5.5. If conclusion (5.8) holds for one pair (r_1, r_2) satisfying $0 < r_1 < r_2 < \infty$, then it holds for any pair (ρ_1, ρ_2) satisfying $0 < \rho_1 < \rho_2 < \infty$ and $r_1 \geqslant \rho_1$. To see this, start by noticing that the case $r_1 \geqslant \rho_1$ and $r_2 \leqslant \rho_2$ is clear. On the other hand, if $r_1 \geqslant \rho_1$ and $r_2 > \rho_2$, then

$$\int_{B_{\rho_1}} |f_n(y)|^2 \, dy \int_{\mathbb{R}^2 \setminus B_{\rho_2}} |f_n(z)|^2 \, dz$$

$$\leqslant \int_{B_{r_1}} |f_n(y)|^2 \, dy \int_{\mathbb{R}^2 \setminus B_{r_2}} |f_n(z)|^2 \, dz \quad + \int_{B_{\rho_1}} |f_n(y)|^2 \, dy \int_{B_{r_2} \setminus B_{\rho_2}} |f_n(z)|^2 \, dz,$$

which tends to zero, as $n \to \infty$, by (5.8) and Lemma 5.3. Moreover, in view of the uniform bound with respect to $y \in \mathbb{R}^2$ from part (c) of Lemma 5.1, the proof given below can be adapted to the case of balls centered at any point $y \in \mathbb{R}^2$, not necessarily the origin.

Proof of Lemma 5.4. Let $r_1 > 0$ be given. If

$$\int_{B_{r_1}} |f_n(y)|^2 \, dy \to 0, \ \text{as} \ n \to \infty,$$

then the conclusion follows at once since $\|f_n\|_{L^2} \leqslant 1$. Therefore no generality is lost in assuming, possibly after passing to a subsequence, that

$$\delta := \inf_{n \in \mathbb{N}} \int_{B_{r_1}} |f_n(y)|^2 \, dy > 0. \tag{5.9}$$

It suffices to show that

$$\int_{\mathbb{R}^2 \setminus B_{r_2}} |f_n(z)|^2 \, dz \to 0, \ \text{as} \ n \to \infty. \tag{5.10}$$

Take $r_3 > r_2$. From Lemma 5.3 and inequality (5.9), we know that

$$\int_{B_{r_3} \setminus B_{r_2}} |f_n(z)|^2 \, dz \to 0, \ \text{as} \ n \to \infty. \tag{5.11}$$

Decompose

$$f_n = f_n \mathbb{1}_{B_{r_2}} + f_n \mathbb{1}_{\mathbb{R}^2 \setminus B_{r_3}} + f_n \mathbb{1}_{B_{r_3} \setminus B_{r_2}} =: F_n + G_n + H_n,$$

and note that
$$f_n\sigma * f_n\sigma = F_n\sigma * F_n\sigma + G_n\sigma * G_n\sigma + 2F_n\sigma * G_n\sigma + R_n,$$
where, in view of inequality (1.6) and estimate (5.11), the remainder term R_n satisfies
$$\|R_n\|_{L^2(\mathbb{R}^3)} \leqslant C\|H_n\|_{L^2(\mathbb{R}^2)} \to 0, \text{ as } n \to \infty.$$
The key step is to bound the quantity $\|F_n\sigma * G_n\sigma\|_{L^2}^2$. We have the pointwise inequality
$$|(F_n\sigma * G_n\sigma)(\xi,\tau)|^2 \leqslant \big(|F_n|^2\sigma * |G_n|^2\sigma\big)(\xi,\tau)\big(\mathbb{1}_{B_{r_2}}\sigma * \mathbb{1}_{\mathbb{R}^2\setminus B_{r_3}}\sigma\big)(\xi,\tau),$$
which follows from an application of the Cauchy–Schwarz inequality as before. As a consequence,
$$\|F_n\sigma * G_n\sigma\|_{L^2(\mathbb{R}^3)}^2 \leqslant \rho^2(r_2,r_3)\|F_n\|_{L^2(\mathbb{R}^2)}^2\|G_n\|_{L^2(\mathbb{R}^2)}^2,$$
where the function ρ is given by
$$\rho(r_2,r_3) := \big\|\mathbb{1}_{B_{r_2}}\sigma * \mathbb{1}_{\mathbb{R}^2\setminus B_{r_3}}\sigma\big\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{2}}.$$
For large values of r_3, the sets B_{r_2} and $\mathbb{R}^2 \setminus B_{r_3}$ interact weakly as discussed in §5.1. Part (a) of Lemma 5.1 implies that, for each fixed $r_2 > 0$,
$$\rho(r_2,r_3) \to 0, \text{ as } r_3 \to \infty.$$
Two applications of Plancherel's Theorem, together with the triangle inequality, imply the following bound for the inner product:
$$|\langle F_n\sigma * F_n\sigma, G_n\sigma * G_n\sigma\rangle_{L^2}| \leqslant \|F_n\sigma * G_n\sigma\|_{L^2}^2.$$
It follows that there exists an absolute constant $C < \infty$, which can be explicitly computed but whose exact numerical value is unimportant for our purposes, for which
$$\|f_n\sigma * f_n\sigma\|_{L^2}^2 \leqslant \|F_n\sigma * F_n\sigma\|_{L^2}^2 + \|G_n\sigma * G_n\sigma\|_{L^2}^2 + C\rho(r_2,r_3) + o_n(1)$$
$$\leqslant \frac{\pi}{2}(\|F_n\|_{L^2}^4 + \|G_n\|_{L^2}^4) + C\rho(r_2,r_3) + o_n(1)$$
$$= \frac{\pi}{2}\|f_n\|_{L^2}^4 - \pi\|F_n\|_{L^2}^2\|G_n\|_{L^2}^2 + C\rho(r_2,r_3) + o_n(1),$$
Here, we used the sharp inequality (1.6), and orthogonality considerations. The function $o_n(1)$ may depend on r_3, but satisfies $o_n(1) \to 0$, as $n \to \infty$, for each fixed r_3, and is allowed to change from line to line. Taking $n \to \infty$ in the previous chain of inequalities, we conclude
$$\limsup_{n\to\infty} \|F_n\|_{L^2}^2\|G_n\|_{L^2}^2 \leqslant C\rho(r_2,r_3).$$
Consequently,
$$\limsup_{n\to\infty} \|G_n\|_{L^2}^2 \leqslant \frac{C}{\delta}\rho(r_2,r_3),$$
where δ was defined in (5.9), and therefore,
$$\limsup_{n\to\infty}\big(\|G_n\|_{L^2}^2 + \|H_n\|_{L^2}^2\big) \leqslant \frac{C}{\delta}\rho(r_2,r_3),$$

which is equivalent to

$$\limsup_{n\to\infty} \int_{\mathbb{R}^2\setminus B_{r_2}} |f_n(z)|^2 \,\mathrm{d}z \leqslant \frac{C}{\delta}\rho(r_2, r_3).$$

Since the left-hand side of this inequality is independent of r_3, and the right-hand side tends to 0 as $r_3 \to 0$, conclusion (5.10) must hold. The proof of the lemma is now complete. □

We have collected all the ingredients necessary to the proof of Theorem 1.5.

Proof of Theorem 1.5. Let $\{f_n\} \subset L^2(\mathbb{R}^2)$ be any extremizing sequence for inequality (1.6). Take any subsequence, and slightly abuse notation by again calling it $\{f_n\}$. If this subsequence $\{f_n\}$ does not concentrate at infinity, then there exists a further sub-subsequence, still denoted by $\{f_n\}$, and a number $r_0 < \infty$, such that

$$\inf_{n\in\mathbb{N}} \int_{B_{r_0}} |f_n(y)|^2 \,\mathrm{d}y > 0.$$

From Lemma 5.4, we conclude

$$\int_{\mathbb{R}^2\setminus B_{2r_0}} |f_n(y)|^2 \,\mathrm{d}y \to 0, \text{ as } n \to \infty.$$

It follows that

$$\int_{B_{2r_0}} |f_n(y)|^2 \,\mathrm{d}y \to 1, \text{ as } n \to \infty. \tag{5.12}$$

As a consequence of Lemma 5.2, $\|\mathbb{1}_{B_{2r_0}}\sigma * \mathbb{1}_{B_{2r_0}}\sigma\|_{L^\infty} = \|\sigma * \sigma\|_{L^\infty}$, and the supremum is achieved inside the ball $\bar{B}_{2r_0}$. In particular, case (i) holds, and the set

$$E := \{y \in \mathbb{R}^2 | H(\phi)(y) = 0\}$$

is nonempty. For $\varepsilon \in (0, 1)$, let $N_\varepsilon(E)$ denote the open ε-neighborhood of E, and consider the set

$$Y := \{(y, z) \in B_{3r_0} \times B_{3r_0} | y \in N_\varepsilon(E), z \in N_\varepsilon(E), |y - z| < \varepsilon\}.$$

Let $X := (\bar{B}_{3r_0} \times \bar{B}_{3r_0}) \setminus Y$. We claim that there exists $\delta = \delta(\varepsilon) > 0$, such that

$$\frac{(\sigma * \sigma)(y + z, \psi(y) + \psi(z))}{\|\sigma * \sigma\|_{L^\infty}} \leqslant 1 - \delta, \tag{5.13}$$

for every $(y, z) \in X$. This follows from the compactness of the set X, together with the fact that, if $(y, z) \in X$, then the point $(y + z, \psi(y) + \psi(z))$ is away from the portion of the boundary of the support where the convolution $\sigma * \sigma$ attains its essential supremum in a quantifiable way that depends only on ε. Since inequality (5.13) holds for every $(y, z) \in X$, Lemma 5.2 implies

$$\int_X |f_n(y)|^2 |f_n(z)|^2 \,\mathrm{d}y \,\mathrm{d}z \to 0, \text{ as } n \to \infty.$$

In light of (5.12), it then follows that

$$\int_Y |f_n(y)|^2 |f_n(z)|^2 \,\mathrm{d}y \,\mathrm{d}z \to 1, \text{ as } n \to \infty. \tag{5.14}$$

The remaining of the proof coincides with the second part of the proof of [26, Proposition 6.3], with minor modifications only. We include it for the convenience of the reader. We seek to locate a sequence $\{y_n\} \subset \mathbb{R}^2$ along which concentration occurs. Fubini's Theorem, and the fact that $\|f_n\|_{L^2} \leqslant 1$, together imply

$$\int_Y |f_n(y)|^2 |f_n(z)|^2 \, \mathrm{d}y \, \mathrm{d}z = \int_{B_{3r_0} \cap N_\varepsilon(E)} |f_n(y)|^2 \left(\int_{B_{3r_0} \cap N_\varepsilon(E) \cap B_\varepsilon(y)} |f_n(z)|^2 \, \mathrm{d}z \right) \mathrm{d}y$$

$$\leqslant \|f_n \mathbb{1}_{B_{3r_0} \cap N_\varepsilon(E)}\|_{L^2}^2 \sup_{y \in B_{3r_0} \cap N_\varepsilon(E)} \int_{B_{3r_0} \cap N_\varepsilon(E) \cap B_\varepsilon(y)} |f_n(z)|^2 \, \mathrm{d}z \leqslant 1.$$

From (5.14), it then follows that

$$\lim_{n \to \infty} \sup_{y \in B_{3r_0} \cap N_\varepsilon(E)} \int_{B_{3r_0} \cap N_\varepsilon(E) \cap B_\varepsilon(y)} |f_n(z)|^2 \, \mathrm{d}z = 1.$$

This implies the existence of a function $N : (0,1) \to \mathbb{N}$, such that

$$\sup_{y \in B_{3r_0} \cap N_\varepsilon(E)} \int_{\{|z-y| \leqslant \varepsilon\}} |f_n(z)|^2 \, \mathrm{d}z \geqslant 1 - \frac{\varepsilon}{2}, \text{ for every } n \geqslant N(\varepsilon).$$

Hence, there exists a sequence $\{y_n^\varepsilon\}_{n \geqslant N(\varepsilon)} \subset \bar{B}_{3r_0} \cap E$, such that

$$\int_{\{|z-y_n^\varepsilon| \leqslant 2\varepsilon\}} |f_n(z)|^2 \, \mathrm{d}z \geqslant 1 - \varepsilon, \text{ for every } n \geqslant N(\varepsilon).$$

Here, we exchanged the neighborhood $N_\varepsilon(E)$ for the set E, at the expense of an extra ε in the domain of integration. We proceed to construct the sequence $\{y_n\}$ via a diagonal process. Take $\varepsilon_k = \frac{1}{k+2}$. We obtain a strictly increasing sequence

$$N_k := \max\{N(\varepsilon_j) \mid 1 \leqslant j \leqslant k\} + k,$$

and sequences $\{y_n^k\}_{n \geqslant N_k}$, satisfying

$$\int_{\{|z-y_n^k| \leqslant \frac{2}{k}\}} |f_n(z)|^2 \, \mathrm{d}z \geqslant 1 - \frac{1}{k},$$

for every $k \geqslant 1$ and $n \geqslant N_k$. For each $n \geqslant N_1$, let $\ell_n := \sup\{k \in \mathbb{N} \mid N_k \leqslant n\}$. This is a finite number since the sequence $\{N_k\}$ is strictly increasing. Further note that $n \geqslant N_{\ell_n}$. Define

$$y_n := \begin{cases} y_n^{\ell_n}, & \text{if } n \geqslant N_1, \\ y_0, & \text{if } n < N_1, \end{cases}$$

where $y_0 \in E$ is arbitrary, but fixed. It is then clear that

$$\int_{\{|z-y_n| \leqslant \frac{2}{\ell_n}\}} |f_n(z)|^2 \, \mathrm{d}z \geqslant 1 - \frac{1}{\ell_n},$$

for every $n \geqslant N_1$, which implies that $\{f_n\}$ concentrates along the sequence $\{y_n\}$ since $\ell_n \to \infty$, as $n \to \infty$. The statement about subsequences of $\{f_n\}$ follows by compactness of the set $E \cap \bar{B}_{3r_0}$, since every subsequence of $\{y_n\}$ has a further sub-subsequence that converges to a point in $E \cap \bar{B}_{3r_0}$. $\qquad \square$

5.3. **Some consequences.** The methods of the proof of Theorem 1.5 specialize to at least two distinct situations of interest. The first one is a direct consequence of the statement of Theorem 1.5.

Corollary 5.6. *Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a nonnegative, twice continuously differentiable, strictly convex function, such that*

(i) *$H(\phi)(y) \neq 0$, for every $y \in \mathbb{R}^2$, and*
(ii) *There exists a sequence $\{y_n\} \subset \mathbb{R}^2$ with $|y_n| \to \infty$, such that $H(\phi)(y_n) \to 0$, as $n \to \infty$.*

Then any extremizing sequence for inequality (1.6) concentrates at infinity.

An example of a function that satisfies the hypotheses of the preceding corollary is $\phi(y_1, y_2) = e^{y_1} + e^{y_2}$, $(y_1, y_2) \in \mathbb{R}^2$. The next result shows that extremizing sequences will not concentrate at spatial infinity if a suitable nondegeneracy condition is placed on the function ϕ.

Corollary 5.7. *Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a nonnegative, twice continuously differentiable, strictly convex function, such that the set $E := \{y \in \mathbb{R}^2 | H(\phi)(y) = 0\}$ is nonempty. Suppose that there exist $r_0 \geqslant 0$ and a function $\Theta : \mathbb{R}^2 \to [0, \infty)$ satisfying $\inf_{\{|y|>r\}} \Theta(y) > 0$, for every $r > r_0$, and such that the matrix*

$$H(\phi)(y) - \Theta(y)I \tag{5.15}$$

is positive semidefinite, for every $y \in \mathbb{R}^2$. Then every extremizing sequence $\{f_n\} \subset L^2(\mathbb{R}^2)$ for inequality (1.6) concentrates along a sequence of points in E. Moreover, given any subsequence of $\{f_n\}$, there exist a point $y_0 \in E \cap \bar{B}_{r_0}$ and a sub-subsequence which concentrates at y_0.

Condition (5.15) implies the existence of a constant $\delta > 0$, for which

$$(\sigma * \sigma)(y + z, \psi(y) + \psi(z)) \leqslant (1 - \delta)\|\sigma * \sigma\|_{L^\infty},$$

for every $(y, z) \in (\mathbb{R}^2)^2 \setminus (B_{3r_0} \times B_{3r_0})$ such that $\langle y, z \rangle \geqslant 0$, and Lemma 5.2 can then be invoked to preclude concentration at infinity. Further note that (5.15) is fulfilled by the functions $\phi = |\cdot|^p$, for each $p > 2$. In this case, we can take $r_0 = 0$, and so concentration can only occur at the origin.

In the case of extremizing sequences concentrating at infinity, we can further refine the analysis as follows. Let $\{f_n\} \subset L^2(\mathbb{R}^2)$ be a sequence such that $\|f_n\|_{L^2} \to 1$, as $n \to \infty$. We say that the sequence $\{f_n\}$ satisfies the *splitting condition* if the following holds. There exists $\alpha \in (0, 1)$ such that, for every $\varepsilon > 0$, there exist $r > 0$, $n_0 \geqslant 1$, and sequences $\{y_n\} \subset \mathbb{R}^2$, $\{r_n\} \subset \mathbb{R}$, with $r_n \to \infty$, as $n \to \infty$, such that the functions $g_{n,1} := f_n \mathbb{1}_{B_r(y_n)}$ and $g_{n,2} := f_n \mathbb{1}_{\mathbb{R}^2 \setminus B_{r_n}(y_n)}$ satisfy

$$\|f_n - (g_{n,1} + g_{n,2})\|_{L^2}^2 \leqslant \varepsilon, \; |\|g_{n,1}\|_{L^2}^2 - \alpha| \leqslant \varepsilon, \; |\|g_{n,2}\|_{L^2}^2 - (1 - \alpha)| \leqslant \varepsilon, \tag{5.16}$$

for every $n \geqslant n_0$. The following result holds.

Proposition 5.8. *Under the hypotheses of Theorem 1.2, let $\{f_n\} \subset L^2(\mathbb{R}^2)$ be any extremizing sequence for inequality (1.6). Then $\{f_n\}$ does not satisfy the splitting condition.*

Sketch of proof. Aiming at a contradiction, suppose that the extremizing sequence $\{f_n\}$ satisfies the splitting condition for a given $\alpha \in (0,1)$. Let $\varepsilon > 0$, and suppose that there exist $r > 0$, $\{y_n\} \subset \mathbb{R}^2$, and $\{r_n\} \subset \mathbb{R}$, for which condition (5.16) holds. Decompose $f_n = g_{n,1} + g_{n,2} + h_n$, where $\|h_n\|_{L^2}^2 \leqslant \varepsilon$. Part (c) of Lemma 5.1 implies the uniform estimate

$$\|g_{n,1}\sigma * g_{n,2}\sigma\|_{L^2}^2 \leqslant \rho(r,r_n)^2 \|g_{n,1}\|_{L^2}^2 \|g_{n,2}\|_{L^2}^2,$$

where the function

$$\rho(r,r_n) := \sup_{y \in \mathbb{R}^2} \|\mathbb{1}_{B_r(y)}\sigma * \mathbb{1}_{\mathbb{R}^2 \setminus B_{r_n}(y)}\sigma\|_{L^\infty}^{\frac{1}{2}}$$

satisfies $\rho(r,r_n) \to 0$, as $n \to \infty$. By an argument similar to the one following (5.11) in the proof of Lemma 5.4, we obtain

$$\limsup_{n \to \infty} \|g_{n,1}\|_{L^2}^2 \|g_{n,2}\|_{L^2}^2 \leqslant C \limsup_{n \to \infty} \rho(r,r_n) + C\varepsilon^{\frac{1}{2}},$$

for a universal constant $C < \infty$. We conclude that

$$(1 - \alpha - \varepsilon)(\alpha - \varepsilon) \leqslant C\varepsilon^{\frac{1}{2}},$$

which yields the desired contradiction if ε is chosen small enough, depending on α. $\qquad\square$

We finish this chapter by reformulating some of our conclusions in the language of the original concentration-compactness principle of Lions, according to which three scenarios may occur: (I) *compactness*, (II) *vanishing*, or (III) *dichotomy*. See [22, Lemma I.1] for the precise definitions. Up to extraction of subsequences, an extremizing sequence for inequality (1.6) which satisfies condition (I) with respect to a bounded sequence will concentrate at a point. An extremizing sequence which satisfies condition (II), or condition (I) with respect to an unbounded sequence, will concentrate at infinity. Condition (III) is only possible if neither of the supports of the split sequence remains in a bounded region, in which case the extremizing sequence again concentrates at infinity. Furthermore, if condition (III) occurs, then condition (II) must also occur. In this case, no fixed positive fraction of the L^2 mass of an extremizing sequence $\{f_n\}$ can remain on any ball of fixed radius, in the limit as $n \to \infty$. To see this, note that the proof of [22, Lemma I.1] implies that condition (III) could otherwise be upgraded to the splitting condition considered above, which in light of Proposition 5.8 does not hold for any extremizing sequence of inequality (1.6).

6. Sharp Strichartz inequalities

In this chapter, we consider a number of sharp instances of the Strichartz inequalities (1.12). All cases will follow a common pattern which we now illustrate by focusing on a particular example. With this purpose in mind, let $\mu = 1$ and consider a function ϕ as in the statement of Theorem 1.2. In this case, inequality (1.12) can be restated as

$$\|\mathcal{F}(f(1 + |\cdot|^2)^{\frac{1}{4}}\sigma_\phi)\|_{L^4(\mathbb{R}^3)} \lesssim \|f\|_{L^2(\mathbb{R}^2)}, \tag{6.1}$$

where the projection measure $\sigma = \sigma_\phi$ is defined in (1.2), and the subscript emphasizes that we are no longer taking $\phi = |\cdot|^4$ as in (1.16). Inequality (6.1) can be rewritten in sharp convolution form as

$$\|f\sqrt{w}\sigma_\phi * f\sqrt{w}\sigma_\phi\|_{L^2(\mathbb{R}^3)} \leqslant \mathcal{S}_\phi^2 \|f\|_{L^2(\mathbb{R}^2)}^2,$$

with weight $w = (1 + |\cdot|^2)^{\frac{1}{2}}$ and optimal constant $\mathcal{S}_\phi$. The usual Cauchy–Schwarz argument implies

$$\|f\sqrt{w}\sigma_\phi * f\sqrt{w}\sigma_\phi\|_{L^2(\mathbb{R}^3)}^2 \leqslant \|w\sigma_\phi * w\sigma_\phi\|_{L^\infty(\mathbb{R}^3)} \|f\|_{L^2(\mathbb{R}^2)}^4, \tag{6.2}$$

whence the upper bound

$$\mathcal{S}_\phi^4 \leqslant \|w\sigma_\phi * w\sigma_\phi\|_{L^\infty(\mathbb{R}^3)}. \tag{6.3}$$

On the other hand, recall formulae (2.4) and (5.1), the boundary values of the convolution measure $w\sigma_\phi * w\sigma_\phi$ are given by

$$\big(w\sigma_\phi * w\sigma_\phi\big)(\xi, 2\psi(\xi/2)) = \frac{\pi w^2(\xi/2)}{\sqrt{\det(H(\psi)(\xi/2))}}, \tag{6.4}$$

where we set $\psi = |\cdot|^2 + \phi$ as usual. A slight modification of Lemma 4.2 then yields the lower bound

$$\mathcal{S}_\phi^4 \geqslant \sup_{\xi \in \mathbb{R}^2} \frac{\pi w^2(\xi)}{\sqrt{\det(H(\psi)(\xi))}}. \tag{6.5}$$

Inequalities (6.3) and (6.5) provide upper and lower bounds for the value of the optimal constant $\mathcal{S}_\phi$. If these bounds happen to coincide, then this determines the value of $\mathcal{S}_\phi$. In this case, if the supremum in (6.3) is achieved only at the boundary of the support of the convolution measure, then extremizers are seen not to exist as before. In other cases, the following result will be useful in revealing some instances in which inequality (6.5) may be strict.

Lemma 6.1. *Given a strictly convex function $\Psi : \mathbb{R}^2 \to \mathbb{R}$, consider the measure $\nu(y,t) = \boldsymbol{\delta}\big(t - \Psi(y)\big)\,\mathrm{d}y\,\mathrm{d}t$. Let E denote the support of the convolution measure $\nu * \nu$. Given $s > 0$ and a nonnegative function w on $\mathbb{R}^2$, let $f_s(y) = e^{-s\Psi(y)}\sqrt{w(y)}$. Then the following inequality holds, for every $f_s \in L^2(\mathbb{R}^2)$ for which $f_s\sqrt{w}\nu * f_s\sqrt{w}\nu \in L^2(\mathbb{R}^3)$:*

$$\frac{\|f_s\sqrt{w}\nu * f_s\sqrt{w}\nu\|_{L^2(\mathbb{R}^3)}^2}{\|f_s\|_{L^2(\mathbb{R}^2)}^4} \geqslant \frac{\|f_s\|_{L^2(\mathbb{R}^2)}^4}{\int_E e^{-2s\tau}\,\mathrm{d}\xi\,\mathrm{d}\tau}. \tag{6.6}$$

In particular,

$$\sup_{0 \neq f \in L^2(\mathbb{R}^2)} \frac{\|f\sqrt{w}\nu * f\sqrt{w}\nu\|_{L^2(\mathbb{R}^3)}^2}{\|f\|_{L^2(\mathbb{R}^2)}^4} \geqslant \sup_{s>0} \frac{\|f_s\|_{L^2(\mathbb{R}^2)}^4}{\int_E e^{-2s\tau}\,\mathrm{d}\xi\,\mathrm{d}\tau}.$$

Proof. For simplicity set $s = 1$, the general case being similar. Note that the function $f(y) = e^{-\Psi(y)}\sqrt{w(y)}$ coincides with $e^{-t}\sqrt{w(y)}$ on the support of the measure ν. Therefore, the following identities hold:

$$(f\sqrt{w}\nu * f\sqrt{w}\nu)(\xi, \tau) = e^{-\tau}(w\nu * w\nu)(\xi, \tau),$$
$$(f^2\nu * f^2\nu)(\xi, \tau) = e^{-\tau}(f\sqrt{w}\nu * f\sqrt{w}\nu)(\xi, \tau).$$

Together with

$$\int_{\mathbb{R}^3} (f^2\nu * f^2\nu)(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau = \|f\|_{L^2}^4,$$

the preceding identities and the Cauchy–Schwarz inequality then imply

$$\|f\|_{L^2}^4 = \int_{\mathbb{R}^3} e^{-\tau}(f\sqrt{w}\nu * f\sqrt{w}\nu)(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

$$\leqslant \left(\int_E e^{-2\tau}\,\mathrm{d}\xi\,\mathrm{d}\tau\right)^{\frac{1}{2}}\|f\sqrt{w}\nu * f\sqrt{w}\nu\|_{L^2},$$

from which (6.6) easily follows. This completes the proof of the lemma. □

6.1. **Quartic perturbations.** We consider a slight generalization of inequality (1.16), given for $a \geqslant 0$ by

$$\|\mathcal{F}(f(1+a|\cdot|^2)^{\frac{1}{4}}\sigma)\|_{L^4(\mathbb{R}^3)} \lesssim \|f\|_{L^2(\mathbb{R}^2)},$$

where the measure σ is again given by $\sigma(y,t) = \boldsymbol{\delta}(t-|y|^2-|y|^4)\,\mathrm{d}y\,\mathrm{d}t$. This inequality can be equivalently rewritten in sharp form as

$$\|f\sqrt{w_a}\sigma * f\sqrt{w_a}\sigma\|_{L^2(\mathbb{R}^3)} \leqslant \mathcal{S}_a^2\|f\|_{L^2(\mathbb{R}^2)}^2, \tag{6.7}$$

with weight $w_a = (1+a|\cdot|^2)^{\frac{1}{2}}$ and optimal constant $\mathcal{S}_a$. With the notation just introduced, we have the following result, which specialized to $a = 1$ yields Theorem 1.7.

Theorem 6.2. *If $0 \leqslant a \leqslant 2$, then the value of the optimal constant for inequality* (6.7) *is given by $\mathcal{S}_a^4 = \frac{\pi}{2}$. Moreover, extremizers for inequality* (6.7) *do not exist, and extremizing sequences concentrate at the origin. If $a > 2$, then the following estimates hold:*

$$\max\left\{\frac{\pi}{2}, \frac{a\sqrt{2\pi}}{8}\Gamma\left(\frac{3}{4}\right)^2\right\} \leqslant \mathcal{S}_a^4 \leqslant \frac{a\pi}{4}. \tag{6.8}$$

Proof. For every $a \geqslant 0$, the trivial estimate

$$\|f\sigma * f\sigma\|_{L^2} \leqslant \||f|\sqrt{w_a}\sigma * |f|\sqrt{w_a}\sigma\|_{L^2}$$

and Theorem 1.2 together imply that $\mathcal{S}_a^4 \geqslant \frac{\pi}{2}$. This lower bound coincides with the value of the right-hand side of (6.5) in the special case when $\phi = |\cdot|^4$. It follows that

$$\frac{\pi}{2} \leqslant \mathcal{S}_a^4 \leqslant \|w_a\sigma * w_a\sigma\|_{L^\infty}, \tag{6.9}$$

for every $a \geqslant 0$. We are thus reduced to studying the convolution measure $w_a\sigma * w_a\sigma$. Formulae (2.11) and (5.1) imply

$$(w_a\sigma * w_a\sigma)(\xi,\tau) = \int_{\mathbb{S}^1} \frac{w_a(\xi/2+\alpha\omega)w_a(\xi/2-\alpha\omega)}{\left\langle\omega, \frac{\nabla\psi(\xi/2+\alpha\omega)-\nabla\psi(\xi/2-\alpha\omega)}{\alpha}\right\rangle}\,\mathrm{d}\mu_\omega, \tag{6.10}$$

where $\psi = |\cdot|^2 + |\cdot|^4$, and the function $\alpha = \alpha(\xi,\tau,\omega)$ is given by (2.3). A straightforward computation shows that the numerator of the integrand in (6.10) equals

$$w_a(\xi/2+\alpha\omega)w_a(\xi/2-\alpha\omega) = \left((1+a(|\xi/2|^2+\alpha^2))^2 - a^2\alpha^2\langle\xi,\omega\rangle^2\right)^{\frac{1}{2}}, \tag{6.11}$$

while the denominator equals

$$\left\langle \omega, \frac{\nabla\psi(\xi/2 + \alpha\omega) - \nabla\psi(\xi/2 - \alpha\omega)}{\alpha} \right\rangle = 4\big(1 + 2(|\xi/2|^2 + \alpha^2) + \langle\xi,\omega\rangle^2\big). \quad (6.12)$$

We split the analysis in two cases.

Case 1: $0 \leqslant a \leqslant 2$. To compare (6.11) and (6.12), note that the inequality

$$\Big(1 + a\big(|\xi/2|^2 + \alpha^2\big)\Big)^2 - a^2\alpha^2\langle\xi,\omega\rangle^2 \leqslant \Big(1 + 2\big(|\xi/2|^2 + \alpha^2\big) + \langle\xi,\omega\rangle^2\Big)^2 \quad (6.13)$$

holds for every $a \in [0,2]$, $\xi \in \mathbb{R}^2$, $\omega \in \mathbb{S}^1$ and $\alpha \geqslant 0$. Moreover, necessary and sufficient conditions for equality in (6.13) to hold for every $\omega \in \mathbb{S}^1$ are $\xi = 0$ when $a = 2$, and $\xi = 0$ and $\alpha = 0$ when $a < 2$. It follows that $\frac{1}{4}$ is an upper bound for the integrand in (6.10). Therefore, for every $(\xi,\tau) \in \mathbb{R}^{2+1}$,

$$(w_a\sigma * w_a\sigma)(\xi,\tau) \leqslant \frac{\pi}{2}. \quad (6.14)$$

Moreover, this inequality turns into an equality if and only if $(\xi,\tau) = (0,0)$ when $a < 2$, and if and only if $\xi = 0$ when $a = 2$. To justify this, note that

$$(w_a\sigma * w_a\sigma)(0,\tau) = \int_{\mathbb{R}^2} \boldsymbol{\delta}\big(\tau - 2(|y|^2 + |y|^4)\big) w_a^2(y)\,\mathrm{d}y = \frac{\pi}{2}\left(\frac{a}{2} + \frac{1 - \frac{a}{2}}{\sqrt{2\tau+1}}\right)\mathbb{1}_{\{\tau \geqslant 0\}}(\tau),$$

which specializes to

$$(w_2\sigma * w_2\sigma)(0,\tau) = \frac{\pi}{2}\mathbb{1}_{\{\tau \geqslant 0\}}(\tau).$$

As a consequence of estimates (6.9) and (6.14), we conclude that $\mathcal{S}_a^4 = \frac{\pi}{2}$, for every $0 \leqslant a \leqslant 2$. Nonexistence of extremizers is a consequence of inequality (6.14) being strict at almost every point, as in the proof of Theorem 1.2. Concentration at the origin can likewise be established in an analogous manner. We point out that the normalized sequence $\{f_n/\|f_n\|_{L^2}\}$, where $f_n(y) = \exp(-n(|y|^2 + |y|^4))$, is extremizing for inequality (6.7).

Case 2: $a > 2$. Start by noting that the inequality

$$\Big(1 + a\big(|\xi/2|^2 + \alpha^2\big)\Big)^2 - a^2\alpha^2\langle\xi,\omega\rangle^2 \leqslant \frac{a^2}{4}\Big(1 + 2\big(|\xi/2|^2 + \alpha^2\big) + \langle\xi,\omega\rangle^2\Big)^2$$

holds for every $a > 2$, $\xi \in \mathbb{R}^2$, $\omega \in \mathbb{S}^1$ and $\alpha \geqslant 0$. It follows that

$$(w_a\sigma * w_a\sigma)(\xi,\tau) \leqslant \frac{a\pi}{4},$$

yielding the upper bound $\mathcal{S}_a^4 \leqslant \frac{a\pi}{4}$. On the other hand, along the boundary of the support of $w_a\sigma * w_a\sigma$, we have that

$$(w_a\sigma * w_a\sigma)(\xi, 2\psi(\xi/2)) = \frac{\pi w_a^2(\xi/2)}{\sqrt{\det(H(\psi)(\xi/2))}} = \frac{\pi(1 + a|\xi/2|^2)}{2\sqrt{(1 + 2|\xi/2|^2)(1 + 6|\xi/2|^2)}},$$

and therefore

$$\mathcal{S}_a^4 \geqslant \sup_{r \geqslant 0} \frac{\pi(1 + ar)}{2\sqrt{(1 + 2r)(1 + 6r)}} \geqslant \max\left\{\frac{\pi}{2}, \frac{a\pi}{4\sqrt{3}}\right\}.$$

This yields the preliminary bounds

$$\max\left\{\frac{\pi}{2}, \frac{a\pi}{4\sqrt{3}}\right\} \leqslant \mathcal{S}_a^4 \leqslant \frac{a\pi}{4}. \tag{6.15}$$

The lower bound can be sharpened by invoking Lemma 6.1. With this purpose in mind, let $f_s(y) = e^{-s\psi(y)}\sqrt{w_a(y)}$. Its L^2 norm is given by

$$\|f_s\|_{L^2}^2 = \int_{\mathbb{R}^2} e^{-2s(|y|^2+|y|^4)}(1+a|y|^2)^{\frac{1}{2}}\,\mathrm{d}y = \pi \int_0^\infty e^{-2s(r+r^2)}(1+ar)^{\frac{1}{2}}\,\mathrm{d}r.$$

On the other hand, letting E denote the support of the measure $\sigma * \sigma$,

$$\int_E e^{-2s\tau}\,\mathrm{d}\xi\,\mathrm{d}\tau = \int_{\mathbb{R}^2}\Big(\int_{2(|\frac{\xi}{2}|^2+|\frac{\xi}{2}|^4)}^\infty e^{-2s\tau}\,\mathrm{d}\tau\Big)\,\mathrm{d}\xi = \frac{2\pi}{s}\int_0^\infty e^{-4s(r+r^2)}\,\mathrm{d}r.$$

It follows that

$$\mathcal{S}_a^4 \geqslant \sup_{s>0} \frac{\pi s}{2} \frac{\big(\int_0^\infty e^{-2s(r+r^2)}(1+ar)^{\frac{1}{2}}\,\mathrm{d}r\big)^2}{\int_0^\infty e^{-4s(r+r^2)}\,\mathrm{d}r}.$$

The limit as $s \to 0^+$ of the expression inside this supremum is easily calculated via a change of variables $u = \sqrt{s}r$, yielding

$$\mathcal{S}_a^4 \geqslant \frac{a\pi}{2} \frac{\big(\int_0^\infty e^{-2u^2}u^{\frac{1}{2}}\,\mathrm{d}u\big)^2}{\int_0^\infty e^{-4u^2}\,\mathrm{d}u} = \frac{a\sqrt{2\pi}}{8}\Gamma\Big(\frac{3}{4}\Big)^2.$$

Since $\frac{\sqrt{2\pi}}{8}\Gamma\big(\frac{3}{4}\big)^2 > \frac{\pi}{4\sqrt{3}}$, this indeed sharpens the lower bound in (6.15), and the proof is complete. $\qquad\square$

Remark 6.3. We can consider more general perturbations $\Psi = |\cdot|^2 + |\cdot|^4 + \phi$, with ϕ as in the statement of Theorem 1.2, satisfying $H(\phi)(0) = 0$. These correspond to perturbations of the cases considered in Theorem 6.2. Letting $\sigma_\Psi(y,t) = \boldsymbol{\delta}\big(t - \Psi(y)\big)\,\mathrm{d}y\,\mathrm{d}t$, a similar analysis reveals that, for every $a \in [0,2]$, the sharp inequality

$$\|f\sqrt{w_a}\sigma_\Psi * f\sqrt{w_a}\sigma_\Psi\|_{L^2(\mathbb{R}^3)}^2 \leqslant \frac{\pi}{2}\|f\|_{L^2(\mathbb{R}^2)}^4 \tag{6.16}$$

holds, extremizers do not exist, and extremizing sequences concentrate at the origin.

6.2. **Convolutions of pure powers.** In this section, we study the convolution of the projection measure

$$\nu_p(y,t) = \boldsymbol{\delta}\big(t - |y|^p\big)\,\mathrm{d}y\,\mathrm{d}t, \tag{6.17}$$

where $p \geqslant 2$ and $(y,t) \in \mathbb{R}^{2+1}$. A scaling argument shows that there exists a unique possible Strichartz estimate in $L^4(\mathbb{R}^3)$, namely

$$\|\mathcal{F}(f|\cdot|^{\frac{p-2}{4}}\nu_p)\|_{L^4(\mathbb{R}^3)} \lesssim \|f\|_{L^2(\mathbb{R}^2)}. \tag{6.18}$$

As before, the analysis of the sharp form of inequality (6.18) leads to the study of the convolution measure $w\nu_p * w\nu_p$, with weight $w = |\cdot|^{\frac{p-2}{2}}$. We record its main properties in the following result, which should be compared to Proposition 2.1.

Proposition 6.4. *Given $p \geqslant 2$, let $w = |\cdot|^{\frac{p-2}{2}}$. Let ν_p be the measure defined by (6.17). Then the following assertions hold for the convolution measure $w\nu_p * w\nu_p$:*

(a) *It is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^3$.*

(b) *Its support, denoted E_p, is given by*

$$E_p = \{(\xi, \tau) \in \mathbb{R}^{2+1} : \tau \geqslant 2^{1-p}|\xi|^p\}.$$

(c) *Its Radon–Nikodym derivative, also denoted by $w\nu_p * w\nu_p$, defines a bounded continuous function in the interior of the set E_p.*

(d) *It is radial in ξ, and homogeneous of degree zero in the sense that*

$$(w\nu_p * w\nu_p)(\lambda\xi, \lambda^p\tau) = (w\nu_p * w\nu_p)(\xi, \tau), \text{ for every } \lambda > 0.$$

(e) *It extends continuously to the boundary of E_p, except at the point $(\xi, \tau) = (0,0)$, with values given by*

$$(w\nu_p * w\nu_p)(\xi, 2^{1-p}|\xi|^p) = \frac{\pi}{p\sqrt{p-1}}, \text{ if } \xi \neq 0.$$

(f) *If $p > 2$, then the maximum value of $w\nu_p * w\nu_p$ is only attained along the vertical axis $\{(0, \tau) : \tau > 0\}$, where it equals $\frac{\pi}{p}$.*

Proof. Properties (a) and (b) follow as in the proof of Proposition 2.1. Property (d) is also straightforward to check. We then start by showing that $w\nu_p * w\nu_p$ defines a continuous function inside its support. Reasoning as in (2.5), we have that

$$(w\nu_p * w\nu_p)(\xi, \tau) = \int_{\mathbb{R}^2} \boldsymbol{\delta}\left(\tau - |\tfrac{\xi}{2} + y|^p - |\tfrac{\xi}{2} - y|^p\right)|\tfrac{\xi}{2} + y|^{\frac{p-2}{2}}|\tfrac{\xi}{2} - y|^{\frac{p-2}{2}} \, \mathrm{d}y.$$

Write $\tau = \lambda|\xi|^p$ with $\xi = 2e_1$, where e_1 denotes the first canonical vector. Changing to polar coordinates with polar axis parallel to e_1, we obtain

$$(w\nu_p * w\nu_p)(\xi, \lambda|\xi|^p) = \int_0^{2\pi}\int_0^\infty \boldsymbol{\delta}\left(2^p\lambda - 2 - \varphi_\theta(r)\right)((r^2+1)^2 - 4r^2\cos^2\theta)^{\frac{p-2}{4}} r \, \mathrm{d}r \, \mathrm{d}\theta,$$

where the function

$$\varphi_\theta(r) := (r^2 + 1 + 2r\cos\theta)^{\frac{p}{2}} + (r^2 + 1 - 2r\cos\theta)^{\frac{p}{2}} - 2 \qquad (6.19)$$

is convex in the variable r for each fixed θ, with unique global minimum at $r = 0$ as a result of Lemma 3.1. A change of variables $s = \varphi_\theta(r)$ yields

$$(w\nu_p * w\nu_p)(\xi, \lambda|\xi|^p) = \int_0^{2\pi}\int_0^\infty \boldsymbol{\delta}\left(2^p\lambda - 2 - s\right)\frac{((r^2+1)^2 - 4r^2\cos^2\theta)^{\frac{p-2}{4}} r}{\varphi_\theta'(r)} \, \mathrm{d}s \, \mathrm{d}\theta$$

$$= \mathbb{1}_{\{\lambda \geqslant 2^{1-p}\}}(\lambda) \int_0^{2\pi} ((r^2+1)^2 - 4r^2\cos^2\theta)^{\frac{p-2}{4}}\left(\frac{r}{\varphi_\theta'(r)}\right) \, \mathrm{d}\theta, \qquad (6.20)$$

where $r = \varphi_\theta^{-1}(2^p\lambda - 2)$, and φ_θ' denotes the derivative of the function φ_θ with respect to r. A calculation shows that

$$\varphi_\theta'(r) = pr\left((r^2 + 1 + 2r\cos\theta)^{\frac{p-2}{2}} + (r^2 + 1 - 2r\cos\theta)^{\frac{p-2}{2}}\right)$$
$$+ p\cos\theta\left((r^2 + 1 + 2r\cos\theta)^{\frac{p-2}{2}} - (r^2 + 1 - 2r\cos\theta)^{\frac{p-2}{2}}\right), \qquad (6.21)$$

which only vanishes at $r = 0$. Using part (d), we see that $(w\nu_p * w\nu_p)(\xi, \tau) = (w\nu_p * w\nu_p)(e_1, |\xi|^{-p}\tau)$, for every $\xi \neq 0$. Therefore continuity of the convolution measure at a point (ξ, τ) in the interior of the set E_p, for $\xi \neq 0$, follows from that of

$(w\nu_p * w\nu_p)(e_1, \lambda)$, for $\lambda > 2^{1-p}$. The latter is seen to hold via the Implicit Function Theorem, given that r is a differentiable function of λ and θ. As for continuity along the positive τ-axis, note that, given $\tau > 0$ and a sequence $(\xi_n, \tau_n) \to (0, \tau)$, as $n \to \infty$, with $\xi_n \neq 0$, for every n, we have

$$(w\nu_p * w\nu_p)(\xi_n, \tau_n) = (w\nu_p * w\nu_p)(e_1, \tfrac{\tau_n}{|\xi_n|^p}) \to \int_0^{2\pi} \frac{\mathrm{d}\theta}{2p} = \frac{\pi}{p}, \text{ as } n \to \infty. \quad (6.22)$$

Here we used that $\lambda_n := \frac{\tau_n}{|\xi_n|^p} \to \infty$, and that $r = r(\lambda_n, \theta) \to \infty$ for each fixed θ, as $n \to \infty$. Boundedness is a consequence of the inequality

$$2pr(r^2 + 1 + 2r\cos\theta)^{\frac{p-2}{4}}(r^2 + 1 - 2r\cos\theta)^{\frac{p-2}{4}} \leqslant \varphi'_\theta(r), \quad (6.23)$$

which holds for every $r \geqslant 0$, $\theta \in [0, 2\pi]$ and $p \geqslant 2$. To verify (6.23), recall expression (6.21) for φ'_θ, and note that, as long as $p \geqslant 2$,

$$p\cos\theta\big((r^2 + 1 + 2r\cos\theta)^{\frac{p-2}{2}} - (r^2 + 1 - 2r\cos\theta)^{\frac{p-2}{2}}\big) \geqslant 0,$$

for every $r \geqslant 0$ and $\theta \in [0, 2\pi]$. This concludes the verification of (c). We can continuously extend the value of the function $(w\nu_p * w\nu_p)(\xi, \lambda|\xi|^p)$ to $\lambda = 2^{1-p}$ by noting that $r \to 0^+$ as $\lambda \to (2^{1-p})^+$. This yields the following value for the extension:

$$(w\nu_p * w\nu_p)(\xi, 2^{1-p}|\xi|^p) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{\varphi''_\theta(0)} = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2p(1 + (p-2)\cos^2\theta)} = \frac{\pi}{p\sqrt{p-1}}.$$

Note that this coincides with the value predicted by the analogous of formula (6.4). Property (e) is now proved. Finally, if $p > 2$, then a discussion of the cases of equality in (6.23) reveals that the strict inequality

$$(w\nu_p * w\nu_p)(\xi, \tau) < \frac{\pi}{p}\mathbb{1}_{\{\tau \geqslant 2^{1-p}|\xi|^p\}}(\xi, \tau) \quad (6.24)$$

holds for every (ξ, τ) with $\xi \neq 0$. Moreover, the value along the τ-axis was already calculated in (6.22), is alternatively given by

$$(w\nu_p * w\nu_p)(0, \tau) = \int_{\mathbb{R}^2} \boldsymbol{\delta}\big(\tau - 2|y|^p\big)|y|^{p-2}\,\mathrm{d}y = \frac{\pi}{p}\mathbb{1}_{\{\tau > 0\}}(\tau),$$

and therefore equals the maximum value. This concludes the verification of (f) and the proof of the proposition. $\qquad \square$

Remark 6.5. The boundedness of $w\nu_p * w\nu_p$ given by part (c) of Proposition 6.4 implies the validity of the Strichartz estimate (6.18). Moreover, parts (e) and (f) imply that the optimal constant $\mathcal{Q}_p$ for the corresponding sharp inequality in convolution form,

$$\|f\sqrt{w}\nu_p * f\sqrt{w}\nu_p\|_{L^2(\mathbb{R}^3)} \leqslant \mathcal{Q}_p^2\|f\|_{L^2(\mathbb{R}^2)}^2, \quad (6.25)$$

satisfies

$$\frac{\pi}{p\sqrt{p-1}} \leqslant \mathcal{Q}_p^4 \leqslant \frac{\pi}{p}. \quad (6.26)$$

Contrary to the case of the quartic perturbation studied in §6.1, this does not determine $\mathcal{Q}_p$ since the upper and lower bounds do not coincide for $p > 2$.

In order to sharpen the lower bound in (6.26), we will use the following straightforward consequence of Lemma 6.1.

Corollary 6.6. *Given $p \geqslant 2$, let $\nu_p(y,t) = \boldsymbol{\delta}\big(t - |y|^p\big)\,\mathrm{d}y\,\mathrm{d}t$, and $w = |\cdot|^{\frac{p-2}{2}}$. Then the following estimate holds for the function $f(y) = \exp(-|y|^p)|y|^{\frac{p-2}{4}}$:*

$$\frac{\|f\sqrt{w}\nu_p * f\sqrt{w}\nu_p\|_{L^2(\mathbb{R}^3)}^2}{\|f\|_{L^2(\mathbb{R}^2)}^4} \geqslant \frac{\pi}{p2^{1-\frac{2}{p}}}\frac{\Gamma\big(\frac{1}{2}+\frac{1}{p}\big)^2}{\Gamma\big(\frac{2}{p}\big)}. \tag{6.27}$$

6.3. **The pure quartic.** In this section, we consider the case $p = 4$ of (6.25). Let $\nu = \nu_4$ be given by (6.17). The next result records the additional simplifications which appear in the integral formula (6.20) for the convolution $|\cdot|\nu * |\cdot|\nu$.

Corollary 6.7. *Let $\nu(y,t) = \boldsymbol{\delta}\big(t - |y|^4\big)\,\mathrm{d}y\,\mathrm{d}t$. Then the following integral formula holds, for every $\xi \neq 0$ and $\lambda \geqslant \frac{1}{8}$:*

$$(|\cdot|\nu * |\cdot|\nu)(\xi, \lambda|\xi|^4)$$
$$= \frac{1}{4\sqrt{2}}\int_0^{2\pi}\left(\frac{\lambda + \cos^2\theta + 2\cos^4\theta - 2(2\lambda + \cos^2\theta + \cos^4\theta)^{\frac{1}{2}}\cos^2\theta}{2\lambda + \cos^2\theta + \cos^4\theta}\right)^{\frac{1}{2}}\mathrm{d}\theta. \tag{6.28}$$

Additionally,

$$(|\cdot|\nu * |\cdot|\nu)(0, \tau) = \frac{\pi}{4}\mathbb{1}_{\{\tau>0\}}(\tau), \text{ and } (|\cdot|\nu * |\cdot|\nu)(\xi, \tfrac{|\xi|^4}{8}) = \frac{\pi}{4\sqrt{3}}, \text{ if } \xi \neq 0.$$

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. In view of [19, Theorem 4.1], the existence of extremizers for inequality (1.14) follows from the strict inequality $\mathcal{Q}^4 > \frac{\pi}{4\sqrt{3}}$. In order to establish it, consider the function $f(y) = \exp(-|y|^4)|y|^{\frac{1}{2}}$. Invoking Corollary 6.6, we have that

$$\mathcal{Q}^4 \geqslant \frac{\|f|\cdot|^{\frac{1}{2}}\nu * f|\cdot|^{\frac{1}{2}}\nu\|_{L^2}^2}{\|f\|_{L^2}^4} \geqslant \frac{\pi}{4\sqrt{2}}\frac{\Gamma\big(\frac{3}{4}\big)^2}{\Gamma\big(\frac{1}{2}\big)} = \frac{\sqrt{2\pi}}{8}\Gamma\big(\tfrac{3}{4}\big)^2 > \frac{\pi}{4\sqrt{3}}, \tag{6.29}$$

as desired. The upper bound $\mathcal{Q}^4 \leqslant \frac{\pi}{4}$ holds in view of part (f) of Proposition 6.4 for $p = 4$. That this upper bound is strict follows from the existence of extremizers, and the fact that the pointwise inequality $|\cdot|\nu * |\cdot|\nu < \frac{\pi}{4}$ is strict almost everywhere, as quantified by (6.24). $\qquad\square$

Remark 6.8. A direct calculation shows that the function $f(y) = \exp(-|y|^4)|y|^{\frac{1}{2}}$ satisfies

$$\frac{\|f|\cdot|^{\frac{1}{2}}\nu * f|\cdot|^{\frac{1}{2}}\nu\|_{L^2(\mathbb{R}^3)}^2}{\|f\|_{L^2(\mathbb{R}^2)}^4} = \frac{2}{\sqrt{\pi}\Gamma(\frac{3}{4})^2}\int_0^{2\sqrt{2}}(|\cdot|\nu * |\cdot|\nu)^2(e_1, t^{-2})\,\mathrm{d}t. \tag{6.30}$$

Invoking formula (6.28) for the convolution $|\cdot|\nu * |\cdot|\nu$, the integral on the right-hand side of (6.30) can be evaluated numerically. With precision 5×10^{-6}, one checks that

$$\frac{\|f|\cdot|^{\frac{1}{2}}\nu * f|\cdot|^{\frac{1}{2}}\nu\|_{L^2(\mathbb{R}^3)}^2}{\|f\|_{L^2(\mathbb{R}^2)}^4} \approx 0.489333. \tag{6.31}$$

Note that the lower bound $\frac{\sqrt{2\pi}}{8}\Gamma(\frac{3}{4})^2 \approx 0.470508$ obtained in (6.29) already amounts to about 96% of the value in (6.31). This indicates that the Cauchy–Schwarz argument from Lemma 6.1 is quite sharp for $p = 4$. We expect the same argument to work for other values of p as well, and remark on that in the next section.

6.4. **Other pure powers.** In this section, we briefly comment on how to approach the problem of existence of extremizers for inequality (6.25) in the case of pure powers other than the quartic. Given $p > 2$, let $\mathcal{Q}_p$ be the optimal constant in inequality (6.25). The following result provides a partial replacement for Theorem 1.6 when $p \neq 4$.

Proposition 6.9. *There exists $p_0 > 5$ such that, for every $p \in (2, p_0)$,*

$$\frac{\pi}{p\sqrt{p-1}} < \mathcal{Q}_p^4 \leqslant \frac{\pi}{p}.$$

Sketch of proof. The upper bound holds in view of part (f) of Proposition 6.4. Invoking Corollary 6.6 as before, we obtain the lower bound (6.27). We are thus reduced to showing that

$$\frac{\pi}{p2^{1-\frac{2}{p}}} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{p}\right)^2}{\Gamma\left(\frac{2}{p}\right)} > \frac{\pi}{p\sqrt{p-1}},$$

or equivalently

$$\Gamma\left(\frac{1}{2} + \frac{1}{p}\right)^2 > \frac{2^{1-\frac{2}{p}}}{\sqrt{p-1}}\Gamma\left(\frac{2}{p}\right). \tag{6.32}$$

Figure 1 below illustrates the validity of this inequality inside the claimed range. $\square$
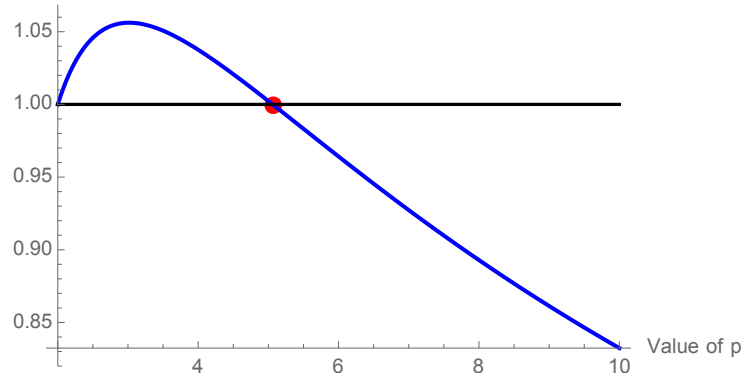

FIGURE 1. Plot of the ratio $\frac{\text{LHS}}{\text{RHS}}$ of inequality (6.32), for $2 < p < 10$. The p-coordinate of the intersection (red) point has been numerically determined and equals 5.061147 (6 d.p.).

Let p_0 be the exponent promised by Proposition 6.9. For every $p \in (2, p_0)$, extremizing sequences for inequality (6.25) are seen not to concentrate at a point of the boundary, except possibly at the origin. Then, arguments similar to the ones from [7, 19] can presumably establish the existence of extremizers, provided that a "cap bound" holds,

together with a principle quantifying the weak interaction between distant caps, in the spirit of Lemma 5.1. As a final remark, we record the following generalization of formula (6.30) for generic values of $p > 2$,

$$\frac{\|f\sqrt{w}\nu_p * f\sqrt{w}\nu_p\|_{L^2(\mathbb{R}^3)}^2}{\|f\|_{L^2(\mathbb{R}^2)}^4} = \frac{p}{2\pi}\frac{\Gamma\left(\frac{2}{p}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{p}\right)^2}\int_0^{2^{2-\frac{2}{p}}}(w\nu_p * w\nu_p)^2(e_1, t^{-\frac{p}{2}})\,\mathrm{d}t,$$

which could be of interest for further numerical explorations.

Acknowledgments

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