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CONE UNRECTIFIABLE SETS AND NON-DIFFERENTIABILITY OF LIPSCHITZ FUNCTIONS

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ABSTRACT

We provide sufficient conditions for a set $E \subset \mathbb{R}^n$ to be a non-universal differentiability set, i.e. to be contained in the set of points of non-differentiability of a real-valued Lipschitz function. These conditions are motivated by a description of the ideal generated by sets of non-differentiability of Lipschitz self-maps of $\mathbb{R}^n$ given by Alberti, Csörnyei and Preiss, which eventually led to the result of Jones and Csörnyei that for every Lebesgue null set E in $\mathbb{R}^n$ there is a Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^n$ not differentiable at any point of E, even though for $n > 1$ and for Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R}$ there exist Lebesgue null universal differentiability sets. Among other results, we show that the new class of Lebesgue null sets introduced here contains all uniformly purely unrectifiable sets and gives a quantified version of the result about non-differentiability in directions outside decomposability bundle with respect to a Radon measure.

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1. Introduction and main results

A recent surge of interest in validity of Rademacher's theorem on almost everywhere differentiability of Lipschitz maps of $\mathbb{R}^n$ to $\mathbb{R}^m$ arose from several new results and approaches. For infinite-dimensional Banach spaces there were successful attempts to obtain its analogues for the notion of Gâteaux derivative, for results and references see [6, Chapter 6], and some results for the stronger notion of Fréchet derivative to which a recent monograph [17] is devoted. In another direction, Pansu [20] obtained an almost everywhere result for Lipschitz maps between Carnot groups, and Cheeger [7] generalised Rademacher's theorem to Lipschitz functions on metric measure spaces.

Here we contribute to this research in the direction started by a result of [22] that, in terms of the size of differentiability sets of real-valued Lipschitz functions on $\mathbb{R}^2$, Rademacher's theorem is not sharp: there is a Lebesgue null set in $\mathbb{R}^2$ containing points of differentiability of any real-valued Lipschitz function on $\mathbb{R}^2$. Following [12, 13], where it was shown how unexpectedly small such sets may be, they are now called universal differentiability sets. The analogues of universal differentiability sets were recently introduced and investigated in the Heisenberg group [21].

The present paper introduces cone unrectifiable sets, which are a novel class of Lebesgue null sets, wider than that of uniformly purely unrectifiable sets (see Definition 1.7 and Remark 1.8) and shows that cone unrectifiable sets arise naturally in the study of sets in which Lipschitz functions may have no points of differentiability. As an application of our results, we strengthen and quantify the result of [4, Theorem 4.1] on non-differentiability with respect to arbitrary Radon measures, and demonstrate that the uniformly purely unrectifiable sets are non-universal differentiability sets in the strongest possible sense, see Theorem 1.13. Our main result is the following theorem.

Theorem 1.1: *If $E \subset \mathbb{R}^n$ is cone unrectifiable, then there is a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ that is non-differentiable at any point of E.*

The non-differentiability sets of Lipschitz maps $\mathbb{R}^n \to \mathbb{R}^m$, $m \geq n$ were first completely described in geometric measure theoretical terms in [3] (see [1, 2] for a published less formal description), and then [8] showed that this description gives precisely the Lebesgue null sets in $\mathbb{R}^n$. Hence Rademacher's theorem is sharp for maps into spaces of the same or higher dimension. This result was complemented in [24] where it was proved that whenever $m < n$, there is a Lebesgue null set in $\mathbb{R}^n$ containing points of differentiability of any Lipschitz map $\mathbb{R}^n \to \mathbb{R}^m$. We will return to the description originally introduced in [3] later as it forms the main starting point of what we do in the present paper.

The problem we address in this paper stems from the above results: can one give a geometric measure theoretical description of non-differentiability sets of Lipschitz maps of $\mathbb{R}^n$ to $\mathbb{R}$? Notice that this is a question about sets and not about measures: if we are given a σ-finite Borel measure μ in $\mathbb{R}^n$ that is singular with respect to the Lebesgue measure, we may use [3] and [8] to find a Lipschitz μ-almost everywhere non-differentiable mapping $f = (f_1, \dots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ and observe that for a random choice of $\alpha_i \in (0,1)$ the real-valued function $\sum_{i=1}^n \alpha_i f_i$ is Lipschitz and μ-almost everywhere non-differentiable. This argument appears both in [3] and [4], and moreover [4] simplifies the general arguments of [3] in the special case of differentiability μ-almost everywhere; notice also that in this case even the results of [8] may be demonstrated by a more accessible proof given in [11] (which is based on different ideas).

A further question (not addressed in the present paper) one may wish to consider is to give a description of sets on which a *typical* Lipschitz mapping is non-differentiable. The first such result was obtained in [25] for $n = 1$, and a recent paper [19] deals with a more restrictive full non-differentiability of typical Lipschitz functions $f : \mathbb{R}^n \to \mathbb{R}^m$ with $m \geq n$.

We will now state our results and explain them in more detail. Their proofs will be given in Section 3. The short Section 4 contains two examples whose meaning will also be discussed here.

The main concept that we use is based on the notion of *width* that has been, together with several variants, introduced in [3].

Definition 1.2: Suppose $e \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in (0,1]$. We let $C_{e,\alpha}$ be the cone $\{x \in \mathbb{R}^n : \langle x, e\rangle \geq \alpha \|x\| \|e\|\}$ and $\Gamma_{e,\alpha}$ the set of Lipschitz curves such that $\gamma'(t) \in C_{e,\alpha}$ for almost every $t \in \mathbb{R}$. The (e,α)-*width* of an open set $G \subset \mathbb{R}^n$ is defined by

$$w_{e,\alpha}(G) = \sup\{\mathcal{H}^1(G \cap \gamma(-\infty,\infty)) : \gamma \in \Gamma_{e,\alpha}\}, \tag{1.1}$$

and of any $E \subset \mathbb{R}^n$ by

$$w_{e,\alpha}(E) = \inf\{w_{e,\alpha}(G) : G \supset E, G \text{ is open}\}. \tag{1.2}$$

For the sake of completeness, when $e = 0$ or $\alpha > 1$ we let $w_{e,\alpha}(E) = 0$ for every $E \subset \mathbb{R}^n$. Of course, these cases have no bearing on what we do.

Notice that, as [3] points out, for constructions of Lipschitz functions, where values of $w_{e,\alpha}$ matter only for arbitrarily small α, the number α in Definition 1.2 may be replaced by any quantity or function that may attain arbitrarily small positive values. For example [4] replaces it by $\cos\alpha$, which is a bound on the angle between $\gamma'(t)$ and e and so is geometrically natural, but for us has the disadvantage that values of α that matter, namely those for which the cone $C_{e,\alpha}$ is close to a half-space, are close to $\pi/2$ rather than to zero.

Many variants of Definition 1.2 that are easily seen or shown to be equivalent to the one given here may be found in [23, Definition 1.1 and Remark 1.2]. A rather useful variant is that $\Gamma_{e,\alpha}$ may be defined as the collection of $\gamma \in C^1(\mathbb{R}, \mathbb{R}^n)$ satisfying $\|\gamma'(t))\| = 1$ and $\gamma'(t) \in C_{e,\alpha}$ for every t.

Perhaps the most interesting modification of Definition 1.2 comes from a so far unpublished result of Máthé and allows taking Borel sets G both in (1.1) and (1.2). It is not exactly equivalent with ours, but has the properties that a set of (e, β) width zero according to Máthé has (e, α) width zero according to the above definition for every $\alpha > \beta$, and a set of (e, α) width zero according to the above definition has (e, α) width zero according to Máthé. We have not used this, since our constructions, like that of [3], use that width is determined by open sets, and so the only difference would be that an appropriate version of Definition 1.2 would be called Máthé's Theorem.

Terms like "cone null" have been used for sets that are defined with the help of the notion of width. We follow this trend in our main notion, introduced in Definition 1.7. Before coming to it, we recall the main starting motivation behind what we do, namely the following definition from [2] and a special case of their result (proved in [3]) which is most relevant for us.

Definition 1.3 (see [2, Definition 1.11]): A map τ of a subset E of $\mathbb{R}^n$ to the Grassmanian $G(k, n)$ is said to be a *k-dimensional tangent field* of E if

$$w_{e,\alpha}\{x \in E : \tau(x) \cap C_{e,\alpha} = \{0\}\} = 0 \text{ for every } e \in \mathbb{R}^n \text{ and } \alpha > 0. \tag{1.3}$$

Obviously, if E is a k-dimensional embedded C^1 submanifold of $\mathbb{R}^n$, its tangent field $\tau(x)$ satisfies (1.3). However, the following theorem proved in [2, 3] shows that many non-smooth sets admit a k-dimensional tangent field. Before stating it, we notice that Definition 1.3 uses the value α in two different meanings and so it is sensitive on the choice of the notion of width. As a more detailed discussion of this will appear in [3], we just point out that the use of Máthé's width and the width from Definition 1.2 are equivalent. The only case to treat is when Definition 1.2 holds in Máthé's sense. Assuming $w_{e,\alpha_k}\{x \in E : \tau(x) \cap C_{e,\alpha_k} = \{0\}\} = 0$ in Máthé's sense for all $k \geq 1$, where $0 < \alpha_k < \alpha < 1$ and $\alpha_k \to \alpha$, we conclude that in the sense of Definition 1.2 we have $w_{e,\alpha}\{x \in E : \tau(x) \cap C_{e,\alpha_k} = \{0\}\} = 0$ for all $k \geq 1$. Hence writing $\{x \in E : \tau(x) \cap C_{e,\alpha} = \{0\}\} = \bigcup_{k=1}^{\infty}\{x \in E : \tau(x) \cap C_{e,\alpha_k} = \{0\}\}$, we get $w_{e,\alpha}\{x \in E : \tau(x) \cap C_{e,\alpha} = \{0\}\} = 0$.

Theorem 1.4 (see [2, Theorem 1.12]): *A set $E \subset \mathbb{R}^n$ is contained in a non-differentiability set of a Lipschitz map $\mathbb{R}^n \to \mathbb{R}^m$ for some, or any, $m \geq n$ if and only if it admits an $(n-1)$-dimensional tangent field. If $n = 2$, this holds if and only of E is Lebesgue null.*

As already mentioned, in the last assertion of Theorem 1.4 the assumption $n = 2$ was removed in [8]. Notice also that the general case of Theorem 1.4 says that the existence of k-dimensional tangent fields is similarly related to the existence of functions that at every point of the set can be differentiable in the direction of linear subspaces of dimension at most k only.

Based on these results, we conjecture that sets of non-differentiability of real-valued Lipschitz functions may be described as those for which there is an $(n-1)$-dimensional tangent field satisfying conditions that make it in some sense closer to being "genuinely" $(n-1)$-dimensional. We do not have a more precise conjecture, but a simple consequence of our main results, Theorem 1.1 and Theorem 1.9, is that sets for which there exists a continuous $(n-1)$-dimensional tangent field are indeed sets of non-differentiability of real-valued Lipschitz functions.

Since for the real-valued case only the tangent fields of codimension one are relevant, we base our approach on an obvious variant of Definition 1.3 that uses the normal fields instead of tangent fields. More interestingly, having in mind conditions similar to continuity of the normal field, we can define the normal vectors pointwise, while in general no pointwise definition of tangent fields from Definition 1.3 is known. A highly interesting exception to this is the special case when we are interested in measures supported by a set admitting a k-dimensional tangent field where [4] provides an interesting pointwise definition of the tangent field at almost every point.

Since our "normal vectors" are not exactly those orthogonal to the tangent field from Definition 1.3, we do not actually call them "normal vectors" and instead use just notation $\mathcal{N}(E,x)$ for their collection.

Definition 1.5: For $x \in E \subset \mathbb{R}^n$ let

$$\mathcal{N}(E,x) := \{e \in \mathbb{R}^n : (\forall \varepsilon > 0)(\exists r > 0) w_{e,\varepsilon}(B(x,r) \cap E) = 0\}.$$

Remark 1.6: Although we will not use it directly, we remark that $\mathcal{N}(E,x)$ is a linear subspace of $\mathbb{R}^n$ for any $x \in E$. This follows from results on "joining cones" in [3], but we describe a quick argument based on the result of Máthé. Since $\lambda\mathcal{N}(E,x) = \mathcal{N}(E,x)$ for each $\lambda \in \mathbb{R}$, we conclude that every nonzero e from the linear span of $\mathcal{N}(E,x)$ can be written as $e = \sum_{i=1}^{k} e_i$ where $e_i \in \mathcal{N}(E,x) \setminus \{0\}$. Suppose $\varepsilon > 0$ is fixed and $\gamma \in \Gamma_{e,\varepsilon}$ belongs to $C^1(\mathbb{R})$ and satisfies $\|\gamma'(t)\| = 1$ for all $t \in \mathbb{R}$ (cf. remarks after Definition 1.2). Let $\alpha = \frac{1}{2}\varepsilon\|e\|/\sum_i \|e_i\|$ and find $\delta > 0$ such that $w_{e_i,\alpha}(E \cap B(x,\delta)) = 0$ for each i. From Definition 1.2 we see that there is a Borel (in fact G_δ) set $G \supset E$ such that $w_{e_i,\alpha}(G \cap B(x,\delta)) = 0$ for every i. Fix now any $t \in \mathbb{R}$ and notice that there exists an i such that $\langle \gamma'(t), e_i \rangle \geq 2\alpha\|e_i\|$. By continuity of γ' there is a $\tau > 0$ such that for this particular i we have $\langle \gamma'(s), e_i \rangle > \alpha\|e_i\|$ whenever $s \in (t-\tau, t+\tau)$. Hence $w_{e_i,\alpha}(G \cap B(x,\delta)) = 0$ for this i implies $|\gamma^{-1}(G \cap B(x,\delta)) \cap (t-\tau, t+\tau)| = 0$. Finally, existence of such $\tau > 0$ for each $t \in \mathbb{R}$ allows us to conclude that $|\gamma^{-1}(G \cap B(x,\delta))| = 0$. As this holds for every $\gamma \in \Gamma_{e,\varepsilon}$, we get $w_{e,\varepsilon}(G \cap B(x,\delta)) = 0$.

Definition 1.7: A set $E \subset \mathbb{R}^n$ satisfying $\mathcal{N}(E,x) \neq \{0\}$ for every $x \in E$ is said to be *cone unrectifiable.*

Remark 1.8: Of course any cone unrectifiable set is Lebesgue null. A basic example of cone unrectifiable sets $E \subset \mathbb{R}^n$ is provided by those for which $\mathcal{N}(E,x) = \mathbb{R}^n$ for every $x \in E$. Such sets are called *uniformly purely unrectifiable.* By the result of András Máthé alluded to above these are precisely those sets that are contained in a Borel 1-purely unrectifiable set, i.e., in a Borel set B whose intersection with any C^1 curve has one-dimensional Hausdorff measure zero. The arguments used to prove Remark 1.6 simplify their definition in another way: E is uniformly purely unrectifiable if and only if there is $0 < \eta < 1$ such that $w_{e,\eta}(E) = 0$ for every unit vector e (this is used as a definition of a uniformly purely unrectifiable set in [9]). In Example 4.4 we point out that a similar simplification of the notion of cone unrectifiable sets is false: given any

$e \neq 0$ and $\eta \in (0,1)$, we construct a set E which does not satisfy the conclusions of Theorem 1.1 but is of (e,η)-width zero.

We are now ready to state the main results of this paper. First, we state Theorem 1.1 again:

THEOREM (Theorem 1.1): *If $E \subset \mathbb{R}^n$ is cone unrectifiable, then there is a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ that is non-differentiable at any point of E.*

There are various ways of stating that non-differentiability of a function f at a given point x is rather strong. The most usual one is by comparing the upper and lower directional derivatives of f at x defined by

$$D^+ f(x;y) := \limsup_{t \searrow 0} \frac{f(x+ty) - f(x)}{t}$$

and

$$D_+ f(x;y) := \liminf_{t \searrow 0} \frac{f(x+ty) - f(x)}{t},$$

respectively. An even stronger non-differentiability statement is obtained by showing that close to x, f may be approximated by many linear functions. Our next result shows that the non-differentiability statement of Theorem 1.1 may be strengthened in this way.

THEOREM 1.9: *For every cone unrectifiable set $E \subset \mathbb{R}^n$ and every $\varepsilon > 0$ there are a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ with $\operatorname{Lip}(f) \leq 1+\varepsilon$ and a continuous function $u : E \to \{z \in \mathbb{R}^n : \|z\| \leq \varepsilon\}$ such that*

$$\liminf_{r \searrow 0} \sup_{\|y\| \leq r} \frac{|f(x+y) - f(x) - \langle e + u(x), y\rangle|}{r} = 0 \tag{1.4}$$

whenever $x \in E$ and $e \in \mathcal{N}(E,x)$ has $\|e\| \leq 1$. In particular,

$$D^+ f(x;y) - D_+ f(x;y) \geq 2 \sup\Big\{\langle e, y\rangle : e \in \mathcal{N}(E,x),\ \|e\| \leq 1\Big\}$$

whenever $x \in E$ and $y \in \mathbb{R}^n$.

Additionally, if E is contained in $\{x : \omega(x) > 0\}$, where $\omega : \mathbb{R}^n \to [0,\infty)$ is continuous, then f may be chosen in such a way that $|f| \leq \omega$.

For a set E admitting an $(n-1)$-dimensional continuous tangent we obviously have $\mathcal{N}(E,x) \supset \tau(x)^\perp \neq \{0\}$. Hence such sets are cone unrectifiable and so are sets of non-differentiability of a real valued Lipschitz function. More interestingly, having countably many cone unrectifiable sets, we may try to add the functions obtained from Theorem 1.9 to get a function non-differentiable at the points of the union. However, addition of non-differentiable functions may create new points of differentiability. To solve this problem we employ the idea that Zahorski [27] used in his precise description of non-differentiability sets of Lipschitz functions on the real line as $G_{\delta\sigma}$-sets of measure zero; it is based on the simple observation that the sum of a differentiable and a non-differentiable function is non-differentiable. For this we need the function f obtained in Theorem 1.9 to be differentiable outside E, in other words, to have that E coincides with the set of points where f is not differentiable. This is however not possible in general as shown in Example 4.2. In the following two Corollaries we solve this difficulty by making a special assumption that the sets we consider are F_σ.

Corollary 1.10: *Suppose $E = \bigcup_k E_k \subset \mathbb{R}^n$, where E_k are disjoint cone unrectifiable F_σ sets, and let $\mathcal{N}_x := \mathcal{N}(E_k, x) \cap \overline{B}(0,1)$ when $x \in E_k$. Then there is a Lipschitz $f : \mathbb{R}^n \to \mathbb{R}$ such that*

- *f is differentiable at every $x \in \mathbb{R}^n \setminus E$;*
- *for every $x \in E$ there is $c > 0$ such that for every $y \in \mathbb{R}^n$,*

$$D^+ f(x;y) - D_+ f(x;y) \geq c \sup_{e \in \mathcal{N}_x} \langle e, y \rangle,$$

so, in particular, f is not differentiable at x.

If we are not interested in estimates of the difference of the upper and lower derivatives, Corollary 1.10 gives the following more naturally sounding statement.

Corollary 1.11: *For any $E \subset \mathbb{R}^n$ that is a countable union of cone unrectifiable F_σ sets there is a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ that is non-differentiable at any point of E and is differentiable at any point of its complement E^c.*

The next Corollary 1.12 contains the constructions of μ-almost everywhere non-differentiable functions from [3] and [4, Theorem 4.1]. Given a Radon measure μ in $\mathbb{R}^n$, these authors assign to μ-a.a. points a linear subspace $T(x)$ of $\mathbb{R}^n$ that in certain sense represents the directions of curves on which μ is "seen". For [3], the definition of "seen" is exactly the assumption of Corollary 1.12 while [4] bases the definition on a related but different property and shows in [4, Lemma 7.5] that the assumption of Corollary 1.12 is satisfied. Hence Corollary 1.12 gives a quantified generalisation of [4, Theorem 4.1]. It is, however, important to point out that both these references define the linear space $T(x)$ which is in a natural sense smallest, and this allows them to obtain also a counterpart to Corollary 1.12 that every Lipschitz function is μ-a.e. differentiable in the direction of $T(x)$. We also notice that the constructions of μ-almost everywhere non-differentiable Lipschitz functions have been strengthened in a different direction by [18] where the authors produce functions that μ-a.e. admit any blow-up behaviour permitted by the differentiability results.

Corollary 1.12: *Let μ be a Radon measure on $\mathbb{R}^n$ and T a μ-measurable map of $\mathbb{R}^n$ to $\bigcup_{m=0}^n G(n,m)$ such that for every unit vector e and $0<\alpha<1$, the set $\{x : C_{e,\alpha} \cap T(x) = \{0\}\}$ is the union of a μ-null set and a set E with $w_{e,\alpha}(E)=0$. Then there is a real valued Lipschitz function f on $\mathbb{R}^n$ such that for μ-a.e. $x \in \mathbb{R}^n$ there is $c>0$ such that $D^+ f(x,v) - D_+ f(x,v) \geq c\operatorname{dist}(v, T(x))$ for every $v \in \mathbb{R}^n$.*

Our final result deals with the *uniformly purely unrectifiable* sets introduced in Remark 1.8. For such sets the statement of Theorem 1.9 concerning upper and lower derivatives is shown in [3]. We prove a stronger result, namely that these sets are non-universal differentiability sets in the strongest possible sense, which corresponds to having $\varepsilon = 0$ in Theorem 1.9. However, in Example 4.1 we demonstrate that such an improvement is specific to the case of uniformly purely unrectifiable sets even when $E \subset \mathbb{R}^2$ is compact, for each $x \in E$ the set $\mathcal{N}(E,x)$ is a one-dimensional linear subspace of $\mathbb{R}^2$ and the map $x \mapsto \mathcal{N}(E,x)$ is continuous.

Theorem 1.13: *For every uniformly purely unrectifiable set $E \subset \mathbb{R}^n$ there is a real valued 1-Lipschitz function f on $\mathbb{R}^n$ such that*

$$\liminf_{r \searrow 0} \sup_{\|y\| \leq r} \frac{|f(x+y) - f(x) - \langle e, y\rangle|}{r} = 0 \tag{1.5}$$

for every $x \in E$ and $e \in \mathbb{R}^n$ with $\|e\| \leq 1$. In particular, $D^+ f(x;y) = \|y\|$ and $D_+ f(x;y) = -\|y\|$ for every $x \in E$ and $y \in \mathbb{R}^n$.

2. Technical lemmas

We will work in the space $\mathbb{R}^n$ equipped with the Euclidean norm $\|\cdot\|$. Most of the notation we use is standard; the open and closed balls will be denoted by $B(x,r)$ and $\overline{B}(x,r)$, respectively. Since we will often need to use the distance of a point to the complement of an open set, we will simplify the notation for it: for an open $G \subset \mathbb{R}^n$ we let

$$\rho_G(x) := \operatorname{dist}(x, \mathbb{R}^n \setminus G). \tag{2.1}$$

As usual, the Lipschitz constant of a real-valued function f defined on a set $E \subset \mathbb{R}^n$ is the smallest constant $\operatorname{Lip}(f, E) \in [0, \infty]$, or just $\operatorname{Lip}(f)$ when $E = \mathbb{R}^n$, such that $|f(y) - f(x)| \le \operatorname{Lip}(f, E)\|y - x\|$ for all $x, y \in E$, and functions with $\operatorname{Lip}(f) \le c$ will be termed c-Lipschitz. The space of Lipschitz functions on $\mathbb{R}^n$, those for which $\operatorname{Lip}(f) < \infty$, will be denoted by $\operatorname{Lip}(\mathbb{R}^n)$. We will also use the pointwise Lipschitz constants defined by $\operatorname{Lip}_x(f) := \limsup_{y \to x} |f(y) - f(x)|/\|y - x\|$ and often use the following well known fact.

Lemma 2.1: *For any $f : \mathbb{R}^n \to \mathbb{R}$, it holds that $\operatorname{Lip}(f) = \sup_{x \in \mathbb{R}^n} \operatorname{Lip}_x(f)$.*

Proof. It suffices to handle the case $n = 1$ when it follows, for example, from the considerably more general Theorem 4.5 in [26, Chapter IX]. ∎

The following lemma allows us to modify the functions we are constructing so that they become smooth on suitable subsets of $\mathbb{R}^n$. A similar lemma is proved in [3], and in [4], although the authors of the latter paper could have used more general [5, Theorem 1] or [16, Corollary 16]. We need a slightly more technical variant of results from these references.

Recall that for any collection $\mathcal{B}$ of open sets in $\mathbb{R}^n$ there is a C^∞ partition of unity of order n subordinated to it, that is a collection of C^∞ functions $\varphi_k : \mathbb{R}^n \to [0, 1]$, $k = 1, 2, \ldots$, such that

- each $\operatorname{spt}(\varphi_k)$ is a compact subset of some $B \in \mathcal{B}$,
- $\sum_k \varphi_k(x) = 1$ for every $x \in \bigcup \mathcal{B}$,
- for each $x \in \bigcup \mathcal{B}$ there is $r > 0$ such that $B(x, r) \cap \operatorname{spt}(\varphi_k) \ne \emptyset$ for at most $n + 1$ values of k.

Lemma 2.2: *Suppose $H \subset \mathbb{R}^n$ is open, $g : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, $\Phi : H \to \mathbb{R}^n$ and $\xi : H \to [0, \infty)$ are continuous and bounded, and $\|g'(x) - \Phi(x)\| \le \xi(x)$ for almost every $x \in H$. Then for every continuous $\omega : H \to (0, \infty)$ there is a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ such that*

(i) *$f(x) = g(x)$ for $x \notin H \cap \{\xi > 0\}$ and $|f(x) - g(x)| \le \omega(x)$ for $x \in H$;*
(ii) *$f \in C^1(H)$ and $\|f'(x) - \Phi(x)\| \le \xi(x)(1 + \omega(x))$ for $x \in H$;*
(iii) $\operatorname{Lip}(f) \le \max(\operatorname{Lip}(g), \sup_{x \in H}(\|\Phi(x)\| + \xi(x)(1 + \omega(x))))$.

Proof. Let $U := H \cap \{\xi > 0\}$, extend ξ and ω to (possibly discontinuous) functions defined on all of $\mathbb{R}^n$ by letting $\xi(x) = \omega(x) = 0$ for $x \notin H$ and let $\omega_0(x) := \frac{1}{2}\min(1, \xi(x)\omega(x), \omega(x), \rho_U^2(x))$. Let $\mathcal{B}$ be the family of balls $B(x,r)$ such that $x \in U$ and $r < \omega_0(x)$. Choose $(\varphi_k)_{k\geq 1}$ forming a locally finite C^∞ partition of unity on U subordinate to $\mathcal{B}$, and denote $m_k = 1 + \|\varphi_k'\|_\infty$.

As, for example, in [14, Appendix C.4], let η be the standard C^∞-smooth mollifier in $\mathbb{R}^n$ and define $\eta_s(x) := \eta(x/s)/s^n$. For each k choose $s_k > 0$ small enough so that the convolution $f_k = g * \eta_{s_k}$ satisfies for every $x \in \operatorname{spt}(\varphi_k)$,

- $|f_k(x) - g(x)| \leq 2^{-k-1} m_k^{-1} \omega_0(x)$;
- $\|f_k'(x) - \Phi(x)\| \leq \xi(x) + \omega_0(x)$.

Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = \sum_k f_k(x)\varphi_k(x)$ for $x \in U$ and $f(x) = g(x)$ for $x \notin U$. Since each $f_k\varphi_k$ is in $C^1(U)$, we have $f \in C^1(U)$. Also, for all $x \in \mathbb{R}^n$,

$$|f(x) - g(x)| \leq \omega_0(x) \tag{2.2}$$

since for $x \notin U$ both sides are zero, and for $x \in U$,

$$|f(x) - g(x)| \leq \sum_k |f_k(x) - g(x)|\varphi_k(x) \leq \sum_k \omega_0(x)\varphi_k(x) \leq \omega_0(x).$$

Since $\omega_0 \leq \omega$ and $\omega_0(x) = 0$ for $x \notin U$, (i) holds.

We show that f is differentiable at every $x \in H$ and

$$\|f'(x) - \Phi(x)\| \leq \xi(x) + 2\omega_0(x). \tag{2.3}$$

To see this for $x \in U$, we use $\sum_k \varphi_k(x) = 1$ and $\sum_k \varphi_k'(x) = 0$ to infer that

$$f'(x) - \Phi(x) = \sum_k (f_k'(x) - \Phi(x))\varphi_k(x) + \sum_k (f_k(x) - g(x))\varphi_k'(x),$$

hence

$$\begin{aligned}
\|f'(x) - \Phi(x)\| &\leq \sum_k \|f_k'(x) - \Phi(x)\|\varphi_k(x) + \sum_k |f_k(x) - g(x)|\|\varphi_k'(x)\| \\
&\leq \sum_k (\xi(x) + \omega_0(x))\varphi_k(x) + \sum_k 2^{-k-1}\omega_0(x) \\
&\leq \xi(x) + 2\omega_0(x).
\end{aligned}$$

To see (2.3) for $x \in H \setminus U$, we infer from the assumptions on g, Φ and ξ that g is differentiable at x and $g'(x) = \Phi(x)$. Since $(f-g)'(x) = 0$ because (2.2) gives $|f(y) - g(y)| \leq \omega_0(y) \leq \rho_U^2(y) \leq \|y - x\|^2$ for all $y \in \mathbb{R}^n$, we get that f is differentiable at x and $f'(x) = g'(x) = \Phi(x)$.

Clearly, (2.3) and the inequality $2\omega_0(x) \leq \xi(x)\omega(x)$ show the second statement of (ii).

To prove (iii), we infer from (2.2) that $\mathrm{Lip}_x(f) \le \mathrm{Lip}(g)$ for $x \in \mathbb{R}^n \setminus U$, and from (2.3) that

$$\mathrm{Lip}_x(f) \le \sup_{y\in U}(\|\Phi(y)\| + \xi(y) + 2\omega_0(y)) \le \sup_{y\in H}(\|\Phi(y)\| + \xi(y)(1+\omega(y)))$$

for $x \in U$. Thus (iii) holds by Lemma 2.1 and, since its right side is finite, we also see that f is Lipschitz.

We already know that f is differentiable at every $y \in H$ and f' is continuous at every $y \in U$. If $y \in H \setminus U$, (2.3) shows that $\lim_{x\to y}(f'(x) - \Phi(x)) = 0$. Since Φ is continuous at y, it follows that f' is continuous at y. Hence $f \in C^1(H)$, which is the last statement we needed to prove. ∎

The next simple Lemma is used to show that the functions we construct may be approximated by linear ones in the way required in equation (1.4) of our main result, Theorem 1.9.

LEMMA 2.3: *Suppose that $H \subset \mathbb{R}^n$ is open, $g : \mathbb{R}^n \to \mathbb{R}$ belongs to $C^1(H)$, $\omega : \mathbb{R}^n \to [0,\infty)$ is continuous and strictly positive on H, and $\eta \in (0,1]$. Then there is a function $\xi : \mathbb{R}^n \to [0,\infty)$ such that*

(i) *$\xi \in C(\mathbb{R}^n, [0,\infty)) \cap C(H, (0,\infty))$ and $\xi \le \frac{1}{2}\omega$;*

(ii) *if $x \in H$ and $h : \mathbb{R}^n \to \mathbb{R}$ satisfies $|h - g| \le 2\xi$, there is $0 < r < \omega(x)$ such that $|h(x+y) - h(x) - \langle g'(x), y\rangle| \le \eta r$ whenever $\|y\| \le r$.*

Proof. Let Ψ be the set of functions $\psi : \mathbb{R}^n \to [0,\infty)$ satisfying $\mathrm{Lip}(\psi) \le 1$, $0 \le \psi \le \frac{1}{2}\min(\rho_H, \omega, 1)$, and $\|g'(y) - g'(z)\| \le \frac{1}{2}\eta$ whenever $x \in H$ and $\max(\|y - x\|, \|z - x\|) < \psi(x)$. Since $0 \in \Psi$, $\varphi(x) := \sup\{\psi(x) : \psi \in \Psi\}$ is well-defined. We also have $\varphi \in \Psi$ since for any x, y, z satisfying $x \in H$ and $\max(\|y - x\|, \|z - x\|) < \varphi(x)$ there is $\psi \in \Psi$ such that $\max(\|y - x\|, \|z - x\|) < \psi(x)$ and hence $\|g'(y) - g'(z)\| \le \frac{1}{2}\eta$.

Let $x \in H$. Since both ρ_H and ω are continuous and strictly positive at x, there is $\varepsilon > 0$ such that $\frac{1}{2}\min(\rho_H, \omega, 1) > \varepsilon$ on $B(x, \varepsilon)$. Then the function $\psi_{\varepsilon,x}(y) := \max(0, \varepsilon - \|y - x\|)$ satisfies $\psi_{\varepsilon,x} = 0$ outside $B(x,\varepsilon)$ and $0 \le \psi_{\varepsilon,x}(y) \le \varepsilon \le \frac{1}{2}\min(\rho_H(y), \omega(y), 1)$ for $y \in B(x, \varepsilon)$. Hence $\psi_{\varepsilon,x}$ belongs to Ψ and we infer that $\varphi(x) \ge \psi_{\varepsilon,x}(x) = \varepsilon > 0$. Consequently, φ is strictly positive on H. Furthermore,

$$|g(x+y) - g(x) - \langle g'(x), y\rangle| \le \|y\| \sup_{z\in B(x,\|y\|)} \|g'(z) - g'(x)\| \le \tfrac{1}{2}\eta\|y\|$$

whenever $x \in H$ and $\|y\| < \varphi(x)$.

Letting $\xi(x) := \frac{1}{12}\eta\varphi(x)$, we see that (i) holds. To prove (ii), given $x \in H$, we let $r := \varphi(x)$, observe that $0 < r < \omega(x)$ and use that $\mathrm{Lip}(\xi) \le \frac{1}{12}\eta$ and $\xi(x) = \frac{1}{12}\eta r$ to estimate

$$\begin{aligned}
&|h(x+y) - h(x) - \langle g'(x), y\rangle| \\
&\qquad\qquad \le 2\xi(x+y) + 2\xi(x) + |g(x+y) - g(x) - \langle g'(x), y\rangle| \\
&\qquad\qquad \le 4\xi(x) + 2\mathrm{Lip}(\xi)\|y\| + \tfrac{1}{2}\eta\|y\| \\
&\qquad\qquad \le \tfrac{1}{3}\eta r + \tfrac{1}{6}\eta\|y\| + \tfrac{1}{2}\eta\|y\| \le \eta r
\end{aligned}$$

whenever $\|y\| < r = \varphi(x)$, and so whenever $\|y\| \le \xi(x)$. ∎

The following Lemmas 2.4 and 2.6 modify corresponding lemmas from [3] in a way suitable for our applications. A special version of Lemma 2.4, which does not suffice for our purposes, can be found also in [4, Lemmas 4.12–4.14]. Since [3] is not yet available, we provide full proofs.

Lemma 2.4: *Given $\varepsilon > 0$ there is $\vartheta \in (0,1)$ such that the following holds. For every $E \subset \mathbb{R}^n$, every unit vector $e \in \mathbb{R}^n$ such that $w_{e,\vartheta}(E) = 0$ and every continuous $\omega : \mathbb{R}^n \to [0,\infty)$ which is strictly positive on E, there is a Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$ such that $0 \le g \le \omega$, $\mathrm{Lip}(g) \le 1 + \varepsilon$ and there is an open set $H \supset E$ contained in $\{\omega > 0\}$ such that $\|g'(x) - e\| \le \varepsilon$ for Lebesgue almost all $x \in H$.*

Proof. Let $\vartheta = \sin\beta$, where $0 < \beta < \pi/2$, be such that $\tan\beta < \varepsilon/2$. Denote $G := \{x : \omega(x) > 0\}$ and choose $\varphi_k \in C^\infty(\mathbb{R}^n)$, $k \ge 1$, with compact support contained in G that form a locally finite partition of unity on G. Let $\varepsilon_k > 0$ be such that $\sum_k \varepsilon_k\|\varphi_k'\| < \varepsilon/2$ and $\varepsilon_k\varphi_k(x) \le 2^{-k}\min(1, \rho_G^2(x), \omega(x))$ for each $k \ge 1$ and all $x \in \mathbb{R}^n$.

Using values ε_k which we have just defined, find open sets G_k such that $G \supset G_k \supset E$ and $w_{e,\vartheta}(G_k) < \varepsilon_k$. For each $x \in \mathbb{R}^n$ we put

$$g_k(x) := \sup\Big\{\mathcal{H}^1\big(G_k \cap \gamma(-\infty, b]\big) - s : \gamma \in \Gamma_{e,\vartheta},\, s \ge 0,\, \gamma(b) = x + se\Big\} \tag{2.4}$$

and show that

(i) $0 \le g_k(x) \le \varepsilon_k$;
(ii) $|g_k(x+y) - g_k(x)| \le \|y\|\tan\beta$ when y is perpendicular to e;
(iii) $g_k(x) \le g_k(x+re) \le g_k(x) + r$ for every $r > 0$;
(iv) $g_k(x+re) = g_k(x) + r$ when $[x, x+re] \subset G_k$;
(v) g_k is a Lipschitz function and $\mathrm{Lip}(g_k) \le 1 + \tan\beta$;

(vi) $\|g_k'(x) - e\| \le \tan\beta$ for almost every $x \in G_k$.

The first inequality in (i) is obvious by considering in (2.4), $s = 0$ and any $\gamma \in \Gamma_{e,\vartheta}$ with $\gamma(b) = x$, and the second is immediate from $w_{e,\vartheta}(G_k) < \varepsilon_k$.

If $y \ne 0$ is orthogonal to e, and γ, b, s come from (2.4), we let $r := \|y\|$ and $\hat{y} := y/r$ and redefine γ on (b, ∞) by $\gamma(b+t) = \gamma(b) + (t\cot\beta)\hat{y} + te$ for $t > 0$. Using (2.4) for $g_k(x+y)$ with $b' := b + r\tan\beta$ and $s' := s + r\tan\beta$, we get

$$g_k(x+y) \ge g_k(x) - r\tan\beta = g_k(x) - \|y\|\tan\beta.$$

To get a lower estimate for $g_k(x)$ apply the above to the vector $-y$ added to $x + y$:

$$g_k(x) = g_k(x+y-y) \ge g_k(x+y) - \|y\|\tan\beta.$$

This verifies (ii).

Now consider $x' = x + re$ where $r > 0$. Since any γ used for x' may be used for x with $\gamma(b) = x + (r+s)e$, we get $g_k(x) \ge g_k(x') - r$. For the rest of (iii) and for (iv), note that as any γ used in (2.4) for x may be redefined by letting $\gamma(b+t) = x + se + te$ for $t \ge 0$, we get

$$g_k(x') \ge \mathcal{H}^1(G_k \cap \gamma(-\infty, b+r]) - s \ge \mathcal{H}^1(G_k \cap \gamma(-\infty, b]) - s$$

for all γ satisfying (2.4), so $g_k(x') \ge g_k(x)$, and this verifies (iii). If $[x, x'] = [x, x+re] \subset G_k$ and $r \le s$, the same argument shows that

$$g_k(x') \ge \mathcal{H}^1(G_k \cap \gamma(-\infty, b+s]) - (s-r) \ge \big(\mathcal{H}^1(G_k \cap \gamma(-\infty, b]) - s\big) + r,$$

and if $r > s$, then

$$\begin{aligned} g_k(x') \ge \mathcal{H}^1(G_k \cap \gamma(-\infty, b+r]) &= \mathcal{H}^1(G_k \cap \gamma(-\infty, b+s]) + (r-s) \\ &\ge \big(\mathcal{H}^1(G_k \cap \gamma(-\infty, b]) - s\big) + r \end{aligned}$$

for all such γ. Hence in both cases $g_k(x') \ge g_k(x) + r$, which, together with (iii), implies equality in (iv).

The statements (ii)–(iv) imply that g_k is Lipschitz and for almost every x, $0 \le Dg_k(x;e) \le 1$, the equality $Dg_k(x;e) = 1$ is satisfied for $x \in G_k$ and $|Dg_k(x;y)| \le \|y\|\tan\beta$ for y perpendicular to e. This gives both (v) and (vi).

Let $g := \sum_{k=1}^{\infty} g_k\varphi_k$. Since by (i) one has $0 \le g_k\varphi_k \le 2^{-k}\min(1, \rho_G^2, \omega)$ for every $k \ge 1$, we conclude that $0 \le g \le \omega$ and $\mathrm{Lip}_x(g) = 0$ for $x \notin G$. Since the sum defining g is locally finite, g is locally Lipschitz on G and by (v) and (i) for almost every $x \in G$,

$$\|g'(x)\| \le \sum_k \|g_k'(x)\|\varphi_k(x) + \sum_k g_k(x)\|\varphi_k'(x)\| \le 1 + \tan\beta + \sum_k \varepsilon_k \|\varphi_k'\| \le 1 + \varepsilon.$$

Hence $\mathrm{Lip}_x(g) \le 1 + \varepsilon$ for every $x \in G$, and we infer from Lemma 2.1 that $\mathrm{Lip}(g) \le 1 + \varepsilon$.

Let $H := \bigcap_k U_k$, where $U_k := (G \setminus \mathrm{spt}(\varphi_k)) \cup G_k$ are open. Then $E \subset \bigcap_k G_k \subset H \subset G$ and H is open because the complements of the U_k in G are closed in G and their collection is locally finite in G since $G \setminus U_k \subset \mathrm{spt}(\varphi_k)$. Finally, by (vi) for almost every $x \in H$,

$$\|g'(x) - e\| \le \sum_k \|g_k'(x) - e\|\varphi_k(x) + \sum_k g_k(x)\|\varphi_k'(x)\| \le \tan\beta + \sum_k \varepsilon_k \|\varphi_k'\| < \varepsilon. \qquad \blacksquare$$

Definition 2.5: Since we will need to use Lemma 2.4 for several values of ε at the same time, we introduce a function $\vartheta : (0,\infty) \to (0,\infty)$ such that $\vartheta(\sigma)$ is the value of ϑ from Lemma 2.4 for $\varepsilon = \frac{1}{7}\sigma$.

LEMMA 2.6: *Suppose $E \subset \mathbb{R}^n$, the functions $\omega : \mathbb{R}^n \to [0,\infty)$ and $\varphi : \mathbb{R}^n \to [0,1]$ are continuous, $\omega > 0$ on E, $e \in \mathbb{R}^n$, $\sigma > 0$ and $w_{e,\vartheta(\sigma)}(E \cap \{\varphi > 0\}) = 0$. Then there exist functions $f : \mathbb{R}^n \to \mathbb{R}$ and $\psi : \mathbb{R}^n \to [0,1]$ and an open set $H \subset \mathbb{R}^n$ such that*

(i) *$E \subset H \subset \{x : \omega(x) > 0\}$ and $f \in \mathrm{Lip}(\mathbb{R}^n) \cap C^1(H)$;*
(ii) *$|f(x)| \le \omega(x)\|e\|$ for all $x \in \mathbb{R}^n$ and $f(x) = 0$ when $\varphi(x) = 0$;*
(iii) *$\|f'(x) - \psi(x)e\| \le \sigma \mathbb{1}_{\{\omega>0\}}(x)\mathbb{1}_{\{\varphi>0\}}(x)\|e\|$ for almost all $x \in \mathbb{R}^n$;*
(iv) *$0 \le \psi(x) \le \varphi(x)\mathbb{1}_{\{\omega>0\}}(x)$ for $x \in \mathbb{R}^n$ and $\psi(x) = \varphi(x)$ for $x \in H$.*

Proof. If $e = 0$ or $\sigma \ge 1$, it suffices to let $f := 0$, $\psi := \varphi$ and $H := \{\omega > 0\}$. So we assume $\|e\| = 1$ and $\sigma < 1$, let $\varepsilon := \sigma/7$ and pick an integer $k \in [6/\sigma, 7/\sigma]$.

Let $\omega_0 := \frac{1}{2}\min(1,\omega)$, $G_0 := \{\omega > 0\}$, $H_0 := G_0 \cap \{\varphi > 0\}$ and, whenever H_{i-1} has been defined for some $i = 1,\dots,k$, let $G_i := H_{i-1} \cap \{\varphi > i/k\}$ and use Lemma 2.4 with continuous $\omega_i(x) = \frac{1}{2}\min(\omega, \rho_{G_i}^2)$, where ρ_{G_i} is defined by (2.1), to find a Lipschitz function $g_i : \mathbb{R}^n \to \mathbb{R}$ and nested open sets $H_i \subset G_i \subset H_{i-1}$ such that for each $1 \le i \le k$,

(a) $\mathrm{Lip}(g_i) \le 1 + \varepsilon$ and $|g_i| \le \frac{1}{2}\min(\omega, \rho_{G_i}^2)$;
(b) $G_i \supset H_i \supset G_i \cap E$ and $\|g_i'(x) - e\| \le \varepsilon$ for a.e. $x \in H_i$.

Let $g := \frac{1}{k}\sum_{i=1}^k g_i$. Then by (a), $\mathrm{Lip}(g) \le 1 + \varepsilon$ and $|g| \le \frac{1}{2}\min(\omega, \rho_{G_1}^2)$. For any $x \in G_0$ find the biggest $j = j(x) \in \{0,1,\dots,k\}$ with $x \in G_j$; since $G_k = \varnothing$, we have $j(x) \le k-1$. Define $\psi(x) = \min((j(x)+2)/k, \varphi(x))$; and for $x \notin G_0$ let $\psi(x) = 0$. Clearly, $0 \le \psi \le \varphi \mathbb{1}_{G_0}$ on $\mathbb{R}^n$, which is the first statement of (iv). For any $x \in H_0$ it holds $\psi(x) \in \left(\frac{j(x)}{k}, \frac{j(x)+2}{k}\right]$, i.e. $0 < \psi(x) - j(x)/k \le 2/k$. Define now

$$H = \bigcup_{j=0}^{k} \{x \in H_j : \varphi(x) < (j+2)/k\}$$

and notice that $\psi(x) = \varphi(x)$ whenever $x \in H$. Indeed, if $x \in H_j$ is such that $\varphi(x) < (j+2)/k$, then $H_j \subset G_j$ implies $j(x) \ge j$, so $\frac{j(x)+2}{k} \ge \frac{j+2}{k} > \varphi(x)$, hence by definition $\psi(x) = \varphi(x)$, and this verifies (iv). Also, $E \subset H$ since $E \subset \bigcup_{j=0}^{k-1}(H_j \setminus G_{j+1})$ from (b), and for $x \in H_j \setminus G_{j+1}$ we have $j(x) = j$ and so $\varphi(x) \le (j+1)/k < (j+2)/k$. Since it is clear that H is open and $\omega > 0$ on H because $H \subset G_0$, we conclude that the first part of (i) is satisfied for E, H and ω. We are now left to define the Lipschitz function f and verify the remaining part of (i), and also (ii) and (iii).

Note that for almost all $x \in G_1$ (where $\varphi > 1/k$), all g_i are differentiable at x and the estimate in (b) is satisfied whenever $x \in H_i$ and $1 \le i \le k$. Consider any such $x \in G_1$. To estimate $g'(x)$, notice that for such x we have $j = j(x) \ge 1$ and

- if $1 \le i < j$, then $x \in H_i$ and so $\|g_i'(x) - e\| \le \varepsilon$ by (b);
- if $i \ge j+1$, then $x \notin G_i$ and so $g_i'(x) = 0$ by (a).

Hence, for almost all $x \in G_1$

$$\begin{aligned} \|g'(x) - \psi(x)e\| &\le \|g'(x) - \tfrac{j-1}{k}e\| + \tfrac{3}{k}\|e\| \\ &\le \tfrac{1}{k}\Big(\sum_{i=1}^{j-1} \|g_i'(x) - e\| + \|g_j'(x)\|\Big) + \tfrac{3}{k} \\ &\le \varepsilon + \tfrac{4}{k} \le \tfrac{5}{k} \le 5\varphi(x). \end{aligned}$$

Since $g'(x) = 0$ outside G_1, we get $\|g'(x) - \psi(x)e\| = \psi(x)$ for $x \notin G_1$. Using that $\psi = 0$ outside H_0 and $\psi \le \varphi \le \frac{1}{k}$ for $x \in H_0 \setminus G_1$ we infer that $\|g'(x) - \psi(x)e\| \le \min(\frac{1}{k}, \varphi(x))$ outside G_1 and conclude $\|g'(x) - \psi(x)e\| \le 5\min(\frac{1}{k}, \varphi(x))$ for almost all $x \in \mathbb{R}^n$.

Using Lemma 2.2 with $\Phi(x) := \psi(x)e$, $\xi(x) := 5\min(\frac{1}{k}, \varphi(x))$ and $\hat{\omega}(x) = \frac{1}{5}\min(\omega(x), \varphi(x), \rho_H^2)$, we find Lipschitz $f : \mathbb{R}^n \to \mathbb{R}$ such that $f \in C^1(H)$, $|f(x) - g(x)| \le \hat{\omega}(x)$ and $\|f'(x) - \psi(x)e\| \le \xi(x)(1 + \hat{\omega}(x))$ for all $x \in H$. Since $f \in C^1(H)$, the remaining condition of (i) is satisfied. Finally, the conditions (ii) and (iii) hold since $|f| \le |f-g| + |g| \le \frac{1}{5}\min(\omega, \varphi) + \frac{1}{2}\min(\omega, \rho_{G_1}^2) \le \omega \mathbb{1}_{\{\varphi > 0\}}$, and $\|f'(x) - \psi(x)e\| \le 6\min(\frac{1}{k}, \varphi(x)) \le \sigma \mathbb{1}_{\{\varphi>0\}}(x)$ and $f' = g' = 0$ and $\psi = 0$ outside $G_0 = \{\omega > 0\}$. ∎

In a rather straightforward way, we will use Lemma 2.6 recursively to obtain the main tool for our construction of a function non-differentiable at points of a given set E.

Lemma 2.7: *Suppose $E \subset H_0 \subset \mathbb{R}^n$, H_0 is open, $f_0 \in \mathrm{Lip}(\mathbb{R}^n) \cap C^1(H_0)$ and $\omega_0 \in C(\mathbb{R}^n, [0,\infty)) \cap C(H_0, (0,\infty))$. Suppose further that for $k \ge 1$ we are given vectors $e_k \in \overline{B}(0,1)$, functions $\varphi_k \in C(\mathbb{R}^n, [0,1])$ and $\sigma_k > 0$ such that $w_{e_k,\vartheta(\sigma_k)}(E \cap \{\varphi_k > 0\}) = 0$. Then for each $j \ge 1$ there are sets $H_j \subset \mathbb{R}^n$ and functions $f_j, \omega_j, \psi_j : \mathbb{R}^n \to \mathbb{R}$ such that*

(i) *H_j is open, $E \subset H_j \subset H_{j-1}$ and $f_j \in \mathrm{Lip}(\mathbb{R}^n) \cap C^1(H_j)$;*
(ii) *$\omega_j \in C(\mathbb{R}^n, [0,\infty)) \cap C(H_j, (0,\infty))$ and $\omega_j \le \frac{1}{2}\min(1, \omega_{j-1}, \rho^2_{H_j})$;*
(iii) *$|f_j - f_{j-1}| \le \omega_{j-1}$ and $f_j(x) = f_{j-1}(x)$ when $\varphi_j(x) = 0$;*
(iv) *if $h : \mathbb{R}^n \to \mathbb{R}$ and $|h - f_j| \le 2\omega_j$ then for every $x \in H_j$ one may find $0 < r < \omega_{j-1}(x)$ such that $\sup_{\|y\|\le r} |h(x+y) - h(x) - \langle f_j'(x), y\rangle| \le \sigma_j r$;*
(v) *$\psi_j : \mathbb{R}^n \to [0,1]$, $0 \le \psi_j \le \varphi_j \mathbb{1}_{H_{j-1}}$ and $\psi_j = \varphi_j$ on H_j;*
(vi) *$\|f_j'(x) - f_{j-1}'(x) - \psi_j(x)e_j\| \le \sigma_j \mathbb{1}_{\{\varphi_j>0\}}(x)$ for every $x \in E$;*
(vii) *$\|f_j'(x) - z\| \le \|f_0'(x) + \sum_{i=1}^j \psi_i(x)e_i - z\| + \sum_{i=1}^j \sigma_i \mathbb{1}_{\{\varphi_i>0\}}(x)$ for any $z \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$.*

Proof. Replacing ω_0 by $\frac{1}{2}\min(1, \omega_0, \rho^2_{H_0})$ if necessary, we may and will assume that $\omega_0 \le \frac{1}{2}\min(1, \rho^2_{H_0})$ and observe that then $H_0 = \{\omega_0 > 0\}$. Assume $j \ge 1$ and an open set $H_{j-1} \supset E$, a function $f_{j-1} \in \mathrm{Lip}(\mathbb{R}^n) \cap C^1(H_{j-1})$, and a function $\omega_{j-1} \in C(\mathbb{R}^n, [0,\infty)) \cap C(E, (0,\infty))$ such that $H_{j-1} = \{\omega_{j-1} > 0\}$, have been already defined; this is certainly the case for $j = 1$. We will now explain how to construct functions f_j, ω_j, ψ_j and sets H_j such that conditions (i)–(vi) of the present lemma are satisfied. Notice that once we construct these objects, we have an open set $H_j \supset E$ and a function $f_j \in \mathrm{Lip}(\mathbb{R}^n) \cap C^1(H_j)$ from (i), and a function $\omega_j \in C(\mathbb{R}^n, [0,\infty)) \cap C(E, (0,\infty))$ satisfying $\omega_j \le \min(1, \rho^2_{H_j})$ for all $x \in \mathbb{R}^n$ from (ii). This will allow us recursively to construct all required objects so that (i)–(vi) hold, and then we will finish the proof by showing that (vii) holds as well.

By Lemma 2.6 find g_j, ψ_j and $H_j \subset \mathbb{R}^n$ such that

(a) H_j is open, $E \subset H_j \subset H_{j-1}$ and $g_j \in \mathrm{Lip}(\mathbb{R}^n) \cap C^1(H_j)$;
(b) $|g_j(x)| \le \omega_{j-1}(x)\|e_j\|$ for all $x \in \mathbb{R}^n$ and $g_j(x) = 0$ when $\varphi_j(x) = 0$;

(c) $\|g_j'(x) - \psi_j(x)e_j\| \le \sigma_j \mathbb{1}_{H_{j-1}}(x)\mathbb{1}_{\{\varphi_j>0\}}(x)\|e_j\|$ for almost all $x \in \mathbb{R}^n$;
(d) $0 \le \psi_j(x) \le \varphi_j(x)\mathbb{1}_{H_{j-1}}(x)$ for $x \in \mathbb{R}^n$ and $\psi_j(x) = \varphi_j(x)$ for $x \in H_j$.

Here we used that $H_{j-1} = \{\omega_{j-1} > 0\}$ to obtain conditions (a)–(d) directly from conditions (i)–(iv) of Lemma 2.6.

Let $f_j := f_{j-1} + g_j$, then (a) and (b) imply (i) and (iii), respectively. By Lemma 2.3 we may find $\xi_j \in C(\mathbb{R}^n, [0,\infty)) \cap C(H_j, (0,\infty))$ having the property that whenever $x \in H_j$ and $h : \mathbb{R}^n \to \mathbb{R}$ satisfies $|h - f_j| \le \xi_j$, there is $0 < r < \omega_{j-1}(x)$ such that $|h(x+y) - h(x) - \langle f_j'(x), y\rangle| \le \eta_j r$ whenever $\|y\| \le r$. Letting $\omega_j := \frac{1}{2}\min(\omega_j, \xi_j, \rho_{H_j}^2)$, we have (ii) and (iv). Clearly, (v) is the same as (d), and (c) implies that

$$\|f_j'(x) - f_{j-1}'(x) - \psi_j(x)e_j\| \le \sigma_j \mathbb{1}_{\{\varphi_j>0\}}(x) \tag{2.5}$$

for almost every $x \in \mathbb{R}^n$. From this, since f_j', f_{j-1}' and $\psi_j = \varphi_j$ are continuous on the open set $H_j \supset E$, we have (vi).

By the recursive use of the above construction we have defined H_j, f_j, ω_j and ψ_j such that (i)–(vi) hold. The last required statement (vii) follows by using (2.5) to estimate, for almost every $x \in \mathbb{R}^n$,

$$\begin{aligned}\|f_j'(x) - z\| &\le \|f_0'(x) + \sum_{i=1}^{j}\psi_i(x)e_i - z\| + \sum_{i=1}^{j}\|f_i'(x) - f_{i-1}'(x) - \psi_i(x)e_i\| \\ &\le \|f_0'(x) + \sum_{i=1}^{j}\psi_i(x)e_i - z\| + \sum_{i=1}^{j}\sigma_i\mathbb{1}_{\{\varphi_i>0\}}(x). \qquad \blacksquare\end{aligned}$$

We will use Lemma 2.7 to prove the two key results, Theorem 1.9 and Theorem 1.13. To prove the former, we will choose the objects required in Lemma 2.7 using the following combination of suitable partitions of unity.

Lemma 2.8: *Suppose $E \subset \mathbb{R}^n$ is cone unrectifiable and $\varepsilon > 0$. Then there exist sequences of positive numbers $\sigma_l > 0$, vectors $e_l \in \overline{B}(0,1)$ and continuous functions $\varphi_l : \mathbb{R}^n \to [0,1]$, such that*

(i) $\sum_{l\ge1}\sigma_l\mathbb{1}_{\mathrm{spt}(\varphi_l)} \le \varepsilon$;
(ii) $w_{e_l,\vartheta(\sigma_l)}(E \cap \{\varphi_l > 0\}) = 0$ *for each* $l \ge 1$;
(iii) *if* $x \in E$, $e \in \mathcal{N}(E,x)$ *and* $\|e\| \le 1$, *then for every* $\eta > 0$ *there are arbitrarily large* l *such that* $\sigma_l < \eta$, $\|e - e_l\| < \eta$ *and* $\varphi_l(x) = 1$.

Proof. For $x \in E$, $e \in \mathcal{N}(E,x)$ and any $\sigma > 0$ there exists, by definition of the cone unrectifiable set, a radius $\delta(x,e,\sigma) > 0$ such that $w_{e,\vartheta(\sigma)}(E \cap B_{x,e,\sigma}) = 0$, where $B_{x,e,\sigma} = B(x, \delta(x,e,\sigma))$.

We may suppose $\varepsilon = 1/p$ for some $p \in \mathbb{N}$ (so that $1/\varepsilon$ is a positive integer). For each $i \geq 1$ we let $\varepsilon_i := 2^{-i}\varepsilon$ and $\tau_i := 3^{-n}\varepsilon_i^{n+1}(n+1)^{-1}$. For each pair of $i \geq 1$ and $j = 1, \dots, 3^n\varepsilon_i^{-n}$ choose $e_{i,j} \in \overline{B}(0,1)$ such that $\overline{B}(0,1) \subset \bigcup_j B(e_{i,j}, \varepsilon_i)$ for every fixed $i \geq 1$. Let

$$E_{i,j} := \{x \in E : (\exists e \in \mathcal{N}(E,x)) \|e - e_{i,j}\| < \varepsilon_i\},$$

so that of course $\bigcup_j E_{i_0,j} = E$ for each fixed $i_0 \geq 1$. For each pair (i_0, j_0) find a partition of unity $\{\varphi_{i_0,j_0,k} : k \geq 1\}$ of order n subordinated to

$$\{B_{y,u,\sigma} : y \in E_{i_0,j_0},\, u \in \mathcal{N}(E,y),\, \|u - e_{i_0,j_0}\| < \varepsilon_{i_0}, \sigma = \tau_{i_0}\}.$$

Order the triples (i,j,k) into a single sequence $(i(l), j(l), k(l))$, and let $\varphi_l := \min\big(1, (n+1)\varphi_{i(l),j(l),k(l)}\big)$ and $\sigma_l := \tau_{i(l)}$. Also, observing that $\operatorname{spt}(\varphi_l) = \operatorname{spt}(\varphi_{i(l),j(l),k(l)})$, find $y_l \in E_{i(l),j(l)}$ and $e_l \in \mathcal{N}(E, y_l)$ such that $\operatorname{spt}(\varphi_l) \subset B_{y_l,e_l,\sigma_l}$. Notice for future reference that $\|e_l - e_{i(l),j(l)}\| < \varepsilon_{i(l)}$.

We show that the Lemma holds with the σ_l, e_l and φ_l defined above.

To prove (i), observe that for each fixed $i_0 \geq 1$ and $x_0 \in \mathbb{R}^n$ there are at most $3^n\varepsilon_{i_0}^{-n}(n+1)$ pairs (j,k) for which $x_0 \in \operatorname{spt}(\varphi_{i_0,j,k})$. Notice also that σ_l is constant and equal τ_{i_0} over all l with the same value of $i(l) = i_0$. Hence

$$\sum_l \sigma_l \mathbb{1}_{\operatorname{spt}(\varphi_l)}(x_0) \leq \sum_i 3^n\varepsilon_i^{-n}(n+1)\tau_i \leq \sum_i \varepsilon_i \leq \varepsilon.$$

The statement (ii) is immediate from $w_{e_l,\vartheta(\sigma_l)}(E \cap B_{y_l,e_l,\sigma_l}) = 0$ and the inclusion $\operatorname{spt}(\varphi_l) \subset B_{y_l,e_l,\sigma_l}$.

Finally, suppose $x \in E$, $e \in \mathcal{N}(E,x)$, $\|e\| \leq 1$, $\eta > 0$ and $l_0 \in \mathbb{N}$. Let $i_0 > \max\{i(l); l \leq l_0\}$ be such that $\varepsilon_{i_0} < \eta/2$. For any $i > i_0$ there is j such that $\|e - e_{i,j}\| < \varepsilon_i < \varepsilon_{i_0} < \eta/2$. Then $x \in E_{i,j}$ and since the partition of unity $\{\varphi_{i,j,k} : k \geq 1\}$ is of order n, there is k such that $\varphi_{i,j,k}(x) \geq 1/(n+1)$. This implies $\varphi_l(x) = 1$ for l satisfying $(i,j,k) = (i(l), j(l), k(l))$. Then $l > l_0$ and $\sigma_l = \tau_i < \varepsilon_i$, so $\|e - e_l\| \leq \|e - e_{i,j}\| + \|e_{i,j} - e_l\| < 2\varepsilon_i < \eta$, so (iii) holds as well. ∎

Our second use of Lemma 2.7, to prove Theorem 1.13, will be more straightforward: we use it to construct functions that will approximate the required function.

Lemma 2.9: *Suppose $E \subset H \subset \mathbb{R}^n$, E is uniformly purely unrectifiable, H is open, $\omega \in C(\mathbb{R}^n, [0,\infty)) \cap C(H, (0,\infty))$ and $f \in \operatorname{Lip}(\mathbb{R}^n) \cap C^1(H)$. Then for every $e \in \mathbb{R}^n$ and $\eta > 0$ there are $g, \xi : \mathbb{R}^n \to \mathbb{R}$ and an open set $U \subset \mathbb{R}^n$ such that*

(i) *$E \subset U \subset H$, $\xi \in C(\mathbb{R}^n, [0,\infty)) \cap C(U, (0,\infty))$ and $\xi \leq \frac{1}{2}\omega$;*

(ii) $|g-f| \le \omega$, $\mathrm{Lip}(g) \le \max(\mathrm{Lip}(f), \|e\|) + \eta$ *and* $g \in C^1(U)$;

(iii) *if* $x \in E$ *and a function* $h : \mathbb{R}^n \to \mathbb{R}$ *satisfies* $|h-g| \le 2\xi$, *there is* $0 < r < \omega(x)$ *such that* $\sup_{\|y\|\le r} |h(x+y) - h(x) - \langle e, y\rangle| \le \eta r$.

Proof. Let $\sigma = \eta/8(n+1)$. Since $f \in C^1(H)$ and $E \subset H$, for each $x \in E$ there is $\delta_x > 0$ such that $\|f'(y) - f'(z)\| < \frac{1}{4}\eta$ for $y, z \in B_x := B(x, \delta_x)$. Find a partition of unity $\{\gamma_k : k \ge 1\}$ of order n subordinated to $\{B_x : x \in E\}$ and choose $x_k \in E$ such that $\mathrm{spt}(\gamma_k) \subset B_{x_k}$.

Set $H_0 = H$, $\omega_0 = \frac{1}{2}\omega$, $f_0 = f$, $\sigma_k = \sigma$, $e_{2k-1} = -f'(x_k) \in \mathcal{N}(E, x_k)$, $e_{2k} = e \in \mathcal{N}(E, x_k)$, and $\varphi_{2k-1} = \varphi_{2k} = \gamma_k$. Since E is uniformly purely unrectifiable, the hypothesis of Lemma 2.7 is satisfied, and so find f_k, ω_k, H_k and ψ_k, $k \ge 1$, such that the statements (i)–(vii) of Lemma 2.7 hold (we leave out (iv) and (vi) as we do not use them here):

(a) H_k is open, $E \subset H_k \subset H_{k-1}$ and $f_k \in \mathrm{Lip}(\mathbb{R}^n) \cap C^1(H_k)$;

(b) $\omega_k \in C(\mathbb{R}^n, [0, \infty)) \cap C(H_k, (0, \infty))$ and $\omega_k \le \frac{1}{2}\min(1, \omega_{k-1}, \rho^2_{H_k})$;

(c) $|f_k - f_{k-1}| \le \omega_{k-1}$ and $f_k(x) = f_{k-1}(x)$ when $\varphi_k(x) = 0$;

(d) $\psi_k : \mathbb{R}^n \to [0, 1]$, $0 \le \psi_k \le \varphi_k \mathbb{1}_{H_{k-1}}$ and $\psi_k = \varphi_k$ on H_k;

(e) $\|f_k'(x) - z\| \le \|f'(x) + \sum_{i=1}^k \psi_i(x) e_i - z\| + \sum_{i=1}^k \sigma \mathbb{1}_{\{\varphi_i > 0\}}(x)$ for all $z \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$.

By (b) and (c), the sequence of Lipschitz functions (f_k) converges to a function $g : \mathbb{R}^n \to \mathbb{R}$ and $|g - f| \le \omega$. For every x at which $f'(x)$ exists write

$$f'(x) + \sum_{i=1}^{2k} \psi_i(x) e_i = a f'(x) + b e + v, \tag{2.6}$$

where $a = 1 - \sum_{i=1}^k \psi_{2i-1}(x)$, $b = \sum_{i=1}^k \psi_{2i}(x)$, $v = \sum_{i=1}^k \psi_{2i-1}(x)(f'(x) - f'(x_i))$. Using $\sum_i \gamma_i \le 1$ as it is a partition of unity, and (d) to get

$$0 \le \psi_{2i} \le \varphi_{2i} \mathbb{1}_{H_{2i-1}} = \varphi_{2i-1} \mathbb{1}_{H_{2i-1}} \le \psi_{2i-1} \le \varphi_{2i-1} = \gamma_i, \tag{2.7}$$

we see that $a, b \ge 0$, $a + b = 1 + \sum_{i=1}^k (\psi_{2i}(x) - \psi_{2i-1})(x) \le 1$, and

$$\|v\| \le \sum_{i \in A} \gamma_i(x) \|f'(x) - f'(x_i)\|,$$

where $A = \{i : x \in \mathrm{spt}(\gamma_i)\}$. Recall that $\mathrm{spt}(\gamma_i) \subset B_{x_i}$, and by the definition of the ball B_{x_i} we have $\|f'(x) - f'(x_i)\| < \frac{1}{4}\eta$ for $x \in B_{x_i}$, hence $\|v\| < \frac{1}{4}\eta$. Thus we conclude from (2.6) that for almost all $x \in \mathbb{R}^n$ and all $k \ge 1$

$$\Big\| f'(x) + \sum_{i=1}^{2k} \psi_i(x) e_i \Big\| \le \max(\mathrm{Lip}(f), \|e\|) + \eta/4. \tag{2.8}$$

Since for every x there are at most $2(n+1)$ values of i with $\varphi_i(x) \neq 0$, we see that $\sum_{i=1}^{2k} \sigma \mathbb{1}_{\{\varphi_i>0\}}(x) \leq 2(n+1)\sigma = \frac{1}{4}\eta$ for any $k \geq 1$, and infer from (e) with $z = 0$ and (2.8) that for a.e. x,

$$\|f'_{2k}(x)\| \leq \|f'(x) + \sum_{i=1}^{2k} \psi_i(x)e_i\| + \sum_{i=1}^{2k} \sigma \mathbb{1}_{\{\varphi_i>0\}}(x) \leq \max(\mathrm{Lip}(f), \|e\|) + \tfrac{1}{2}\eta.$$

Since, by (a), f_{2k} is Lipschitz, we conclude $\mathrm{Lip}(f_{2k}) < \max(\mathrm{Lip}(f), \|e\|) + \eta$ for each k, and so (ii) holds.

For each $x \in E$ there is a neighbourhood where all but a finite number of the functions φ_k's are zero, so we can find $r_x > 0$ and $k_x \in \mathbb{N}$ such that $B(x, r_x) \cap \mathrm{spt}\varphi_k = \emptyset$ for $k \geq k_x$. Let $U_x := B(x, r_x) \cap H_{k_x}$, where $H_{k_x} \supset E \ni x$ is defined in (a), and define an open set $U := \bigcup_{x \in E} U_x$. As $x \in U_x \subset H_{k_x} \subset H_0 = H$ for any $x \in E$, we conclude that $E \subset U \subset H$, this verifies the first two statements of (i). By (c), $g = f_k$ on $B(x, r_x) \supset U_x$ for every $k \geq k_x$; hence $g \in C^1(U_x)$ by (a) as $U_x \subset H_{k_x}$, and so $g \in C^1(U)$. Thus Lemma 2.3 applied to U, g, ω and $\frac{1}{2}\eta$ provides a continuous function $\xi : \mathbb{R}^n \to [0, \infty)$ such that (i) holds and for every $x \in E \subset U$ and $h : \mathbb{R}^n \to \mathbb{R}$ satisfying $|h - g| \leq 2\xi$, there is $0 < r < \omega(x)$ such that

$$\sup_{\|y\| \leq r} |h(x+y) - h(x) - \langle g'(x), y \rangle| \leq \tfrac{1}{2}\eta r. \tag{2.9}$$

Observe now that for $x \in E$ we have $x \in H_i$ for any $i \geq 1$, hence $\psi_i(x) = \varphi_i(x)$ for any $i \geq 1$ by (d). Together with definition of k_x this implies that $\sum_{i=1}^{k} \psi_{2i-1}(x) = \sum_{i=1}^{k} \varphi_{2i-1}(x) = \sum_{i=1}^{k} \gamma_i(x) = \sum_{i \geq 1} \gamma_i(x) = 1$ for any $k \geq k_x$, hence for such k the constants a, b from (2.6) satisfy $a = 0$ and, similarly, $b = 1$. Using equation (2.6) and recalling that $\|v\| \leq \frac{1}{4}\eta$, we get $\|f'(x) + \sum_{i=1}^{2k} \psi_i(x)e_i - e\| = \|v\| \leq \frac{1}{4}\eta$ for any $k \geq k_x$. With $k = k_x$ we have $g = f_{2k}$ on U_x, hence using (e) with $z = e$ it follows

$$\|g'(x) - e\| = \|f'_{2k}(x) - e\| \leq \|f'(x) + \sum_{i=1}^{2k} \psi_i(x)e_i - e\| + \sigma \sum_{i=1}^{2k} \mathbb{1}_{\{\varphi_i>0\}}(x) \leq \tfrac{1}{2}\eta,$$

and by combining this with (2.9), we obtain (iii). ∎

3. Proofs of main results

Proof of Theorem 1.9. Recall that we are given a cone unrectifiable set $E \subset \mathbb{R}^n$. We are also given $\varepsilon > 0$ and a continuous function $\omega \geq 0$ such that $E \subset \{x : \omega(x) > 0\}$; if ω is not given, we set $\omega = 1$ everywhere on $\mathbb{R}^n$.

We begin by finding numbers $\sigma_k > 0$, vectors $e_k \in \overline{B}(0,1)$ and continuous functions $\varphi_k : \mathbb{R}^n \to [0,1]$, $k = 1, 2, \ldots$, such that

(A) $\sum_k \sigma_k \mathbb{1}_{\mathrm{spt}(\varphi_k)} \le \varepsilon$;
(B) $w_{e_k,\vartheta(\sigma_k)}(E \cap \{\varphi_k > 0\}) = 0$;
(C) if $x \in E$, $e \in \mathcal{N}(E,x)$ and $\|e\| \le 1$, then for every $\eta > 0$ there are arbitrarily large k such that $\sigma_{2k-1} < \eta$, $\|e - e_{2k-1}\| < \eta$ and $\varphi_{2k-1}(x) = 1$;
(D) for every $k \ge 1$, $\varphi_{2k} = \varphi_{2k-1}$ and $e_{2k} = -e_{2k-1}$.

For this, it suffices to take $\hat\sigma_l$, $\hat e_l$ and $\hat\varphi_l$ from Lemma 2.8 with ε replaced by $\varepsilon/2$ and let $\sigma_{2l-1} = \sigma_{2l} := \hat\sigma_l$, $\varphi_{2l-1} = \varphi_{2l} := \hat\varphi_l$, $e_{2l-1} := \hat e_l$ and $e_{2l} := -\hat e_l$.

We set $f_0 := 0$, $H_0 := \{\omega > 0\}$, $\omega_0 := \frac{1}{2}\min(1, \omega, \rho^2_{H_0})$ and use Lemma 2.7 to find $f_j, \omega_j, H_j, \psi_j$, $j = 1, 2, \dots$ such that

(E) H_j is open, $E \subset H_j \subset H_{j-1}$ and $f_j \in \mathrm{Lip}(\mathbb{R}^n) \cap C^1(H_j)$;
(F) $\omega_j \in C(\mathbb{R}^n, [0,\infty)) \cap C(H_j, (0,\infty))$ and $\omega_j \le \frac{1}{2}\min(1, \omega_{j-1}, \rho^2_{H_j})$;
(G) $|f_j - f_{j-1}| \le \omega_{j-1}$ and $f_j(x) = f_{j-1}(x)$ when $\varphi_j(x) = 0$;
(H) if $h : \mathbb{R}^n \to \mathbb{R}$ and $|h - f_j| \le 2\omega_j$ then for every $x \in H_j$ one may find $0 < r < \omega_{j-1}(x)$ such that $\sup_{\|y\| \le r} |h(x+y) - h(x) - \langle f_j'(x), y\rangle| \le \sigma_j r$;
(I) $\psi_j : \mathbb{R}^n \to [0,1]$, $0 \le \psi_j \le \varphi_j \mathbb{1}_{H_{j-1}}$ and $\psi_j = \varphi_j$ on H_j;
(J) $\|f_j'(x) - f_{j-1}'(x) - \psi_j(x) e_j\| \le \sigma_j \mathbb{1}_{\{\varphi_j > 0\}}(x)$ for every $x \in E$;
(K) $\|f_j'(x) - z\| \le \|f_0'(x) + \sum_{i=1}^j \psi_i(x) e_i - z\| + \sum_{i=1}^j \sigma_i \mathbb{1}_{\{\varphi_i > 0\}}(x)$ for all $z \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$.

Notice that (F) implies $\omega_j \le 2^{i-j}\omega_i$ for $j \ge i$, and so also $\omega_j \le 2^{-j}$. Consequently, by (G), f_j converge uniformly to a function $f : \mathbb{R}^n \to \mathbb{R}$ and $|f - f_j| \le \sum_{i=j}^\infty \omega_i \le 2\omega_j$. We show that f has the required properties.

Notice that (I) and (D) imply that

$$\psi_{2i-1}(x) e_{2i-1} + \psi_{2i}(x) e_{2i} = -(\psi_{2i-1}(x) - \psi_{2i}(x)) \mathbb{1}_{H_{2i-2} \setminus H_{2i}}(x) e_{2i},$$

and this vector has norm at most $\mathbb{1}_{H_{2i-2} \setminus H_{2i}}(x)$, as condition (I) implies $0 \le \psi_{2i} \le \varphi_{2i} \mathbb{1}_{H_{2i-1}} = \varphi_{2i-1} \mathbb{1}_{H_{2i-1}} \le \psi_{2i-1} \le \varphi_{2i-1} \le 1$ (cf. (2.7)). Hence (K) with $z = 0$ and (A) give

$$\begin{aligned} \|f_{2k}'(x)\| &= \Big\|\sum_{i=1}^k (\psi_{2i}(x) e_{2i} + \psi_{2i-1}(x) e_{2i-1})\Big\| + \sum_{i=1}^{2k} \sigma_i \mathbb{1}_{\{\varphi_i > 0\}}(x) \\ &\le \sum_{i=1}^k \mathbb{1}_{H_{2i-2} \setminus H_{2i}}(x) + \sum_{i=1}^{2k} \sigma_i \mathbb{1}_{\{\varphi_i > 0\}}(x) \le 1 + \varepsilon \end{aligned}$$

for almost every x. Since (E) shows that f_{2k} is Lipschitz, $\mathrm{Lip}(f_{2k}) \le 1 + \varepsilon$, and we conclude that $\mathrm{Lip}(f) \le 1 + \varepsilon$.

For every $i \geq 1$ and $x \in E \subset H_{2i} \subset H_{2i-1}$, (I), (D) and (J) imply

$$\begin{aligned}
&\|f'_{2i}(x) - f'_{2i-2}(x)\| \\
&\quad = \|(f'_{2i}(x) - f'_{2i-1}(x) - \varphi_{2i}(x)e_{2i}) + (f'_{2i-1}(x) - f'_{2i-2}(x) - \varphi_{2i-1}(x)e_{2i-1})\| \\
&\quad \leq \sigma_{2i}\mathbb{1}_{\{\varphi_{2i}>0\}}(x) + \sigma_{2i-1}\mathbb{1}_{\{\varphi_{2i-1}>0\}}(x).
\end{aligned}$$

Since $\sum_j \sigma_j \mathbb{1}_{\{\varphi_j>0\}}(x) \leq \varepsilon$ by (A), the restrictions of f'_{2k} to E converge pointwise to a function $u : E \to \mathbb{R}^n$ and $\|u(x)\| \leq \varepsilon$ for $x \in E$.

Suppose $x \in E$, $e \in \mathcal{N}(E, x)$, $\|e\| \leq 1$ and $\eta > 0$. By (C) there is k such that $2^{-2k} < \eta$, $\|f'_{2k}(x) - u(x)\| < \frac{1}{4}\eta$, $\|e - e_{2k+1}\| < \frac{1}{4}\eta$, $\sigma_{2k+1} < \frac{1}{4}\eta$ and $\varphi_{2k+1}(x) = 1$. Since $x \in E \subset H_{2k+1}$, the latter immediately implies $\psi_{2k+1}(x) = 1$ by (I). Since $|f - f_{2k+1}| \leq 2\omega_{2k+1}$ and (J) gives $\|f'_{2k+1}(x) - (f'_{2k}(x) + e_{2k+1})\| \leq \sigma_{2k+1}$, we conclude that (H) provides $0 < r < \omega_{2k}(x) \leq 2^{-2k}\omega_0 < \eta$ such that for every $\|y\| \leq r$,

$$\begin{aligned}
&|f(x+y) - f(x) - \langle u(x) + e, y\rangle| \\
&\qquad \leq |f(x + y) - f(x) - \langle f'_{2k+1}(x), y\rangle| + \|f'_{2k+1}(x) - (f'_{2k}(x) + e_{2k+1})\|\|y\| \\
&\qquad\quad + \|f'_{2k}(x) - u(x)\|\|y\| + \|e_{2k+1} - e\|\|y\| \\
&\qquad < (\sigma_{2k+1} + \sigma_{2k+1} + \eta/4 + \eta/4)r < \eta r.
\end{aligned}$$

Since $\eta > 0$ may be arbitrarily small,

$$\liminf_{r\searrow 0} \sup_{\|y\|\leq r} \frac{|f(x + y) - f(x) - \langle e + u(x), y\rangle|}{r} = 0, \tag{3.1}$$

which is the main statement we wished to prove. The estimate of the lower and upper derivatives is an immediate consequence: if $e \in \mathcal{N}(E, x)$ and $\|e\| \leq 1$, we use (3.1) for e and $-e$ to infer

$$D^+f(x; y) - D_+f(x; y) \geq \langle e + u(x), y\rangle - \langle -e + u(x), y\rangle = 2\langle e, y\rangle. \qquad \blacksquare$$

Proof of Corollary 1.10. We are given $E = \bigcup_{k\geq 1} E_k \subset \mathbb{R}^n$ where E_k are disjoint cone unrectifiable F_σ sets, and $\mathcal{N}_x = \mathcal{N}(E_k, x) \cap \overline{B}(0, 1)$ for $x \in E_k$.

Write $E_k = \bigcup_{j\geq 1} H_{k,j}$ where $H_{k,j}$ are closed cone unrectifiable sets, and let $F_{k,j} := \bigcup_{i<j} H_{k,i}$ and $E_{k,j} := H_{k,j} \setminus F_{k,j}$, so that $E_{k,j}$ are pairwise disjoint over all (k, j). Let $c_{k,j} := 2^{-k-j}$ and $\omega_{k,j}(x) := c_{k,j} \min(1, \mathrm{dist}^2(x, F_{k,j}))$. By Theorem 1.9 there are Lipschitz functions $f_{k,j} : \mathbb{R}^n \to \mathbb{R}$ such that $\mathrm{Lip}(f_{k,j}) < 2$,

$|f_{k,j}| \le \omega_{k,j}$ and

$$D^+ f_{k,j}(x;y) - D_+ f_{k,j}(x;y) \ge 2\sup\{\langle e,y\rangle : e \in \mathcal{N}(E_{k,j},x),\ \|e\| \le 1\}$$
$$\ge 2 \sup_{e\in\mathcal{N}_x} \langle e,y\rangle$$

for $x \in H_{k,j}$ and $y \in \mathbb{R}^n$; the last inequality follows from $\mathcal{N}_x \subset \mathcal{N}(E_{k,j},x)$.

Apply Lemma 2.2 to $\omega = \omega_{k,j+1}$, $H = \{\omega_{k,j+1} > 0\}$, $g = f_{k,j}$, $\Phi = 0$ and $\xi = 2$ to find Lipschitz functions $g_{k,j} : \mathbb{R}^n \to \mathbb{R}$ such that $g_{k,j} \in C^1\{\omega_{k,j+1} > 0\}$, $|g_{k,j} - f_{k,j}| \le \omega_{k,j+1}$ and $\mathrm{Lip}(g_{k,j}) \le 3$. We observe that $g_{k,j}$ is differentiable at every $x \notin H_{k,j}$. Indeed, for such an x, if $\omega_{k,j}(x) = 0$, i.e. $x \in F_{k,j} \subset F_{k,j+1}$, then $g_{k,j}(x) = f_{k,j}(x) = 0$ as $\omega_{k,j}(x) = \omega_{k,j+1}(x) = 0$, and $|g_{k,j}(y)| \le 2c_{k,j}\|y-x\|^2 \le \|y-x\|^2$, using upper estimates for $|g_{k,j} - f_{k,j}|$ and $|f_{k,j}|$, and $x \in F_{k,j} \subset F_{k,j+1}$; hence $g'_{k,j}(x) = 0$. If, however, $x \notin H_{k,j}$ and $\omega_{k,j}(x) > 0$, then $x \notin E_{k,j} \cup F_{k,j}$, hence $\omega_{k,j+1}(x) > 0$ and so it follows that $g_{k,j}$ is C^1 on a neighbourhood of x. We also observe that for every $x \in H_{k,j}$ and $y \in \mathbb{R}^n$, we have $x \in F_{k,j+1}$, and therefore $|g_{k,j}(y) - f_{k,j}(y)| \le c_{k,j+1}\|y-x\|^2$ and hence $g_{k,j}(x) = f_{k,j}(x)$ and

$$D^+ g_{k,j}(x;y) - D_+ g_{k,j}(x;y) = D^+ f_{k,j}(x;y) - D_+ f_{k,j}(x;y) \ge 2 \sup_{e\in\mathcal{N}_x} \langle e,y\rangle. \tag{3.2}$$

Summarising, $g_{k,j}$ is differentiable at every $x \notin H_{k,j}$ and is not differentiable at any $x \in H_{k,j}$, moreover, it satisfies (3.2) at such points x.

We let $f := \sum_{(s,t)} c_{s,t} g_{s,t}$ and $h_{k,j} := \sum_{(s,t)\ne(k,j)} c_{s,t} g_{s,t}$. Since for any (s,t), if $x \notin H_{s,t}$, then the function $g_{s,t}$ is differentiable at x, and since we have $\sum_{(s,t)} \mathrm{Lip}(c_{s,t} g_{s,t}) < \infty$, we infer that f is differentiable at any $x \notin \bigcup_{(s,t)} H_{s,t} = E$ and $h_{k,j}$ is differentiable at any $x \in H_{k,j} \cup (\mathbb{R}^n \setminus E)$.

Let $x \in E_k$ and find j such that $x \in H_{k,j}$. Then for every $y \in \mathbb{R}^n$, $D^+ g_{k,j}(x;y) - D_+ g_{k,j}(x;y) \ge 2\sup_{e\in\mathcal{N}_x} \langle e,y\rangle$ by (3.2), and so, since $f = c_{k,j} g_{k,j} + h_{k,j}$ and $h_{k,j}$ is differentiable at x, we conclude that

$$D^+ f(x;y) - D_+ f(x;y) \ge 2c_{k,j} \sup_{e\in\mathcal{N}_x} \langle e,y\rangle. \qquad \blacksquare$$

Proof of Corollary 1.11. We are given a set $E \subset \mathbb{R}^n$ that is a countable union of (not necessarily disjoint) cone unrectifiable F_σ sets. Since each of these F_σ sets is a countable union of closed cone unrectifiable sets, we can write $E = \bigcup_{k=1}^\infty F_k$ where F_k are closed and cone unrectifiable. Hence $E = \bigcup_{k=1}^\infty E_k$ where $E_k := F_k \setminus \bigcup_{j<k} F_j$ are disjoint cone unrectifiable F_σ sets, and it suffices to take the function f obtained from Corollary 1.10 used with these sets E_k. $\blacksquare$

Proof of Corollary 1.12. We are given a Radon measure μ on $\mathbb{R}^n$ and a μ-measurable map $T : \mathbb{R}^n \to \bigcup_{m=0}^n G(n,m)$ such that for every unit vector e and $\alpha \in (0,1)$, the set $\{x : C_{e,\alpha} \cap T(x) = \{0\}\}$, where $C_{e,\alpha} := \{u : |\langle u, e\rangle| \geq \alpha \|u\|\}$, is the union of a μ-null set and a set E with $w_{e,\alpha}(E) = 0$. We show that there are cone unrectifiable F_σ sets E_k such that $\mu(\mathbb{R}^n \setminus \bigcup_k E_k) = 0$ and $T(x)^\perp \subset \mathcal{N}(x, E_k)$ for every $x \in E_k$. Then the function f from Corollary 1.10 will have all the required properties.

By Lusin's Theorem, μ-almost all of $\mathbb{R}^n$ is covered by the union of disjoint closed sets F_k such that for each k, the restriction of T to F_k is continuous. For every rational $\alpha \in (0,1)$ and u from a countable dense subset Q of the unit sphere in $\mathbb{R}^n$ write $\{x : C_{u,\alpha} \cap T(x) = \{0\}\} = Z_{u,\alpha} \cup E_{u,\alpha}$, where $\mu(Z_{u,\alpha}) = 0$ and $w_{u,\alpha}(E_{u,\alpha}) = 0$. Letting E_k be F_σ subsets of $F_k \setminus \bigcup_{u,\alpha} Z_{u,\alpha}$ satisfying $\mu(F_k \setminus E_k) = 0$, we just need to show that $T(x)^\perp \subset \mathcal{N}(x, E_k)$ for $x \in E_k$. For this, assume $x \in E_k$, $e \in T(x)^\perp$ and $\varepsilon \in (0,1)$, and choose $u \in Q$ and rational $\alpha \in (0,1)$ so that $C_{e,\varepsilon} \subset C_{u,\alpha}$ and $C_{u,\alpha} \cap T(x) = \{0\}$. By continuity of T on F_k, there is $r > 0$ such that $C_{u,\alpha} \cap T(y) = \{0\}$ for every $y \in B(x,r) \cap F_k$. Hence $B(x,r) \cap E_k \subset E_{u,\alpha}$ and $w_{e,\varepsilon}(B(x,r) \cap E_k) \leq w_{u,\alpha}(E_{u,\alpha}) = 0$. ∎

Proof of Theorem 1.13. Let E be the given uniformly purely unrectifiable set. Pick a sequence e_k dense in the unit ball of $\mathbb{R}^n$ such that $\|e_k\| \leq 1 - 2^{-k}$.

Let $f_0 = 0$, $H_0 = \mathbb{R}^n$, $\omega_0 = 1$ and $\eta_k = 2^{-k-1}$. When f_{k-1}, H_{k-1} and ω_{k-1} have been defined, we use Lemma 2.9 to find f_k, H_k and $\omega_k := \xi$ such that

(a) $E \subset H_k \subset H_{k-1}$, $\omega_k \in C(\mathbb{R}^n, [0,\infty)) \cap C(U, (0,\infty))$ and $\omega_k \leq \frac{1}{2}\omega_{k-1}$;
(b) $|f_k - f_{k-1}| \leq \omega_{k-1}$, $\mathrm{Lip}(f_k) \leq \max(\mathrm{Lip}(f_{k-1}), \|e_k\|) + \eta_k$ and $f_k \in C^1(H_k)$;
(c) if $x \in E$ and $h : \mathbb{R}^n \to \mathbb{R}$ satisfies $|h - f_k| \leq 2\omega_k$, there is $0 < r < \omega_{k-1}(x)$ such that $\sup_{\|y\|\leq r} |h(x+y) - h(x) - \langle e_k, y\rangle| \leq \eta_k r$.

Notice that $\omega_0 = 1$ and the last inequality in (a) imply $\omega_j \leq 2^{j-k}\omega_k$ and $\omega_k \leq 2^{-k}$ for $j \geq k \geq 0$. From (b) we see by induction that $\mathrm{Lip}(f_k) \leq 1 - 2^{-k-1}$. Hence the inequality $|f_k - f_{k-1}| \leq \omega_{k-1} \leq 2^{-k+1}$ implies that f_k converge to some $f : \mathbb{R}^n \to \mathbb{R}$ with $\mathrm{Lip}(f) \leq 1$.

Given any $x \in E$, $e \in \mathbb{R}^n$ with $\|e\| \leq 1$, and $\varepsilon > 0$, there are arbitrarily large k such that $\|e_k - e\| < \varepsilon$ and $\eta_k < \varepsilon$. Inferring from (b) that $|f - f_k| \leq \sum_{j=k}^\infty \omega_j \leq \sum_{j=k}^\infty 2^{j-k}\omega_k \leq 2\omega_k$, we use (c) to find $0 < r < \omega_{k-1}(x) \leq 2^{-k+1}$ such that $\sup_{\|y\|\leq r} |f(x+y) - f(x) - \langle e_k, y\rangle| \leq \eta_k r < \varepsilon r$. Since $\|e_k - e\| < \varepsilon$, we conclude that $\sup_{\|y\|\leq r} |f(x+y) - f(x) - \langle e, y\rangle| < 2\varepsilon r$. As $\varepsilon > 0$ is arbitrary and k may be arbitrarily large,

$$\liminf_{r\searrow 0} \sup_{\|y\|\le r} \frac{|f(x+y)-f(x)-\langle e,y\rangle|}{r} = 0,$$

which is the statement (1.5) of the Theorem. The estimate of upper and lower derivatives follows by using this with $e = y/\|y\|$ and $e = -y/\|y\|$ to get $D^+f(x;y) \ge \|y\|$ and $D_+f(x;y) \le -\|y\|$, respectively. ∎

4. Examples

The argument behind our first example has already been used many times, starting with [22], to find points of differentiability or almost differentiability of Lipschitz functions. See, e.g., [10, 13] or [4, Example 4.7] for an example showing that in Corollary 1.12 the constant $c = c(x)$ cannot be bounded away from zero.

Example 4.1: *There is a compact set $E \subset \mathbb{R}^2$ and a continuous mapping $x \in E \to e_x \in \{e \in \mathbb{R}^2 : \|e\| = 1\}$ such that $\mathcal{N}(E,x) = \{te_x : t \in \mathbb{R}\}$ for every $x \in E$ and whenever $f : \mathbb{R}^2 \to \mathbb{R}$ has $\mathrm{Lip}(f) \le 1$, there is $x \in E$ such that $\overline{D}f(x,e_x) < 1$. Consequently, in Theorem 1.9 we cannot take $\varepsilon = 0$.*

Proof. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a C^1 function such that $\varphi(-1) = \varphi(1) = 0$, $\varphi'(-1) = \varphi'(1) = 0$ and $\varphi(s) > 0$ for $s \ne \pm 1$. Denote $\varphi_0 = 0$ and $\varphi_k = \varphi/k$, and let

$$E := \{(s,\varphi_k(s)) : s \in [-1,1],\ k = 0, \pm 1, \pm 2, \dots\}.$$

For $x \in E$, $x = (s,\varphi_k(s))$ let u_x and e_x denote the unit vectors in the directions of $(1,\varphi_k'(s))$ and $(-\varphi_k'(s),1)$, respectively. Then $e_x \in \mathcal{N}(E,x)$ and, since $\varphi'(-1) = \varphi'(1) = 0$, the map $x \in E \to e_x$ is continuous.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ has $\mathrm{Lip}(f) \le 1$. Consider any $x \in E \setminus \{(-1,0),(1,0)\}$ such that $a := f'(x,u_x)$ exists and $\overline{D}f(x;e_x) = 1$. Then

$$\begin{aligned}\limsup_{t\to 0} & \frac{f(x+te_x)-f(x-atu_x)}{t} \\ &= \limsup_{t\to 0} \frac{f(x+te_x)-f(x)}{t} - \lim_{t\to 0} \frac{f(x-atu_x)-f(x)}{t} = 1 + a^2.\end{aligned}$$

Hence

$$1 \ge \mathrm{Lip}(f) \ge \limsup_{t\to 0+} \frac{|f(x+te_x)-f(x-atu_x)|}{\|(x+te_x)-(x-atu_x)\|} = \frac{a^2+1}{\sqrt{a^2+1}},$$

which gives $f'(x,u_x) = 0$.

If f is a function satisfying the conclusion of Theorem 1.9 with $\varepsilon = 0$, then for every k, $x = (s, \varphi_k(s))$ satisfies the above assumptions for a.e. $s \in (-1,1)$. Since f is Lipschitz, we infer that $s \to f(s, \varphi_k(s))$ is constant on $[-1,1]$, and hence f is constant on E. Consequently, when $s \in (-1,1)$ and $x = (s, \varphi_0(s))$, $e_x = (0,1)$ and so $\lim_{t\to 0} |(f(x+te_x) - f(x))/t| \le \lim_{t\to 0} \operatorname{dist}(x+te_x, E)/|t| = 0$, as $\operatorname{dist}(x+te_x, E)/|t| \le (k+1)/\big(2k(k+1)\big) = 1/(2k)$ when $|t|$ is between $\varphi(x)/(k+1)$ and $\varphi(x)/k$. This contradicts $\overline{D} f(x; e_x) = 1$. ∎

Our second example is related to Zahorski's description of non-differentiability sets of real-valued functions of a real variable which was already mentioned in the introductory remarks to Corollaries 1.10 and 1.11. Recall first that the set of points of non-differentiability of any real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is easily seen to be of the type $G_{\delta\sigma}$: just write it as

$$\bigcup_{\varepsilon>0} \bigcap_{e\in\mathbb{R}^n} \{x : (\exists r > 0)(\exists u, v \in B(x,r)) |f(x+u) - f(x+v) - \langle e, u-v\rangle| > \varepsilon r\}$$

where ε runs over positive rational numbers and e over elements of a dense countable subset of $\mathbb{R}^n$. The main argument in Zahorski's [27] proof of the converse when $n = 1$ (both in the general and in the Lipschitz case) constructs, for a given G_δ Lebesgue null set $E \subset \mathbb{R}$, a function $f : \mathbb{R} \to \mathbb{R}$ with $\operatorname{Lip}(f) = 1$ which is differentiable at every point of $\mathbb{R} \setminus E$ and at every point of E has upper derivative 1 and lower derivative -1. (For a more modern treatment of this construction see [15].)

While it is not clear what an exact analogy of Zahorski's result for $n > 1$ should be, one may at least hope that its analogy holds for uniformly purely unrectifiable sets, namely that for every uniformly purely unrectifiable $G_{\delta\sigma}$ set $E \subset \mathbb{R}^n$ there is a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ such that E is precisely the set of points at which f in non-differentiable in any direction. We do not know whether this is true or not, but the following example shows that in this situation the argument based on uniform discrepancy between upper and lower derivatives fails in a very strong sense. Recalling that every uniformly purely unrectifiable set is contained in a G_δ uniformly purely unrectifiable set, the example provides a G_δ uniformly purely unrectifiable set such that not only for it, but even for any bigger G_δ uniformly purely unrectifiable set there is no function analogous to the one from Zahorski's main argument.

Example 4.2: *There is a uniformly purely unrectifiable set $A \subset \mathbb{R}^2$ such that for any set $E \supset A$ and any $c > 0$ there is no Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$ such that*

(a) *$D^+ f(x; y) - D_+ f(x; y) \ge c\|y\|$ for every $x \in E$ and $y \in \mathbb{R}^2$;*
(b) *f is differentiable at every point x of $\mathbb{R}^2 \setminus E$.*

Proof. By [9] there is a universal differentiability set $D \subset \mathbb{R}^2$, i.e., a set such that every real-valued Lipschitz function on $\mathbb{R}^2$ has a point of differentiability belonging to D, such that there is a Lipschitz $h : \mathbb{R}^2 \to \mathbb{R}$ for which the set A of points $x \in D$ such that h is differentiable at x, is uniformly purely unrectifiable. Suppose $E \supset A$ and Lipschitz $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy (a) and (b). For a small $\varepsilon \in \big(0, c/(4\mathrm{Lip}(h))\big)$ consider the function $g := f + \varepsilon h$. If $x \in E$, (a) shows that for some $y \in \mathbb{R}$, $D^+ g(x; y) - D_+ g(x; y) \ge (c - 2\varepsilon \mathrm{Lip}(h))\|y\| > 0$. If $x \in D \setminus E$, g is the sum of the function f that is differentiable at x and of the function εh that is non-differentiable at x; hence it is non-differentiable at x. Consequently, the Lipschitz function g has no point of differentiability at D, contradicting that D is a universal differentiability set. ∎

Remark 4.3: The reason for considering a uniform non-differentiability condition such as (a) was explained in the text before the Example. Notice that, if (a) were replaced just by non-differentiability of f at every point of E, the statement of the Example would be false: we would use Theorem 1.13 to find a function g that is non-differentiable at every point of A and define E as the non-differentiability set of g. On the other hand, it is easy to find uniformly purely unrectifiable sets $E \supset A$ for which there is no Lipschitz function non-differentiable exactly at points of E, as such E need not be $G_{\delta\sigma}$. For the set A from [9] which was used in the proof of the Example 4.2 we can take $E = A$ as it is not difficult to see that A is not $G_{\delta\sigma}$, although it is $F_{\sigma\delta}$ since A is the intersection of D with the set of points of differentiability of h and D used in [9] is G_δ. It may be of interest to notice that the fact that A is not a non-differentiability set of any Lipschitz function f may be seen directly from the properties of A, D and h: for any such f the Lipschitz function $f + h$ would be non-differentiable at any $x \in D \setminus A$ as f is differentiable and h is not differentiable at such x; and $f + h$ would be non-differentiable at any $x \in A$ as f is not differentiable and h is differentiable at such x. As in the proof of the Example 4.2, this a contradiction as D is a universal differentiability set.

Our final example is related to the already pointed out fact that E is uniformly purely unrectifiable if and only if there is $0 < \eta < 1$ such that $w_{e,\eta}(E) = 0$ for every unit vector e. When considering general non-differentiability sets, a natural analogy of this statement would say that for any set $E \subset \mathbb{R}^n$ satisfying $w_{e,\eta}(E) = 0$ for some unit vector e and some $0 < \eta < 1$ there is a real-valued Lipschitz function f on $\mathbb{R}^n$ that is non-differentiable at any point of E. We show here that this is false; recall however that [3] shows (directly, not using [8]) that for any such set E there is an $\mathbb{R}^n$-valued Lipschitz function f on $\mathbb{R}^n$ that is non-differentiable at any point of E.

Example 4.4: *For every $\eta \in (0,1)$ and a unit vector $e \in \mathbb{R}^2$ there is a universal differentiability set $E \subset \mathbb{R}^2$ such that $w_{e,\eta}(E) = 0$.*

Proof. Let L_j be an enumeration of all rational lines in $\mathbb{R}^n$, J be the set of those indexes j for which the direction u of L_j satisfies $|\langle u, e\rangle| < \frac{1}{2}\eta$ and $\varepsilon_{i,j} > 0$ be such that $\sum_{i,j} \varepsilon_{i,j} < \infty$. It is easy to see that $E := \bigcap_i \bigcup_{j\in J}\{x : \operatorname{dist}(x, L_j) < \varepsilon_{i,j}\}$ satisfies $w_{e,\eta}(E) = 0$. The fact that E is a universal differentiability set has been often mentioned, but does not seem to be documented in the literature. We therefore explain the argument.

Recall from [12], [13] or [22] that, given any Lipschitz $g : \mathbb{R}^n \to \mathbb{R}$, a procedure leading to a point of differentiability of g may be described as follows. One starts with an arbitrary $\delta_0 > 0$ and (x_0, e_0) from the set D of pairs (x, u) where $x \in \mathbb{R}^n$, u is a unit vector, and there is $j = j(x, u)$ such that $x \in L_j$ and u is the direction of L. Recursively, when (x_k, e_k) has been defined, one first chooses an arbitrarily small $\delta_{k+1} > 0$ and then $(x_{k+1}, e_{k+1}) \in D$ satisfying rather delicate conditions about which we need to know only that $x_{k+1} \in B(x_k, \delta_{k+1})$, $Dg(x_{k+1}, e_{k+1}) \geq Dg(x_k, e_k)$ and that they imply that the sequence x_k converges to a point of differentiability of g.

Returning to our set E, given any Lipschitz $f : \mathbb{R}^n \to \mathbb{R}$, choose $(x_0, e_0) \in D$ so that $|\langle e_0, e\rangle| < \frac{1}{4}\eta$ and let $g(x) := f(x) + c\langle x, e_0\rangle$ with $c > 64\mathrm{Lip}(f)/\eta^2$; the choice of such large c guarantees that $Dg(x, u) \geq Dg(x_0; e_0)$ implies $0 \leq 1 - \langle u, e_0\rangle \leq \frac{1}{c}(Df(x; u) - Df(x_0; e_0)) \leq 2\mathrm{Lip}(f)/c \leq \frac{1}{32}\eta^2$, so that $\|u - e_0\| \leq \frac{1}{4}\eta$, hence $|\langle u, e\rangle| \leq \|u - e_0\| + |\langle e_0, e\rangle| < \frac{1}{2}\eta$. This will imply that in the recursive construction $j_k := j(x_k, e_k) \in J$, and so we can choose δ_{k+1} such that $\overline{B}(x_k, \delta_{k+1}) \subset B(L_{j_k}, \varepsilon_{k,j_k}) \cap B(x_k, \delta_k)$. Hence the limit of the x_k, which is a differentiability point of g and so of f, belongs to E. ∎

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