Frozen colourings of bounded degree graphs

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Abstract

Let G be a graph with maximum degree Δ and k be an integer. The k-recolouring graph of G is the graph whose vertices are proper k-colourings of G and where two colourings are adjacent iff they differ on exactly one vertex. Feghali, Johnson and Paulusma showed that the $(\Delta + 1)$-recolouring graph is composed by a unique connected component and (possibly many) isolated vertices, also known as frozen colourings of G.

Motivated by its applications to sampling, we study the proportion of frozen colourings of connected graphs. Our main result is that the probability a proper colouring is frozen is exponentially small on the order of the graph. The obtained bound is tight up to a logarithmic factor on Δ in the exponent. We briefly discuss the implications of our result on the study of the Glauber dynamics on $(\Delta+1)$-colourings. Additionally, we show that frozen colourings may exist even for graphs with arbitrary large girth. Finally, we show that typical Δ-regular graphs have no frozen colourings.

Keywords: Graph colourings, Recolouring graph, Random colourings, Glauber dynamics.

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1 Introduction

We denote by $G = (V, E)$ a graph on the set of vertices $[n] = \{1, \ldots, n\}$ and by $\Delta = \Delta(G)$ its maximum degree. A *proper k-colouring* of G is a function $\sigma : V(G) \to [k]$ such that, for every edge $xy \in E$, we have $\sigma(x) \neq \sigma(y)$. Throughout this paper, all the colourings will be proper. The *chromatic number* $\chi(G)$ of a graph G is the smallest integer k such that G admits a k-colouring.

The recolouring framework consists in finding step-by-step transformations between two proper colourings such that all intermediate states are also proper colourings. The *k-recolouring graph of G*, denoted by $\mathcal{C}_k(G)$ and defined for any $k \geq \chi(G)$, is the graph whose vertices are k-colourings of G and where two k-colourings are *adjacent* if and only if they differ on exactly one vertex.

In the last few years, the structural properties of reconfiguration graphs have received quite a lot of attention. Other frameworks in which reconfiguration problems have also been studied are boolean satisfiability [2,7], independent sets [8], dominating sets [1,12], list edge-colouring, Kempe chains [5] and many others (see [13] for a survey).

2 Glauber dynamics and frozen colourings

The study of the recolouring graph was initially motivated by problems arising from statistical physics in the context of Glauber dynamics. Glauber dynamics is a Markov chain over configurations of spin systems of graphs. This is a very general framework and "k-colourings of a graph" is a particular case known as the "antiferromagnetic Potts Model at zero temperature". For every graph

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$G = (V, E)$ with $V = [n]$ and $k \in \mathbb{N}$, the *Glauber dynamics* for k-colourings of G is a chain with state space the set of k-colourings of G. If X_t is a k-colouring of G, then the chain transitions are defined as

1) choose $v \in [n]$ and $c \in [k]$ uniformly at random;
2) for every $u \neq v$, set $X_{t+1}(u) = X_t(u)$;
3) if $c \notin X_t(N(v))$, then set $X_{t+1}(v) = c$; otherwise set $X_{t+1}(v) = X_t(v)$.

The Glauber dynamics for k-colourings is ergodic if and only if $\mathcal{C}_k(G)$ is connected. It is easy to check that $\mathcal{C}_k(G)$ is connected for every $k \geq \Delta + 2$. If the chain is ergodic, then the stationary distribution is uniform over the set of k-colourings of G. In this case, the corresponding Markov Chain Monte Carlo (MCMC) algorithm gives a valid method to sample k-colourings of G.

The mixing time of an ergodic Markov chain is the number of steps needed to be "close" to its stationary distribution. A chain is rapidly mixing if its mixing time is polylogarithmic on its size. As the number of colourings of G is typically exponential on n, Glauber dynamics is rapidly mixing if its mixing time is polynomial in n. In such a case, it provides a Fully Polynomial Randomized Approximation Scheme (FPRAS) to approximately compute the number of k-colourings of G. For a survey on that topic, we refer the interested reader to [10].

A well-known conjecture in the area is that Glauber dynamics mixes in $O(n \log n)$ steps provided that $k \geq \Delta + 2$. Jerrum [9] and independently Salas and Sokal [11], showed that the Glauber dynamics for k-colourings is rapidly mixing for every $k \geq 2\Delta$. This result was improved by Vigoda [14] to every $k \geq 11\Delta/6$ using a path coupling argument on the flip dynamics, a Markov chain on k-colourings based on Kempe recolourings. See [6] for a survey on the topic.

For $k = \Delta + 1$, the chain is non-necessarily ergodic on Δ-regular graphs (e.g. on cliques). An obvious obstruction is the existence of frozen $(\Delta + 1)$-colourings; that is, colourings where every colour appears in the closed neighbourhood of every vertex. Frozen colourings correspond to fixed points in the dynamics, and thus, their existence makes the chain non-ergodic. Feghali et al. [5] showed that if $k = \Delta+1$ and $\Delta \geq 3$, frozen colourings are actually the only obstruction for the ergodicity of the chain. That is, the recolouring graph is composed by a unique component containing all non-frozen colourings and (possibly many) frozen colourings.

This raises the following natural questions: (1) what is the size of the non-frozen component and, (2) given it is non-empty, does the Glauber dynamics rapidly mixes there.

3 Results of the paper

The main result of this paper is that frozen colourings are rare among proper colourings.

Theorem 3.1 *Let $\Delta, n \in \mathbb{N}$, with $3 \leq \Delta \leq n-2$ and let G be a connected graph on n vertices and maximum degree Δ. Let σ be a colouring chosen uniformly at random among all proper $(\Delta+1)$-colourings of G. Then*

$$\mathbb{P}(\sigma \text{ is frozen}) \leq (6/7)^{\frac{n}{\Delta+1}} .$$

The proof of this theorem is based on a careful counting of the frozen and non-frozen extensions of partial colourings of G. Note that the condition $\Delta \leq n-2$ is necessary since every $(\Delta+1)$-colouring of a $(\Delta+1)$-clique is frozen.

In the light of the result of Feghali et al. [5] and as $n \to \infty$, Theorem 3.1 implies that the size of the non-frozen component of $\mathcal{C}_{\Delta+1}(G)$ is always exponentially larger than the number of frozen $(\Delta+1)$-colourings. Thus, the Glauber dynamics for $(\Delta+1)$-colourings is extremely likely to start in the non-frozen component, and while non-ergodic, the chain will converge to the uniform distribution on it. This motivates the question of whether the $O(n \log n)$ mixing time conjecture can be extended to the Glauber dynamics for non-frozen $(\Delta+1)$-colourings. Next result answers it in the negative.

Proposition 3.2 *Let $\Delta, k \in \mathbb{N}$ with $\Delta \geq 3$ and $k \geq 5$. Then, there exists a Δ-regular graph G on $n = 2k(\Delta+1)$ vertices that satisfies the following. Let H denote the graph induced by the non-frozen $(\Delta+1)$-colourings of G in $\mathcal{C}_{\Delta+1}(G)$. Then, the lazy random walk on H converges to the uniform distribution on $V(H)$ and*

$$t_{mix}(H) \geq \frac{n^2}{8(\Delta+1)} .$$

Analogously, the proposition states that the Glauber dynamics on the set of non-frozen $(\Delta+1)$-colourings of G mixes in time $\Omega(n^2)$, in contrast to the conjecture for $k \geq \Delta+2$. Nevertheless, it seems reasonable to believe that the chain will mix in polynomial time .

Our result assumes that $\Delta \geq 3$. For $\Delta = 2$, Dyer, Goldberg and Jerrum [4] proved that the Glauber dynamics for 3-colourings of a path P_n of length n has mixing time $\Theta(n^3 \log n)$. One can check that the recolouring graph $\mathcal{C}_3(P_n)$ has diameter $\Omega(n^2)$. Up to our knowledge, for $\Delta \geq 3$, no Δ-regular graph

G is known whose diameter of the non-frozen component in $\mathcal{C}_{\Delta+1}(G)$ has superlinear order.

We believe that the dependence on Δ in Theorem 3.1 is not best possible. However, we can construct graphs where the probability of being frozen is only a factor $\log \Delta$ in the exponent off from our upper bound.

Proposition 3.3 *For every $n_0, \Delta \geq 3$ there exist $n \geq n_0$ and a connected Δ-regular graph G on n vertices such that if σ is chosen uniformly at random among all proper $(\Delta+1)$-colourings of G, then*

$$\mathbb{P}(\sigma \text{ is frozen}) \geq e^{-\frac{3\log(\Delta)}{\Delta}\cdot n} .$$

The graph in Proposition 3.3 contains many cliques of size Δ. Our next result shows the existence of locally sparse Δ-regular graphs with frozen colourings.

Proposition 3.4 *For every $\Delta, g \geq 3$ there exists a connected Δ-regular graph G with girth at least g that has frozen colourings.*

To prove this proposition we use a randomised construction based on random lifts of the $(\Delta+1)$-clique.

We conclude the paper by showing that typical Δ-regular graphs do not admit frozen colourings.

Proposition 3.5 *Let $n, \Delta \geq 3$ and let $G_{n,\Delta}$ be a graph chosen uniformly at random among all simple Δ-regular graphs with vertex set $[n]$. Then there exists $c(\Delta) > 0$ such that, as $n \to \infty$, we have*

$$\mathbb{P}(G_{n,\Delta} \text{ has a frozen colouring}) \leq e^{-c(\Delta)n} .$$

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