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Strict Nash Equilibria in a Duopolistic Market Share Model

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This paper develops a duopolistic discounted marketing model with linear advertising costs and advertised prices for mature markets still in expansion. Generic and predatory advertising effects are combined together in the model. We characterise a class of advertising models with some efficiency for production costs. For such a class of models, advertising investments have a no-free-riding strict Nash equilibrium in pure strategies if discount rates are small. We discuss the entity of this efficiency at varying of parameters of our advertising model. We provide a computational framework in which market shares can be computed at *equilibrium*, too. We analyse market share dynamics for an asymmetrical numerical scenario where one of the two firms is more effective in generic and predatory advertising. Several numerical insights on market share dynamics are obtained. Our computational framework allows for different scenarios in practical applications and it is developed using the *Mathematica* software.
We provide rational insights on how competing firms might ultimately reduce the quality of manufactured goods when they publish the prices at the beginning of marketing campaigns.

Keywords: Advertising models; Nash equilibrium; Generic Advertising; Brand Advertising; Computational Equilibria; Market Shares; Sticky Prices; Supply Chains

1. Introduction

Advertising expenditures for companies may be generally viewed as a form of investment and the main thrust of the advertising literature is to examine optimal strategies which maximise the net present value of future cash flows. When advertising is aimed at increasing product sales, then it is called generic or informative advertising, whilst when advertising is aimed at gaining market shares, it is called brand advertising. When brand advertising is devoted to stealing customers from competitors, it is called predatory advertising. Many advertising models have been built to provide solutions in monopolistic, duopolistic and oligopolistic environments (Vidale and Wolfe, 1957; Sethi, 1973; Deal, 1979; Little, 1979; Sethi, 1983) for many decision variables. Distinct dynamic advertising strategies were never developed prior to the heartbreaking work in (Bass et al., 2005, Table 1) for brand and generic advertising respectively. Even the deep dynamic analysis provided in (Bass et al., 2005) suggests that generic advertising expenditures must be resolved separately from brand advertising ones in order to halt suboptimal advertising. Externalities from simple generic advertising may become significant and may modify brand preferences, as market demand becomes more informed (Kinnucan, 1996; Norman et al., 2008; Rutz and Bucklin, 2011; Brahim et al., 2014). The main controversial *economic* issue is that generic advertising may redistribute market shares, especially in markets that have become strongly differentiated(Chakravarti and Janiszewski, 2004; Brady, 2009; Espinosa and Mariel, 2001; Piga, 1998; Friedman, 1983).
The analysis provided in (Espinosa and Mariel, 2001) deals with a duopoly where generic and predatory advertising are present but separately analyzed. The authors argue that static strategies do not internalise any variations of market shares if firms play a game with time-invariant market shares. Therefore it is expected that Nash equilibria are neither efficient if the advertising is predatory (too high expenditures), nor efficient if advertising is generic (too low expenditures). In the current paper we face a more complex scenario than the static one analysed in (Espinosa and Mariel, 2001). In fact, generic and brand/predatory advertising are unified and their effect on market shares cannot be distinguished at any time for an infinite time horizon. Strategic generic advertising interactions between firms may cause short-run bubble effects but similar predatory effects may cause long-run lowering effects on market shares, especially for asymmetrical market conditions. For instance, different investment strategies may generate different market share dynamics, which yield the same discounted profit to one of the firms. To some extent, the efficiency of such equilibria is not so intuitive. For instance, in (Espinosa and Mariel, 2001) closed-loop equilibria are more efficient than open-loop equilibria if the advertising is only generic while open-loop are more efficient than closed-loop if the advertising is only predatory. Very little attention has been devoted to the efficiency and stability of Nash equilibria in the game theory of advertising. The simple concept of strong Nash equilibria, which incorporates a strong form of efficiency, may be useful for analysis in a duopolistic model such as this. By being complementary to

the dynamic analysis in (Espinosa and Mariel, 2001), we restrict optimal analysis to static expenditure strategies. The stability we require is the following: if a firm unilaterally changes its strategy away from Nash equilibrium strategies, then such a firm has to be *strictly* worse off. We fill the gap in this literature and we investigate the existence of strict Nash equilibria in the context of generic and brand advertising models.

2. Results

In the current paper, we initially develop a duopolistic discounted advertising model for mature markets in expansion. Generic and predatory advertising effects are combined together. We model generic advertising by adopting a diffusion model and we model brand/predatory advertising by adopting the Lanchester model of combat. For the sake of simplicity we allow firms to advertise prices even if we do not model the latter as a decision variable. In fact, prices may be announced before marketing campaigns (Jiang et al., 2014; Grewal et al., 1998; Tenn and Wendling, 2014; Lu et al., 2016). [a] We define a class of advertising models where production costs are lowered by marketing parameters, by following an economic analysis between advertising and quality of products suggested in (Bagwell, 2007). We obtain payoffs in a closed-form using tools from the theory of linear differential systems. Significantly, payoffs are continuous but they are not quasi concave on non-compact subsets. Our method provides for an algebraic analysis of best reply correspondences. We find that a strict Nash equilibrium in pure strategies exists for the class of advertising games with low costs. Interestingly, our equilibria are not free-riding and they cannot be conceptually compared to the ones found in (Krishnamurthy, 2000). The existence of equilibria is not surprising for small discount rates by taking into account Folk Theorem in game theory. However, the efficiency of such equilibria is not so intuitive. Our strict equilibria are not obtained for any level of production costs, as it has been found in the static model provided in (Espinosa and Mariel, 2001). Unfortunately, due to the lack of a closed form expression of pure strategy Nash equilibria we cannot analytically discuss the influence of the parameters on optimal investments strategies. However our algebraic method provides a computational framework in order to integrate such optimal investment strategies (Deal, 1979; Erickson, 1985).

As a further investigation, we focus on market share dynamics generated by strict Nash equilibria in pure strategies. Since generic advertising supports the general standard of the product category, it brings advantages to firms in the market regardless of whether or not they contributed to advertising campaigns (Han et al.,

[a] We prefer to characterise our model in terms of advertised prices to make our results comparable to another literature stream, i.e. optimal pricing in markets with sticky prices (Gorodnichenko and Weber, 2016; Piga, 2000). Our model is comparable to a limit case in this literature, i.e. the rate of price readjustment is null. Recently, static optimal solutions are also provided in advertising models subject to interferences (Baggio and Viscolani, 2014; Viscolani, 2012).

2017; Shapiro, 2018). In the duopolistic model analysed in (Bass et al., 2005), when the asymmetries between the firms increase, there is a larger difference between their generic advertising contributions. However, the weaker firm always invests a non-null amount of money and, then, cheaply but not freely rides the market(Krishnamurthy, 2000). In our numerical scenario we assume the similar scenario adopted in (Bass et al., 2005). One firm is *stronger* if it is endowed with more favourable competitive and generic advertising parameters. Interestingly, we discover that the *weaker* firm is not a cheap rider. We assume that this does not happen because our advertising expenditures are combined together for brand and generic advertising. As an expected result, the weaker firm enlarges its market share due to generic advertising while its market share is affected by the long run effect of brand advertising from its competitor. Our results support those obtained in (Bass et al., 2005). Thanks to our numerical framework we are numerically able to integrate the time at which the weaker firm achieves its maximum market share, if the market in itself is not initially saturated. Same insights cannot been replicated from the results in (Bass et al., 2005) because the two models are similar but different.
The rest of the paper is organised as follows. In Section 3 we provide a background on the modelling from the literature of advertising models. In Section 4, we provide details of our advertising model with low costs. In Section 5 we prove the existence of investment equilibria in pure strategies. In Section 6 we provide an asymmetric numerical scenario for our model and we illustrate market share dynamics if our Nash equilibria investments are implemented. Our proofs are provided in the Appendix.

3. Modelling background

Each firm has a market share $x_i(t) \geq 0$ for time $t \geq 0$. The sum of market shares $x(t) = x_1(t)+x_2(t)$ is less than the maximum market, i.e. 1, and the potential market is $1-x(t)$. The potential market is the demand which is not already contained within the market share of any firm. Generic advertising is only directed to the potential market.
Diffusion models were originally introduced in (Fisher and Pry, 1971). Diffusion models are used to capture the life cycle dynamics of new products or to forecast the demand of markets. Originally, diffusion models do not incorporate any advertising effort and one of the main challenges is to add exogenous influences, most importantly the influence of advertising efforts (Bass et al., 1994). A generic functional form for diffusion advertising models for firm i is $\dot{x_i}(t) = f(x_i(t), u_i(t), t)$ where $u_i(t)$ is the advertising effort and $\dot{x_i}(t)$ is the change in market share for firm i (Dockner and Jørgensen, 1988). For instance, a diffusion model to uninformed consumers is given by $\dot{x_i}(t) = \alpha\, u_i(t)[1 - x_i(t)] + \gamma\, x_i(t)[1 - x(t)]$ for a monopolistic firm i where $\alpha > 0$ is the advertising effectiveness of the firm and γ represents the effectiveness of word-of-mouth advertising in (Jørgensen et al., 2006). With respect to generic advertising, we symmetrically adopt this diffusion model with $\gamma = 0$ and

$\alpha = 1$ for both firms in our duopoly. The Lanchester model has often been used to model the competitive advertising (Fruchter and Kalish, 1997) The Lanchester dynamics capture the competitive market shares' shifts of firms i, j due to investments in advertising by the two market rivals, (Chintagunta and Vilcassim, 1992). The Lanchester model is $\dot{x}_i(t) = \rho_i x_j(t)u_i(t) - \rho_j x_i(t)u_j(t)$ where $\rho_i, \rho_j \in [0, 1]$ denote the effectiveness of brand advertising for firms i, j (Little, 1979). In particular, here we assume that $\rho_i, \rho_j > 0$ for both firms.

4. Our advertising model

We adopt some modifications to the models in the previous Section. We assume that the advertising strategy variable $u_i \in [0, \infty[$ is time-invariant. Moreover, we do not adopt a linear form for investment variable u_i. In spite of mathematical simplifications we add a level of complexity and we shape advertising returns as a functional form $a_i, b_i : [0, \infty[\rightarrow [0, \infty[$ for generic and brand advertising expenditures respectively. We assume these functions satisfy a law of diminishing returns on the investment for both generic and brand advertising (Hanssens et al., 2003; Freimer and Horsky, 2012). It thus follows that

- $a_i(u_i) = \frac{u_i}{u_i+\alpha_i} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is the generic advertising return for i. The coefficient $\alpha_i \in \mathbb{R}^+ \setminus \{0\}$ is called the ineffectiveness of generic advertising returns.

- $b_i(u_i) = \frac{u_i}{u_i+\beta_i} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is the brand advertising return for i. The coefficient $\beta_i \in \mathbb{R}^+ \setminus \{0\}$ is called the ineffectiveness of brand advertising returns.

Functions $a_i(\cdot), b_i(\cdot)$ are increasing and concave and approximate 1 for large investments. The lower α_i and b_i's values are, the more profitable advertising investments are.
Summing up all these features of the diffusion model and of the Lanchester model, the derivative in primary demand $\dot{x}_i$ is modelled by the following differential system with initial conditions

$$\begin{cases} \dot{x}_i(t) = [a_i\left(u_i\right) + a_j\left(u_j\right)][1 - (x_i(t) + x_j(t))] + \rho_i x_j(t) b_i\left(u_i\right) - \rho_j x_i(t) b_j\left(u_j\right) \\ x_i(0) = x_i^0 \end{cases} \tag{1}$$

where $u_i \in \mathbb{R}^+$ is the advertising expenditure of firm i, $x_i\left(t\right) : \mathbb{R}^+ \rightarrow [0, 1]$ is the market share of firm i at time t and $x_i^0 = x_i(0)$ are the initial market shares. We assume linear advertising costs that are, traditionally, used in the Nerlove-Arrow model (Gould, 1976). In the literature, it is widely accepted that firms have different discount rates. Here firms adopt a unique positive discount rate $\rho \neq 0$ (Jørgensen, 1982). Therefore, the discounted flow of profits is

$$\pi^i\left(u_i, u_j\right) = \int_0^\infty e^{-\rho\, t}\left(r_i\, x_i\left(t\right) - u_i\right) dt.$$

where $r_i = p_i - c_i$. The quantity $p_i > 0$ is the marginal advertised price p_i for the good i. The quantity $c_i > 0$ is the marginal cost for good i. We assume that marginal profits r_i are positive. We say that $G = ([0, +\infty[^2, \pi^i, c_i)$ is the advertising game with costs. Classical solution concepts are defined below.

Definition 1 (Nash equilibria, market shares at equilibrium). *Let $(\hat{u}_1, \hat{u}_2)$ be a strict Nash equilibrium in pure strategies for G. Let us substitute $(\hat{u}_1, \hat{u}_2)$ in* (1). *A solution $\hat{x}_i(t)$ of the first order differential system* (1) *is the associated market share of firm i at equilibrium. A pure strategy profile is free-riding if one strategy is null and the remaining strategy is non-null. A pure strategy profile is null when both strategies are null. We say that a pure strategy profile is proper if it is neither null nor free-riding.*

We characterise advertising games where firms have low production costs or high profits by taking into account the fact that prices are advertised in our model.

Definition 2. *The following inequality is satisfied*

$$c_i + c_i^{sh} < p_i \tag{3}$$

where

$$c_i^{sh} = \frac{(\rho_j + \rho_i)}{2\rho_i}\alpha_i + \frac{\rho_j}{\rho_i}\beta_i.$$

We say that $G = ([0, +\infty[^2, \pi^i, c_i)$ is an advertising game with low costs if the above conditions are satisfied. We say that $c_i^{sh} > 0$ is the marketing incentive for firm i.

Marketing incentives are below the advertised prices. Marketing incentives depend on the effectiveness of brand advertising of the competitor in the duopoly.

5. Existence of a strict Nash equilibrium in pure strategies

In this section, we find sufficient conditions which guarantee the existence of static advertising expenditures equilibria for our advertising model. Let $\widehat{U}_i : [0, +\infty[\to 2^{[0,+\infty[}$ be the best reply correspondence for firm i. The set $\widehat{U}_i(u_j)$ collects firm i's best replies to a strategy u_j. We transform system (1) through a change of variables into

$$\begin{cases} \dot{x} &= 2\left\{a_1(u_1) + a_2(u_2)\right\}(1 - x) \\ \dot{w} &= \rho_1 b_1(u_1)(x - w) - \rho_2 b_2(u_2)(x + w) \end{cases} \tag{4}$$

where $x = x_1 + x_2$ and $w = x_1 - x_2$.

Proposition 1. *The solution for system (4) is given by the following formulae*

$$\begin{cases} x(t) = 1 - \left(1 - x_1^0 - x_2^0\right)e^{-2At} & t \geq 0 \\ w(t) = \frac{B^+}{B^-} + \frac{B^+}{(B^+ - 2A)}\left(x_1^0 + x_2^0 - 1\right)\left(e^{-2B^+t} - e^{-At}\right) + e^{-B^+t}C_u & \end{cases} \tag{5}$$

where C_u *is a constant depending on*

$$A = a_1(u_1) + a_2(u_2), \; B^+ = \rho_1 b_1(u_1) + \rho_2 b_2(u_2), \; B^- = \rho_1 b_1(u_1) - \rho_2 b_2(u_2).$$

Here, we provide the following result in which market shares and payoffs are obtained in closed formulas.[b]

Proposition 2. *Assume* $\rho \neq 0$*. Then market shares and payoffs are*

$$x_i(t) = \frac{\rho_i b_i(u_i)}{B^+}\left(1 - e^{-B^+ t}\right) + \frac{\left(x_i^0 - x_j^0 + 1\right)\left(\rho_i b_i(u_i) - A\right)}{\left(B^+ - 2A\right)}\left(e^{-2At} - e^{-B^+ t}\right) + x_i^0 e^{-B^+ t}. \tag{6}$$

$$\pi^i(u_i, u_j) = \frac{r_i}{\left(B^- + \rho\right)}\left(\frac{\rho_i b_i(u_i)}{\rho} + \frac{\left(x_i^0 + x_j^0 - 1\right)\left(\rho_i(u_i) - A\right)}{(2A + \rho)} + x_i^0\right) - \frac{u_i}{\rho}. \tag{7}$$

for $i, j = 1, 2$*, respectively.*

Each payoff on its own variable is defined on a non-compact set $[0, \infty[$ and may lack quasi-concavity. Therefore, we cannot apply classical results for the existence of pure strategy Nash equilibria. We prefer to follow a different approach to the problem of whether a strict Nash equilibrium exists in pure strategies. First we identify an algebraic structure for the firm's best reply strategies, i.e. best reply strategies are zeros of a polynomial equation.

Proposition 3. *Assume* $\rho \neq 0$*. The payoff* $\pi^1(u_1, u_2)$ *is rational in* u_1 *and can be calculated as follows*

$$\pi^1(u_1, u_2) = -\frac{h u_1^3 + (b + c_1 d) u_1^2 + (r_1 - c_1 f) u_1 - r_1}{\rho\left(h u_1^2 + b u_1 + c\right)}$$

where

$$b = (\alpha_1 - \beta_1)\left(\rho^2 + (\theta + 2\vartheta)\rho + 2\theta\vartheta\right) + 2\beta_1(\rho - \theta) + \rho_1\alpha_1(\rho + 2\vartheta)$$
$$c = \alpha_1\beta_1\left(\rho^2 + (\theta + 2\vartheta)\rho + 2\vartheta\theta\right)$$
$$d = x_1^0\rho^2 + \left((\vartheta + 1)\left(1 + x_1^0 - x_2^0\right) + \rho\left(x_1^0 + x_2^0\right)\right)\rho + 2\rho(\vartheta + 1)$$
$$f = x_1^0(\alpha_1 + \beta_2)\rho^2 + \left(\alpha_1\rho_1\left(x_1^0 + x_2^0\right) + \left((\alpha_1 + \beta_1)\vartheta + \beta_1\right)\left(1 + x_1^0 - x_1^0\right)\right)\rho + 2\rho_1\alpha_1\vartheta$$
$$g = \alpha_1\beta_1\left(x_1^0\rho + \vartheta\left(1 + x_2^0 - x_2^0\right)\right)$$
$$h = (\rho + 2\vartheta + 2)(\rho + \rho_1 + \theta)$$

$$\theta = \rho_2\frac{u_2}{u_2 + \beta_2}, \quad \vartheta = \frac{u_2}{u_2 + \alpha_2}. \tag{9}$$

[b]It is worth noting that $\lim_{t\to\infty} x(t) = 1$ if $A \neq 0$. The whole market is asymptotically saturated if u is a proper strategy profile. The solution (5) can be extended on the subset $\{u | (B^+ - 2A)(u) = 0\}$ by continuity.

Proof. The proof follows from a direct calculation. □

Proposition 4. *Assume $\rho \neq 0$. Let $u_2 \geq 0$ be a strategy of firm 2. Assume that a best reply to strategy u_2 exists, i.e. $u_1 > 0$. Then, u_1 is a root of*

$$\begin{aligned} h^2 {u_1}^4 + 2b{u_1}^3 + \left(r_1 h + r_1 d + 2hc + b^2\right) {u_1}^2 + \\ + \left(2bc - 2r_1 dc + 2r_1 gh\right) u_1 + \left(r_1 gb - r_1 fc + c^2\right) = 0 \end{aligned} \tag{10}$$

Proof. Let $\hat{u}_2$ be an investment of firm 2. By simple calculation, we have

$$\begin{aligned} \frac{\partial \pi^1 (u_1, \hat{u}_2)}{\partial u_1} = & -\frac{h^2 u_1^4 + 2hbu_1^3 + \left(r_1 fh - r_1 db + 2hc + b^2\right) u_1^2}{\rho \left(hu_1^2 + bu_1 + c\right)^2} + \\ & + \frac{\left(2c - 2r_1 dc + 2r_1 gh\right) u_1 + \left(r_1 gb - r_1 fc - c^2\right)}{\rho \left(u_1^2 + bu_1 + c\right)^2}. \end{aligned} \tag{11}$$

If there exists a best reply in u_1, trivially u_1 is a root of the above equation. □

We conventionally rank the coefficients on the left side of equation (10) by decreasing order. The first two Lemmas are preliminary to the fundamental Lemma 3.

Lemma 1. *If $\rho \neq 0, \rho \approx 0$ and $r_1 > \dfrac{(\rho_2 + \rho_1)}{2\rho_1}\alpha_1 + \beta_1 \dfrac{\rho_2}{\rho_1}$ then the fourth coefficient is strictly negative, for any $u_2 \geq 0$.*

Lemma 2. *If $\rho \neq 0, \rho \approx 0$ and $r_1 > \beta_1 \dfrac{\rho_2}{\rho_1}$, then the fifth coefficient is strictly negative for any $u_2 \geq 0$.*

Lemma 3. *If $\rho \neq 0, \rho \approx 0$ and $r_1 > \dfrac{(\rho_2 + \rho_1)}{2\rho_1}\alpha_1 + \beta_1 \dfrac{\rho_2}{\rho_1}$ then there exists a unique best reply $u_1 > 0$ for any strategy $u_2 \geq 0$.*

A fundamental property of best reply correspondences is presented below.

Lemma 4. *Under the hypothesis of Lemma 3, best reply correspondences $\widehat{U}_i$ are continuous functions. In addition, $\lim_{u_j \to \infty} \hat{U}_i(u_j) < \infty$.*

By a classical fixed-point argument, we prove the existence of a strict Nash equilibrium in pure strategies.

Theorem 1 (Main Existence Result). *Assume that $G = ([0, +\infty[^2, \pi^i, c_i)$ is an advertising game with low costs. Assume that the discount rate satisfies the following properties: $\rho \neq 0$ and $\rho \approx 0$. Then, a strict Nash equilibrium in proper pure strategies exists for G.*

Proof. The thesis of Lemma 4 is satisfied by hypotheses. By coercivity conditions the image of $\widehat{U}_2$ is a bounded subset in $[0, +\infty[$. Let $K_1 = \widehat{U}_2([0, +\infty[)) \subset]0, +\infty[$ be the image values of $\widehat{U}_2$. Let $\widehat{U}_1|_{K_1} : K_1 \to]0, +\infty[$ be the restricted function to the compact subset K_1. In addition $\widehat{U}_1|_{K_1}(K_1)$ is a compact subset in $]0, +\infty[$ because

K_1 is compact and $\widehat{U}_1|_{K_1}$ is continuous. By the definition of K_1, it follows that $\widehat{U}_1 \circ \widehat{U}_2|_{K_1} : K_1 \to K_1$. In addition, the above function is continuous because it is a composition of continuous functions and K_1 is a compact in $]0, +\infty[$. By Brower's fixed point Theorem $\widehat{U}_1 \circ \widehat{U}_2|_{K_1}$ admits a fixed point. Then there exists a strategy $\widehat{u}_1 \in K_1 \subset]0, +\infty[$ such that $\widehat{U}_1(\widehat{U}_2(\widehat{u}_1)) = \widehat{u}_1$. We define $\widehat{u}_2 := \widehat{U}_2(u_1)$. By Lemma 3, we have that $\widehat{u}_2 \neq 0$. It is straightforward to prove that $\widehat{U}_2(\widehat{U}_1)(\widehat{u}_2)) = \widehat{u}_2$. By definition, $\widehat{u} = (\widehat{u}_1, \widehat{u}_2)$ is a Nash equilibrium in pure strategies. By construction we have that $\widehat{u}$ is a strategy profile with $\widehat{u}_i \neq 0$. Therefore $\widehat{u}$ is a proper strategy profile. By Lemma 3 again, $\widehat{u}$ is a strict Nash equilibrium in pure strategies. □

6. Market shares at equilibrium for an asymmetric advertising situation

In this section we provide a numerical example under some asymmetric conditions for two firms. We assume that firm 1 has higher investment returns for both generic and predatory advertising. Further, we assume that firm 1 is more effective at capturing demand belonging to the market share of the competitor, *ceteribus paribus* their investments. Table 1 contains the numerical parameters for the simulation of our model. Values of parameters of our model are listed in Table 1. It is straightforward to verify that our game is an advertising game with low costs. Using best reply

ineffectiveness of generic advertising return	$\alpha_1 = 5$, $\alpha_2 = 10$
ineffectiveness of brand advertising return	$\beta_1 = 15$, $\beta_2 = 5$
effectiveness of brand advertising	$\rho_1 = 1/25$, $\rho_2 = 1/100$
marketing incentive	$c_1^{sh} = 25/4$, $c_1^{sh} = 45$

Table 1. Parameters of the model are listed.

equation (10), we obtain implicit formulae for best reply functions. We numerically integrate their intersection points, i.e. strict Nash equilibria in proper pure strategies. Interestingly, we find that a strict Nash equilibrium is unique and it is equal to $(5.23, 7.24)$ and it is not cheap-riding. [c] By replacing Nash equilibrium strategies in formula (6), we compute firms' market shares. Figures 1–5 describe market share dynamics. The red, blue and green lines represent the dynamics of market shares of the stronger firm (firm 1) and of the weaker firm (firm 2) and of the whole market demand, respectively. If the market is initially not saturated, firms invest in marketing advertising to improve their market positions. The *weaker* firm, i.e. firm 2, initially increases its market position (blue line) and its market share reaches

[c] We consider the same values of $\rho_i, \beta_i, \alpha_i, r_i$ in Table 1 and we choose higher discount rates. From our numerical implementation, it follows that pure strategy Nash equilibria fail to exist if ρ approximatively becomes higher than a threshold value equal to 0.040821.

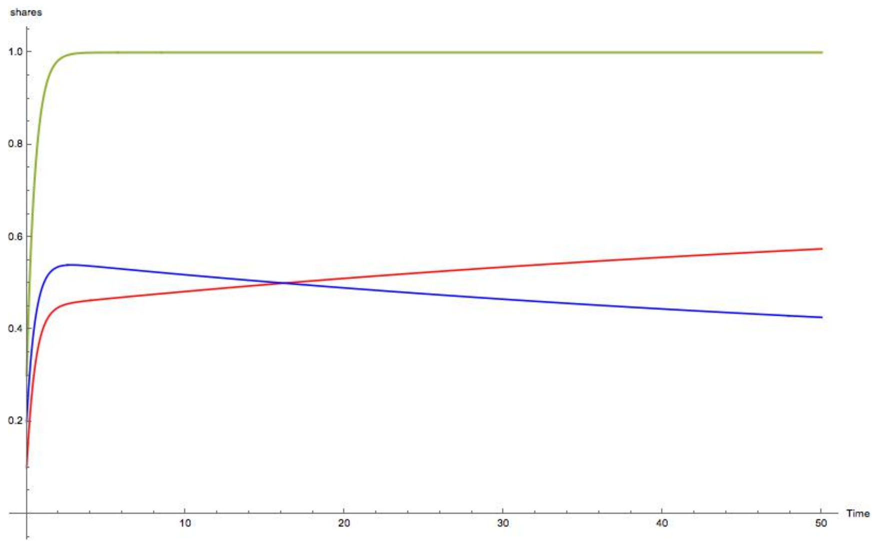

Fig. 1. $x_1^0 = \frac{1}{10}$, $x_2^0 = \frac{1}{5}$. Firm 1 is initially weaker in the market. The market is initially not saturated.

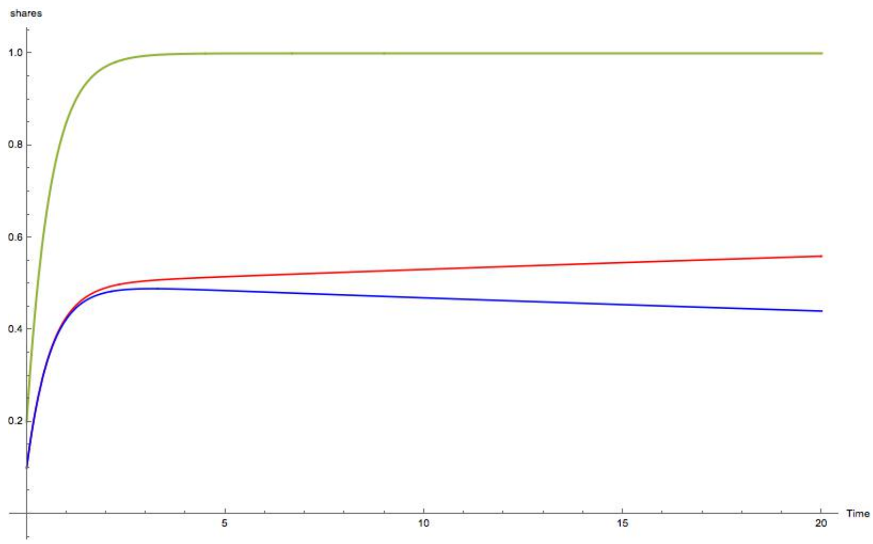

Fig. 2. $x_1^0 = x_2^0 = \frac{1}{10}$. Firm 1,2 equally share the market. The market is initially not saturated.

its maximum at time $t \approx 2$ and, then it decreases because of predatory effects. In Figure 1 the *stronger* firm's market share is initially smaller than the opponent's one and market shares become equal at $t \approx 16$. If the market size is saturated, then

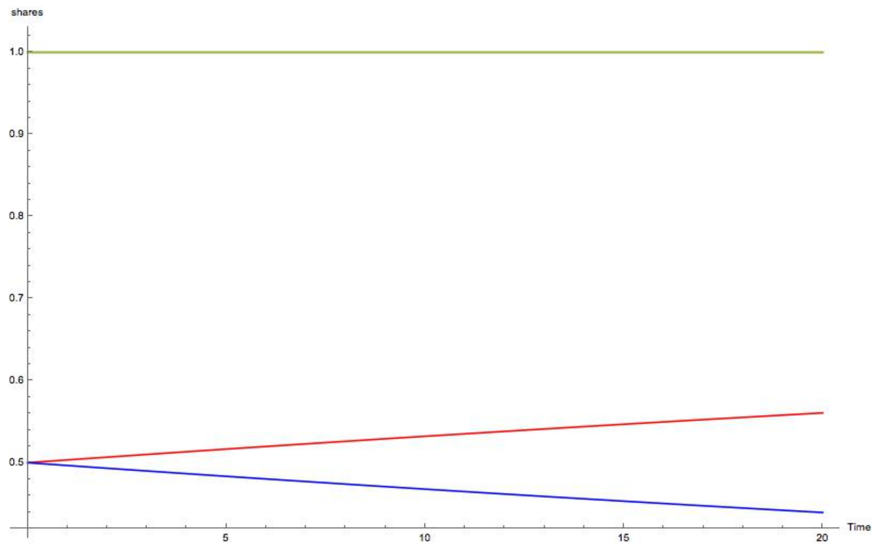

Fig. 3. $x_1^0 = x_2^0 = \frac{1}{2}$. Firm 1,2 equally share the market. The market is initially saturated.

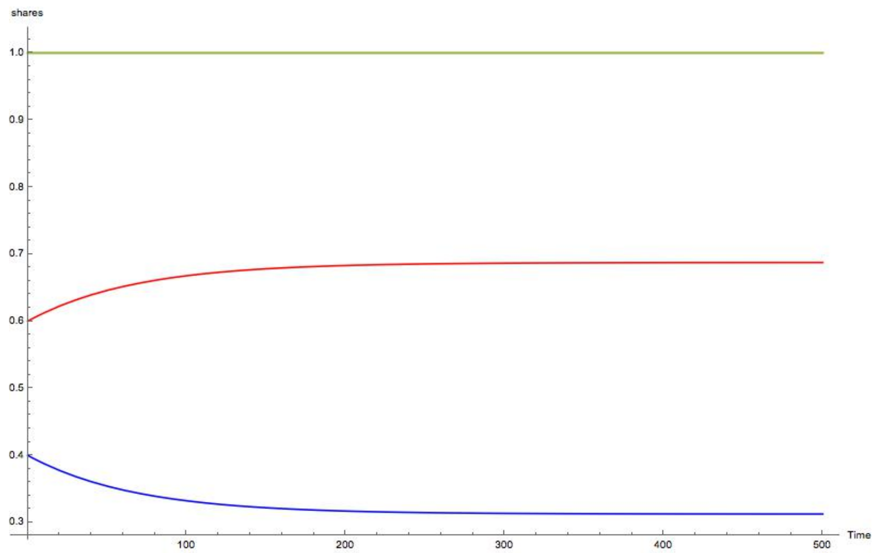

Fig. 4. $x_1^0 = \frac{9}{10}$, $x_2^0 = \frac{1}{10}$. Firm 1 is initially much stronger on the market. The market is initially saturated.

any generic marketing effort is uninfluential in extending the market. Although firm 1 is more effective in marketing campaigns and its initial market position is very dominant, its market share decreases (Figure 5). The latter happens when the

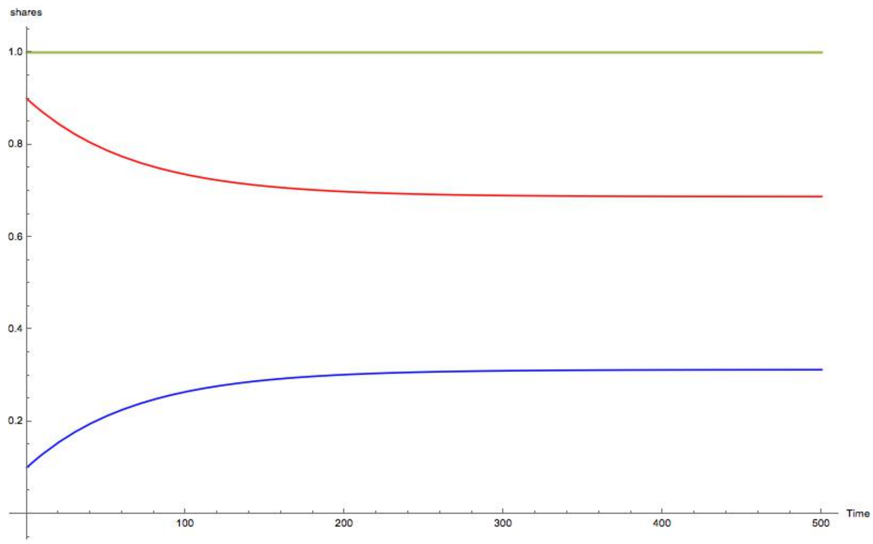

Fig. 5. $x_1^0 = \frac{3}{5}$, $x_2^0 = \frac{2}{5}$. Firm 1 is initially stronger on the market. The market is initially saturated.

difference between two initial market shares is high. If this difference is not high enough, then firm 1 keeps on improving its market share by making the opponent's market position weaker (Figures 3–4).

7. Conclusions, limitations and future work

We develop a novel duopolistic advertising model describing changes to the market shares in a duopoly caused by strategic investments in generic and predatory advertising. We provide the existence of strict optimal static investment strategies for firms under low costs. We discuss marketing incentives due to advertising by varying the parameters of the model. We provide numerical insights on cheap-riding phenomena, optimum points for market shares and stationary market shares in an asymmetrical marketing scenario. We provide rational insights on how competing firms might reduce the quality of manufactured goods when they publish the prices at the beginning of marketing campaigns.

Our work has some limitations. Static strict optimal advertising investments are not in a closed-form formula. Due to the lack of a closed form expression for these equilibria, we cannot study the sensitivity of strict optimal investment strategies to a change in marketing parameters. Our main existence result lies in the assumption that the unique discount rate is small. A natural question arising from this paper is, if firms are given different discount rates, can we extend our main existence result? Future research should consider parameterising this model using advanced econometric methods in order to estimate marketing incentives in our model. The latter will help to measure the quality of products provided by firms competing in both informative and predatory marketing campaigns.

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References

Baggio, A. and Viscolani, B. (2014). An advertising game with multiplicative interference. *Optimization*, 63(9):1401–1418.

Bagwell, K. (2007). The economic analysis of advertising. *Handbook of industrial organization*, 3:1701–1844.

Bass, F. M., Krishnamoorthy, A., Prasad, A., and Sethi, S. P. (2005). Generic and brand advertising strategies in a dynamic duopoly. *Marketing Science*, 24(4):556–568.

Bass, F. M., Krishnan, T. V., and Jain, D. C. (1994). Why the bass model fits without decision variables. *Marketing science*, 13(3):203–223.

Brady, M. P. (2009). Advertising effectiveness and spillover: simulating strategic interaction using advertising. *System Dynamics Review*, 25(4):281–307.

Brahim, N. B. E.-B., Lahmandi-Ayed, R., and Laussel, D. (2014). Advertising spillovers and strategic interaction in media-product markets. *Louvain Economic Review*, 80(2):51–98.

Chakravarti, A. and Janiszewski, C. (2004). The influence of generic advertising on brand preferences. *Journal of consumer research*, 30(4):487–502.

Chintagunta, P. K. and Vilcassim, N. J. (1992). An empirical investigation of advertising strategies in a dynamic duopoly. *Management science*, 38(9):1230–1244.

Deal, K. R. (1979). Optimizing advertising expenditures in a dynamic duopoly. *Operations Research*, 27(4):682–692.

Dockner, E. and Jørgensen, S. (1988). Optimal advertising policies for diffusion models of new product innovation in monopolistic situations. *Management Science*, 34(1):119–130.

Erickson, G. M. (1985). A model of advertising competition. *Journal of Marketing Research*, pages 297–304.

Espinosa, M. P. and Mariel, P. (2001). A model of optimal advertising expenditures in a dynamic duopoly. *Atlantic Economic Journal*, 29(2):135–161.

Fisher, J. C. and Pry, R. H. (1971). A simple substitution model of technological change. *Technological forecasting and social change*, 3:75–88.

Freimer, M. and Horsky, D. (2012). Periodic advertising pulsing in a competitive market. *Marketing Science*, 31(4):637–648.

Friedman, J. W. (1983). Advertising and oligopolistic equilibrium. *The Bell Journal of Economics*, pages 464–473.

Fruchter, G. E. and Kalish, S. (1997). Closed-loop advertising strategies in a duopoly. *Management Science*, 43(1):54–63.

Gorodnichenko, Y. and Weber, M. (2016). Are sticky prices costly? evidence from the stock market. *American Economic Review*, 106(1):165–99.

Gould, J. P. (1976). *Diffusion processes and optimal advertising policy*, pages 169–174. Lecture Notes in Economics and Mathematical Systems. Springer, Berlin, Heidelberg.

Grewal, D., Monroe, K. B., and Krishnan, R. (1998). The effects of price-comparison advertising on buyers' perceptions of acquisition value, transaction value, and behavioral intentions. *The Journal of Marketing*, pages 46–59.

Han, S., Heywood, J. S., and Ye, G. (2017). Informative advertising in a mixed oligopoly. *Review of Industrial Organization*, 51(1):103–125.

Hanssens, D. M., Parsons, L. J., and Schultz, R. L. (2003). *Market response models: Econometric and time series analysis*, volume 12 of *International Series in Quantitative Marketing*. Kluwer Academic Publishers, London.

Jiang, B., Ni, J., and Srinivasan, K. (2014). Signaling through pricing by service providers with social preferences. *Marketing Science*, 33(5):641–654.

Jørgensen, S. (1982). A differential games solution to a logarithmic advertising model. *Journal of the Operational Research Society*, 33(5):425–432.

Jørgensen, S., Kort, P. M., and Zaccour, G. (2006). Advertising an event. *Automatica*, 42(8):1349–1355.

Kinnucan, H. (1996). A note on measuring returns to generic advertising in interrelated markets. *Journal of Agricultural Economics*, 47(1-4):261–267.

Korn, G. and Korn, T. (1968). *Mathematics Handbook for Scientist and Engineers.* McGraw-Hill.

Krishnamurthy, S. (2000). Enlarging the pie vs. increasing one's slice: An analysis of the relationship between generic and brand advertising. *Marketing Letters*, 11(1):37–48.

Little, J. D. (1979). Aggregate advertising models: The state of the art. *Operations Research*, 27(4):629–667.

Lu, L., Gou, Q., Tang, W., and Zhang, J. (2016). Joint pricing and advertising strategy with reference price effect. *International Journal of Production Research*, 54(17):5250–5270.

Norman, G., Pepall, L., and Richards, D. (2008). Generic product advertising, spillovers, and market concentration. *American Journal of Agricultural Economics*, 90(3):719–732.

Piga, C. A. (1998). A dynamic model of advertising and product differentiation. *Review of Industrial Organization*, 13(5):509–522.

Piga, C. A. (2000). Competition in a duopoly with sticky price and advertising. *International Journal of Industrial Organization*, 18(4):595–614.

Rutz, O. J. and Bucklin, R. E. (2011). From generic to branded: A model of spillover in paid search advertising. *Journal of Marketing Research*, 48(1):87–102.

Sethi, S. P. (1973). Optimal control of the vidale-wolfe advertising model. *Operations Research*, 21(4):998–1013.

Sethi, S. P. (1983). Deterministic and stochastic optimization of a dynamic advertising model. *Optimal Control Applications and Methods*, 4(2):179–184.

Shapiro, B. T. (2018). Positive spillovers and free riding in advertising of prescription pharmaceuticals: The case of antidepressants. *Journal of political economy*, 126(1):000–000.

Tenn, S. and Wendling, B. W. (2014). Entry threats and pricing in the generic drug industry. *Review of Economics and Statistics*, 96(2):214–228.

Vidale, M. and Wolfe, H. (1957). An operations-research study of sales response to advertising. *Operations Research*, 5(3):370–381.

Viscolani, B. (2012). Pure-strategy nash equilibria in an advertising game with interference. *European Journal of Operational Research*, 216(3):605–612.

Appendix: Proofs

Proof of Proposition 1. Analysing the first equation of system (4) we can see that it is possible to separate the variables and integrate it as follows

$$\begin{aligned}&\frac{dx}{dt} = 2A\,(1-x)\\ &x\,(t) = 1 - C_x e^{-2At}.\end{aligned}$$

If at $t = 0$ we have $x\,(0) = x_1^0 + x_2^0$, then $C_x = 1 - x_1^0 - x_2^0$, and hence

$$x\,(t) = 1 + \left(x_1^0 + x_2^0 - 1\right) e^{-2At}.$$

For the function $w\,(t)$ we have

$$\frac{dw}{dt} = \rho_1\, b_1\,(u_1)\,(x-w) - \rho_2 b_2\,(u_2)\,(x+w) = B^- x\,(t) - B^+ w\,(t)\,.$$

It is a linear equation of the first order. We integrate it using the standard approach and obtain that the exact solution is given as

$$w\,(t) = e^{-B^+t}\int e^{B^+t} B^- x\,(t)\;dt + e^{-B^+t} C_u.$$

Therefore, for the function $w(t)$ we have

$$\begin{aligned}
w(t) &= e^{-B^- t}\int e^{B^+ t} B^- x(t)\, dt + e^{-B^+ t} C_u \\
&= e^{-B^+ t}\int e^{B^+ t} B^- \left(\left(1+\left(x_1^0 - x_2^0\right) e^{-2t}\right)\right) dt + e^{-B^+ t} C_u \\
&= e^{-B^+ t}\left(B^-\int e^{B^+ t} dt + B^-\left(x_1^0 + x_2^0 - 1\right)\int e^{\left(B^+ - 2A\right)t} dt\right) + e^{-B^+ t} C_u \\
&= e^{-B^- t}\left(\frac{B^-}{B^+} e^{B^- t} + \frac{B^-\left(x_1^0 + x_2^0 - 1\right)}{\left(B^+ - 2A\right)}(e^{\left(B^+ - 2A\right)t} - 1)\right) + e^{-B^+ t} C_u \\
&= \frac{B^-}{B^+} + \frac{B^-\left(x_1^0 + x_2^0 - 1\right)}{\left(B^+ - 2A\right)}(e^{-2At} - e^{-B^- t}) + e^{-B^+ t^2} C_u. \qquad \square
\end{aligned}$$

Proof of Proposition 2. Since $x_1 = \frac{x+w}{2}$ and $x_2 = \frac{x-w}{2}$ we obtain

$$\begin{aligned}
x_1(t) &= \frac{x(t) + w(t)}{2} = \\
&= \frac{1}{2}\left(1 + \left(x_1^0 + x_2^0 - 1\right) e^{-2At} + \frac{B^-}{B^+} + \frac{B^-\left(x_1^0 + x_2^0 - 1\right)}{\left(B^+ - 2A\right)}(e^{-2At} - e^{-B^+ t}) + e^{-B^+ t} C_u\right) \\
&= \frac{1}{2}\left(\frac{B^+ + B^-}{B^+} + \frac{\left(x_1^0 + x_2^0 - 1\right)\left(B^+ + B^- - 2A\right)}{\left(B^+ - 2A\right)} e^{-2At} + (C_u - \frac{\left(x_1^0 + x_2^0 - 1\right)\left(B^-\right)}{\left(B^+ - 2A\right)}) e^{-B^+ t}\right) \\
&= \frac{\rho_1 b_1(u_1)}{B^+} + \frac{\left(x_1^0 + x_2^0 - 1\right)\left(\rho_1 b_1(u_1) - A\right)}{\left(B^+ - 2A\right)} e^{-2At} + \frac{C_u}{2} e^{-B^+ t}.
\end{aligned}$$

Let us now substitute the initial condition and find the constant C_u. If $x_1(0) = x_1^0$ then

$$C_u/2 = 2\left(x_1^0 - \frac{\rho_1 b_1(u_1)}{B^+} - \frac{\left(x_1^0 + x_2^0 - 1\right)\left(-2\rho_1 b_1(u_1) + 2A + B^-\right)}{2\left(B^+ - 2A\right)}\right)$$

Therefore,

$$\begin{aligned}
x_1(t) &= \frac{\rho_1 b_1(u_1)}{B^+} + \frac{\left(x_1^0 + x_2^0 - 1\right)\left(\rho_1 b_1(u_1) - A\right)}{\left(B^+ - 2A\right)} e^{-2At} \\
&+ \left(x_1^0 - \frac{\rho_1 b_1(u_1)}{B^+} - \frac{\left(x_1^0 + x_2^0 - 1\right)\left(2\rho_1 b_1(u_1) - 2A\right)}{2\left(B^+ + 2A\right)}\right) e^{-B^+ t} = \\
&= \frac{\rho_1 b_1(u_1)}{B^+}\left(1 - e^{-B^+ t}\right) + \frac{\left(x_1^0 - x_2^0 - 1\right)\left(\rho_1 b_1(u_1) - A\right)}{\left(B^+ - 2A\right)}\left(e^{-2At} - e^{-B^+ t}\right) + x_1^0 e^{-B^+ t}.
\end{aligned}$$

For $x_2(t)$ we have a similar result.

$$\begin{aligned}
x_2(t) &= \frac{\rho_2 b_2(u_2)}{B^+} + \frac{\left(x_1^0 + x_2^0 - 1\right)(\rho_2 b_2(u_2) - A)}{(B^+ - 2A)} e^{-2At} \\
&\quad - \left(-x_2^0 + \frac{\rho_2 b_2(u_2)}{B^+} + \frac{\left(x_1^0 + x_2^0 - 1\right)(\rho_2 b_2(u_2) - A)}{(B^+ - 2A)}\right) e^{-B^+ t} \\
&= \frac{\rho_2 b_2(u_2)}{B^+}\left(1 - e^{-B^+ t}\right) + \frac{\left(x_1^0 + x_2^0 - 1\right)(\rho_2 b_2(u_2) - A)}{(B^+ - 2A)}\left(e^{-2At} - e^{-B^+ t}\right) + x_2^0 e^{-B^+ t}.
\end{aligned}$$

We therefore have obtained the formulae for the evolution of market shares given the investment rates of firms 1,2. We have

$$\begin{aligned}
\pi^1(u_1, u_2) &= \lim_{T\to\infty} \int_0^T e^{-\rho t}\left(r_1 x_1(t) - u_1\right) dt \\
&= \lim_{T\to\infty} \int_0^T \left(\frac{r_1 \rho_1 b_1(u_1)}{B^+} - u_1\right) e^{-\rho t} dt \\
&\quad + \int_0^T \frac{r_1\left(x_1^0 + x_2^0 - 1\right)(\rho_1 b_1(u_1) - A)}{(B^+ - 2A)} e^{-\rho t} e^{-2At} dt \\
&\quad - \int_0^T r_1\left(\frac{\rho_1 b_1(u_1)}{B^+} + \frac{\left(x_1^0 + x_2^0 - 1\right)(\rho_1 b_1(u_1) - A)}{(B^+ - 2A)} - x_1^0\right) e^{-\rho t} e^{-B^+ t} dt \\
&= \frac{1}{\rho}\left(\frac{r_1 \rho_1 b_1(u_1)}{B^+} - u_1\right) + \frac{r_1\left(x_1^0 + x_2^0 - 1\right)(\rho_1 b_1(u_1) - A)}{(2A + \rho)(B^+ - 2A)} \\
&\quad - \frac{r_1}{(B^+ + \rho)}\left(\frac{\rho_1 b_1(u_1)}{B^+} + \frac{\left(x_1^0 + x_2^0 - 1\right)(\rho_1 b_1(u_1) - A)}{(B^+ - 2A)} - x_1^0\right) \\
&= \frac{r_1}{(B^+ + \rho)}\left(-\tfrac{\rho_1 b_1(u_1)}{\ln\beta} + \frac{\left(x_1^0 + x_2^0 - 1\right)(\rho_1 b_1(u_1) - A)}{(2A + \rho)} + x_1^0\right) - \frac{u_1}{\rho}.
\end{aligned}$$

The expression of $\pi^2(u_1, u_2)$ can be obtained in the same way. So we have derived explicit formulae for payoffs. □

Proof of Lemma 1. We assume that $\rho \neq 0$. Consider the exact form of the fourth coefficient and substitute the values for

$$\theta = \rho_2 \frac{u_2}{u_2 + \beta_2} \quad \text{and} \quad \vartheta = \frac{u_2}{u_2 + \alpha_2}.$$

Multiplying the fourth coefficient by $\frac{(u_2+\alpha_2)^2(u_2+\beta_2)^2}{\alpha_1\beta_1}$ and representing the expression $P(u_2) = \frac{(u_2+\alpha_2)^2(u_2+\beta_2)^2}{\alpha_1\beta_1}(2bc - 2r_1 dc + 2r_1 gh)$ as a polynomial in u_2, we find that the sign of $P(u_2)$ is equal to the sign of the fourth coefficient. We have the following polynomial expression

$$
\begin{aligned}
P\left(u_2\right) = {} & 2\alpha_2^2\rho^2\beta_2^2\Bigg(-2r_1\rho_1\left(1-x_1^0\right)+\overline{\overline{o}}\left(\rho\right)\Bigg) \\
& + u_2\Bigg(2\alpha_2\rho\beta_2\Bigg(-2\rho_1 r_1\left(\beta_2\left(1-x_1^0+x_2^0\right)+\alpha_2\rho_2\right)+\overline{\overline{o}}\left(\rho\right)\Bigg)\Bigg) \\
& + u_2^2\Bigg(-8r_1\rho_1\alpha_2\rho_2\beta_2+\overline{\overline{o}}\left(\rho\right)\Bigg) \\
& + u_2^3\Bigg(8\rho_2\Bigg(\beta_1\alpha_2\rho_2-r_1\rho_1\alpha_2+\rho_1\alpha_1\beta_2-2r_1\rho_1\beta_2\Bigg)+\overline{\overline{o}}\left(\rho\right)\Bigg) \\
& + u_2^4\Bigg(8\rho_2\Bigg(2\rho_2\beta_1-\alpha_1\rho_2-\rho_1\alpha_1-2r_1\rho_1\Bigg)+\overline{\overline{o}}\left(\rho\right)\Bigg)
\end{aligned}
$$

where $f(\rho)=\overline{\overline{o}}\left(\rho\right)$ is such that $\lim_{\rho\to 0}\dfrac{f(\rho)}{\rho}=0$. By hypothesis we know that $\rho\approx 0$, then $\overline{\overline{o}}\left(\rho\right)$ does not contribute to the signs of coefficients. If the following system of equations

$$
\begin{cases}
8\rho_2\left(\beta_1\alpha_2\rho_2+\rho_1\alpha_2-\rho_1\alpha_1\beta_2+2r_1\rho_1\beta_2\right)<0 \\
8\rho_2\left(2\rho_2\beta_1-\alpha_1\rho_2+\rho_1\alpha_1+2r_1\rho_1\right)<0
\end{cases}
$$

or, in equivalent way,

$$
\begin{cases}
r_1>\dfrac{\beta_1\alpha_1\rho_2+\rho_2\alpha_1\beta_1}{\rho_1\left(\alpha_1+2\beta_2\right)} \\
r_1>\dfrac{\rho_2}{\rho_1}\alpha_2+\dfrac{\rho_2+\rho_2}{2\rho_2}\alpha_1.
\end{cases}
\tag{18}
$$

is satisfied, then the number of sign alterations of polynomial equation $P\left(u_2\right)=0$ is zero. Since

$$
\frac{\rho_2}{\rho_1}\beta_1+\frac{\rho_2+\rho_1}{2\rho_1}\alpha_1-\frac{\beta_1\alpha_2\rho_2+\rho_1\alpha_1\beta_2}{\rho_1\left(\alpha_2+2\beta_2\right)}>0
$$

then we conclude that the second condition is stronger than the first in (18). By hypothesis the second condition in (18) is satisfied. Then the first condition in (18) is satisfied. It thus follows there are no positive roots for the equation $P\left(u_2\right)=0$. It is straightforward to verify that $P\left(0\right)<0$. By continuity argument, it thus follows that $P\left(u_2\right)<0$ for any $u_2\geq 0$. Then the third coefficient is strictly negative for any $u_2\geq 0$. □

Proof of Lemma 2. Arguing as in the proof of Lemma 1,we consider

$$
P\left(u_2\right)=\frac{\left(u_2+\alpha_2\right)^2\left(u_2+\beta_2\right)^2}{\alpha_1\beta_1}\left(r_1gb-r_1fc+c^2\right).
$$

Here

$$
\begin{aligned}
P\left(u_2\right) = {} & \rho^3\alpha_2^2\beta_2^2\left(-r_1\left(\alpha_1\rho_1x_2^0-\beta_1x_2^0+\beta_1-\beta_1x_1^0\right)+\overline{\overline{o}}\left(\rho\right)\right)+u_2\rho^2\beta_2\alpha_2 \\
& +\left[-r_1\left[\alpha_2\beta_1\rho_2\left(1-x_1^0-x_2^0\right)+\beta_2\rho_1\alpha_1\left(1-x_1^0-3x_2^0\right)+\alpha_1\alpha_2\rho_2\rho_1\left(x_1^0+x_2^0\right)\right]+\overline{\overline{o}}\left(\rho\right)\right]
\end{aligned}
$$

Therefore we impose the set of following conditions

$$\begin{cases} -r_1\left(\alpha_1\rho_1x_1^0+\beta_1\left(1-x_1^0-x_2^0\right)\right)<0 \\ -r_1\left(\alpha_2\beta_2\rho_2\left(1+x_1^0+x_2^0\right)+\beta_2\rho_1\alpha_2\left(1-x_1^0+3x_2^0\right)+\alpha_1\alpha_2\rho_2\rho_1\left(x_1^0+x_2^0\right)\right)<0 \\ -2\rho_1\alpha_3r_1\beta_2\left(\beta_2\left(1-x_1^0+x_2^0\right)+\alpha_2\rho_2\left(1-x_1^0-x_2^0\right)\right)<0 \\ -4r_1\rho_2\alpha_1\rho_1\beta_2<0 \\ -4\alpha_1\rho_2\left(r_1\rho_1-\beta_1\rho_2\right)<0 \end{cases}$$

which is trivially equivalent to $r_1\rho_1-\beta_1\rho_2>0$. If $r_1>\beta_1\dfrac{\rho_2}{\rho_1}$ then the fifth coefficient is negative for any non negative value of $\hat{u}_2$. □

Proof of Lemma 3. Let $\hat{u}_1$ be a best reply to $\hat{u}_2$. Then $\hat{u}_1>0$ is a positive root of equation (10). According to the Descartes's rule of sign alterations, if the number of sign alterations in the sequence of the coefficients of equations of a polynomial equation is equal to 1, then there exists exactly one positive root of the equation (Korn and Korn, 1968). Since the first and the second coefficients in left-side of equation (10) are positive, we obtain the different cases in Table 2. Cases are reduced by assuming that the third coefficient is not null in Table 2. If this happens the number of sign alterations may be just lower to the previous case, i.e. the third coefficient is not null.

Third coefficient	Fourth coefficient	Fifth coefficient	Sign alterations
>0	>0	>0	0
>0	>0	<0	1
<0	<0	>0	2
$>\mathbf{0}$	$<\mathbf{0}$	$<\mathbf{0}$	**1**
<0	>0	>0	2
<0	<0	<0	0
<0	<0	<0	0
$<\mathbf{0}$	$<\mathbf{0}$	$<\mathbf{0}$	**1**

Table 2. Number of sign alterations in equation (10)

Taking into account our hypothesis, theses of Lemmas 1, 2 hold. Therefore the fourth and the fifth coefficients are strictly negative. It follows from Table 2 that $\hat{u}_1>0$ is the unique best reply to $\hat{u}_2$. If there is exactly one positive root of the numerator of (11), and since the denominator is positive for any $u_1>0$, the derivative of payoffs takes negative values for $u_1>\hat{u}_1$ and positive values for $u_1<\hat{u}_1$. This proves that $\hat{u}_1$ is a maximum point of $\pi^1\left(u_1,\hat{u}_2\right)$. □

Proof of Lemma 4. By hypothesis, the thesis of Lemmas 3 is satisfied. Therefore, best reply multifunctions $\widehat{U}_i$ are functions. In addition, $\widehat{U}_1$ are continuous since payoffs in (7) are continuous. The rest of the proof is just technical and we leave it to the reader. The proof is simply based on convergence properties of a_i, b_i when u converges to ∞. In fact, we have $\lim_{u_2 \to \infty} \theta(u_2) = \rho_2$ and $\lim_{u_2 \to \infty} \vartheta(u_2) = 1$. $\square$