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Lo, Allan

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LONG PROPERLY COLOURED CYCLES IN EDGE-COLOURED GRAPHS.

ALLAN LO

*School of Mathematics, University of Birmingham,
Birmingham, B15 2TT, UK*

ABSTRACT. Let G be an edge-coloured graph. The minimum colour degree $\delta^c(G)$ of G is the largest integer k such that, for every vertex v , there are at least k distinct colours on edges incident to v . We say that G is properly coloured if no two adjacent edges have the same colour. In this paper, we show that, for any $\varepsilon > 0$ and n large, every edge-coloured graph G with $\delta^c(G) \geq (1/2 + \varepsilon)n$ contains a properly coloured cycle of length at least $\min\{n, \lfloor 2\delta^c(G)/3 \rfloor\}$.

1. INTRODUCTION

An *edge-coloured* graph is a graph G with an edge-colouring c of G . We say that G is *properly coloured* if no two adjacent edges of G have the same colour. If all edges have the same (or distinct) colour, then G is *monochromatic* (or *rainbow*, respectively).

Finding properly coloured subgraphs in edge-coloured graphs G has a long and rich history. Grossman and Häggkvist [10] are the first to give a sufficient condition on the existence of properly coloured cycles in edge-coloured graphs with two colours. Later on, Yeo [19] extended the result to edge-coloured graphs with any number of colours. A natural question is to ask what guarantees the existence of properly coloured Hamiltonian paths and cycles.

In particular, the case when G is an edge-coloured K_n has been receiving the most attention. Given $k \in \mathbb{N}$, an edge-coloured graph G is *locally k -bounded* if for all vertices $v \in V(G)$, no colour appears more than k times on the edges incident to v for all vertices v . A conjecture of Bollobás and Erdős [4] states that every locally $(\lfloor n/2 \rfloor - 1)$ -bounded edge-coloured K_n contains a properly coloured Hamilton cycle. There is a series of partial results toward this conjecture by Bollobás and Erdős [4], Chen and Daykin [6], Shearer [17], and Alon and Gutin [1]. In [15] the author showed that the conjecture of Bollobás and Erdős holds asymptotically, that is, for any $\varepsilon > 0$ and n sufficiently large, every locally $(1/2 - \varepsilon)n$ -bounded edge-coloured K_n contains a properly coloured Hamilton cycle. A hypergraph generalisation of finding properly coloured Hamilton cycle in locally k -bounded edge-coloured complete graphs has also been studied by Dudek, Frieze and Ruciński [8] as well as Dudek and Ferrara [7]. Recently, Sudakov and Volec [18] proved that every locally $n/(500r^{3/4})$ -bounded edge-coloured K_n contains all properly coloured graphs with at most r paths of length two. This proved a conjecture of Shearer [17] as well as improves results of

E-mail address: s.a.lo@bham.ac.uk.

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Alon, Jiang, Miller, Pritikin [2] and Böttcher, Kohayakawa and Procacci [5]. For a survey regarding properly coloured subgraphs in edge-coloured graphs, we recommend Chapter 16 of [3]. Also see [11] for a survey regarding monochromatic and rainbow subgraphs in edge-coloured graphs.

Consider an edge-coloured (not necessarily complete) graph G . Given a vertex $v \in V(G)$, the *colour degree* $d_G^c(v)$ is the number of distinct colours of edges incident to v . The *minimum colour degree* $\delta^c(G)$ is the minimum $d_G^c(v)$ over all vertices v in G . Li and Wang [12] showed that every edge-coloured graph G with $\delta^c(G) \geq d$ contains a properly coloured path of length $2d$ or a properly coloured cycle of length at least $2d/3$. In [13], the author improved $2d/3$ to $d+1$, which is best possible. In the same paper, the author conjectured the following.

Conjecture 1.1. *Every edge-coloured connected graph G with $\delta^c(G) \geq d$ contains a properly coloured Hamilton cycle or a properly coloured path of length $\lfloor 3d/2 \rfloor$.*

If this conjecture holds, then the bound is sharp by the following example. Let $d, n \in \mathbb{N}$ with $n \geq 3d/2$. Let c_1, c_2, \dots, c_d be distinct colours. Let X, Y be disjoint sets of vertices such that $X = \{x_1, x_2, \dots, x_d\}$ and $|Y| = n - d$. For each $1 \leq i \leq d$, join x_i to each vertex of Y with colour c_i . For $1 \leq i < j \leq d$, join x_i to x_j with a new distinct colour. Let G be the resulting edge-coloured graph. Note that G has n vertices and $\delta^c(G) = d$. Every properly coloured path in G with both endpoints in Y must contain at least two vertices in X . Thus, every properly coloured path in G is of length at most $|X| + \lfloor |X|/2 \rfloor = \lfloor 3d/2 \rfloor$.

In [14], the author proved that the conjecture holds when $d \geq (2/3 + \varepsilon)n$ for $\varepsilon > 0$ and n large, that is, every edge-coloured graph G on n vertices with $\delta^c(G) \geq (2/3 + \varepsilon)n$ contains a properly coloured Hamilton cycle.

In this paper, we prove the following results.

Theorem 1.2. *For $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that every edge-coloured graph G on $n \geq n_0$ vertices with $\delta^c(G) \geq (1/2 + \varepsilon)n$ contains a properly coloured cycle of length at least $\min\{\lfloor 3\delta^c(G)/2 \rfloor, n\}$.*

Note that Theorem 1.2 implies Conjecture 1.1 when $d \geq (1/2 + \varepsilon)n$ and n large. By analysing the proof of Theorem 1.2, one might be able to prove Conjecture 1.1 when $d \geq n/2$. Therefore, it would be interesting to know whether Conjecture 1.1 hold for $d < n/2$.

2. NOTATION AND SKETCH PROOF

For a graph G , we denote $V(G)$ and $E(G)$ for the vertex set and edge set of G , respectively. Write $|G|$ for $|V(G)|$. For (edge-coloured) graphs G and H , we write $G - H$ for the graph with vertex set $V(G)$ and edge set $E(G) \setminus E(H)$. For $W \subseteq V(G)$, we write $G \setminus W$ for the subgraph of G induced by the vertex set $V(G) \setminus W$, and write $G \setminus H$ for $G \setminus V(H)$. For disjoint $X, Y \subseteq V(G)$, let $G[X]$ be the (edge-coloured) subgraph induced by X and let $G[X, Y]$ be the induced bipartite subgraph with vertex classes X and Y . For a set of edges E , we write $G \cup E$ for the graph with vertex set $V(G) \cup V(E)$ and edge set $E(G) \cup E$. For a singleton set $\{v\}$, we sometimes write v for short.

For an edge-coloured graph G , let $C(G) := \{c(uv) : uv \in E(G)\}$, that is, the set of colours appeared in G . For a vertex $v \in V(G)$, let $C_G(v) := \{c(uv) : u \in N_G(v)\}$. Thus $d_G^c(v) = |C_G(v)|$. For $V \subseteq V(G)$, define $d_G^c(v, V) := |C_{G[V \cup v]}(v)|$. Let $\mathbf{x} = (x, c_x)$ be a pair with vertex $x \in V(G)$ and colour $c_x \in C_G(x)$. We write $N_G(\mathbf{x})$ be the set of vertices $v \in N_G(x)$ such that $c(xv) \neq c_x$. For distinct $x, y \in V(G)$, we denote by

$\text{dist}_G(x, y)$ the shortest distance between x and y . If x and y are not connected, then we say $\text{dist}_G(x, y) = \infty$. If G is known from the context, then we omit G in the subscript.

For a path $P = x_1x_2 \dots x_k$ from x_1 to x_k and a vertex $y \notin V(P)$, we write Py for the path $x_1x_2 \dots x_ky$. If $P' = y_1 \dots y_\ell$ is a path with $y_1 = x_k$ and $V(P) \cap V(P') = \{x_k\}$, then we write PP' for the concatenated path $x_1x_2 \dots x_ky_2 \dots y_\ell$.

An edge-coloured graph G is *critical*, if for every edge uv , $d_G^c(u) > d_{G-uv}^c(u)$ or $d_G^c(v) > d_{G-uv}^c(v)$. Note that if G is critical, then any monochromatic subgraph H of G is a union of vertex-disjoint stars. Since we are only concerning about properly coloured subgraphs, we may assume further that any two vertex-disjoint monochromatic component in G have distinct colours. Thus, from now on, we assume that every monochromatic subgraph H of any critical edge-coloured graph G is a star.

Let F be a directed graph. For $u, v \in V(F)$, we write uv for the directed edge from u to v . For $Z, W \subseteq V(F)$, denote by $e_F(Z, W)$ the number of directed edges from Z to W in F .

The constants in the hierarchies used to state our results are chosen from right to left. For example, if we claim that a result holds whenever $0 < 1/n \ll a \ll b \ll c \leq 1$ (where n is the order of the graph), then there is a non-decreasing function $f : (0, 1] \rightarrow (0, 1]$ such that the result holds for all $0 < a, b, c \leq 1$ and all $n \in \mathbb{N}$ with $b \leq f(c)$, $a \leq f(b)$ and $1/n \leq f(a)$. Hierarchies with more constants are defined in a similar way.

2.1. Sketch proof of Theorem 1.2. Here we present an outline of the proof of Theorem 1.2, which naturally splits into three lemmas. First, we consider the case when G is close to the extremal example in Section 3. More precisely, for $\delta, \varepsilon > 0$, we say that an edge-coloured graph G on n vertices is (δ, ε) -*extremal* if there exist disjoint $A, B \subseteq V(G)$ such that

- (A1) $|A| \geq (\delta - \varepsilon)n$ and $|B| \geq (1 - \delta - \varepsilon)n$;
- (A2) for each $a \in A$, there exists a distinct colour c_a such that there are at least $|B| - \varepsilon n$ vertices $b \in B$ such that $c(ab) = c_a$;
- (A3) for each $b \in B$, $d_G(b) \leq (\delta + \varepsilon)n$ and b has at least $|A| - \varepsilon n$ neighbours $a \in A$ such that $c(ab) = c_a$.

Throughout this paper, we will always assume that $\varepsilon \ll \delta$. In this case, we will find a properly coloured cycle (of the desired length) directly (see Section 3).

Lemma 2.1. *Let $0 < 1/n \ll \varepsilon \ll \delta \leq 1$. Let G be a (δ, ε) -extremal critical edge-coloured graph on n vertices with $\delta^c(G) \geq \delta n$. Then G contains a properly coloured cycle of length $\min\{\lfloor 3\delta n/2 \rfloor, n\}$.*

Note that Lemma 2.1 does not require that $\delta \geq 1/2 + \varepsilon$. Thus Lemma 2.1 implies that Conjecture 1.1 holds if G is (δ, ε) -extremal with $1/n \ll \varepsilon \ll \delta \leq 1$.

If G is not close to the extremal, then we proceed using the *absorption technique* introduced by Rödl, Ruciński and Szemerédi [16], which was used to tackle Hamiltonicity problems in hypergraphs. The absorption technique has been adapted for finding properly coloured Hamilton cycles in [14, 15]. First we find a small ‘absorbing cycle’ C in G using the following lemma, which is proved in Section 4.

Lemma 2.2. *Let $0 < 1/n \ll \gamma \ll \varepsilon < 1/2$. Suppose that G is an edge-coloured graph on n vertices with $\delta^c(G) \geq (1/2 + \varepsilon)n$. Then there exists a properly coloured cycle C of length at most $\varepsilon n/2$ such that for any collection P_1, \dots, P_k of vertex-disjoint properly coloured paths in $G \setminus V(C)$ with $k \leq \gamma n$, there exists a properly coloured cycle with vertex set $V(C) \cup \bigcup_{1 \leq i \leq k} V(P_i)$.*

Remove the vertices of C from G and call the resulting graph G' . Since G is not extremal, neither is G' . (Indeed, if G' is (δ, ε) -extremal with vertex subsets A, B , then G is $(\delta, 2\varepsilon)$ -extremal with vertex subsets A, B as $\varepsilon \ll 1$.) We find vertex-disjoint properly coloured paths by the next lemma (which is implied by Lemma 5.1).

Lemma 2.3. *Let $0 < 1/n \ll \beta \ll \varepsilon \ll 1/2 < \delta$. Suppose that G is a critical edge-coloured graph on n vertices with $\delta^c(G) \geq \delta n + 1$. If G is not (δ, ε) -extremal, then G contains vertex-disjoint properly coloured paths P_1, \dots, P_k with $k \leq 100\beta^{-1}$ covering $\min\{(3\delta + \beta)n/2, n\}$ vertices.*

We now prove Theorem 1.2 using Lemmas 2.1–2.3.

Proof of Theorem 1.2. Without loss of generality, we may assume that G is critical edge-coloured with $\delta^c(G) = \delta n$ and that ε is sufficiently small. Let γ, ε' be such that $1/n \ll \gamma \ll \varepsilon \ll \varepsilon' \ll 1/2 < \delta$.

Apply Lemma 2.2 and obtain a properly coloured cycle C of length at most $\varepsilon n/2$ such that for any collection P_1, \dots, P_k of vertex-disjoint properly coloured paths in $G \setminus V(C)$ with $k \leq \gamma n$, there exists a properly coloured cycle with vertex set $V(C) \cup \bigcup_{1 \leq i \leq k} V(P_i)$.

Let $G' := G \setminus C$, $n' := |G'|$ and $\delta' := (\delta n - |C| - 1)/n'$. Note that $\delta^c(G) \geq \delta' n' + 1$ and $1/n' \ll \varepsilon \ll \varepsilon' \ll 1/2 < \delta'$. If G' is not (δ', ε') -extremal, then apply Lemma 2.3 (with $\varepsilon, \varepsilon', \delta', n'$ playing the roles of $\beta, \varepsilon, \delta, n$) and obtain vertex-disjoint properly coloured paths P_1, \dots, P_k such that $k \leq 100\varepsilon^{-1} \leq \gamma n$ and

$$\bigcup_{i \leq k} |V(P'_i)| \geq \min\{3(\delta - |C| - 1)n + \varepsilon n'\}/2, n - |C|\} \geq \min\{3\delta n/2, n\} - |C|$$

as $|C| \leq \varepsilon n/2 \leq \varepsilon n'$. Thus, by the property of C , there exists a properly coloured cycle C' with vertex set $V(C) \cup \bigcup_{i \leq k} V(P'_i)$. So $|C'| \geq \min\{3\delta n/2, n\}$ as desired.

On the other hand, if G' is (δ', ε') -extremal, then there exist disjoint $A, B \subseteq V(G') = V(G) \setminus V(C)$ satisfying

- (A1) $|A| \geq (\delta' - \varepsilon')n' \geq (\delta - 2\varepsilon')n$ and $|B| \geq (1 - \delta' - \varepsilon')n' \geq (1 - \delta - 2\varepsilon')n$;
- (A2) for each $a \in A$, there exists a colour c_a such that there are at least $|B| - \varepsilon'n' \geq |B| - 2\varepsilon'n$ vertices $b \in B$ such that $c(ab) = c_a$;
- (A3) for each $b \in B$,

$$d_G(b) \leq d_{G'}(b) + |C| \leq (\delta' + \varepsilon')n' + |C| = \delta n - 1 + \varepsilon'n' < (\delta + 2\varepsilon')n$$

and b has at least $|A| - \varepsilon'n' \geq |A| - 2\varepsilon'n$ neighbours $a \in A$ such that $c(ab) = c_a$.

Therefore G is $(\delta, 2\varepsilon')$ -extremal. By Lemma 2.1, G contains a properly coloured cycles of length at least $\min\{\lfloor 3\delta n/2 \rfloor, n\}$. \square

3. EXTREMAL CASE

In this section, we prove Lemma 2.1, that is, Theorem 1.2 when G is critical and (δ, ε) -extremal. We would need the following definition. Let G be an edge-coloured graph on n vertices. Let $A, B \subseteq V(G)$ be disjoint. We say that the ordered pair (A, B) is ε -extremal if the following holds:

- (E1) for each $a \in A$, there exists a distinct colour c_a ;
- (E2) for each $a \in A$, there are at least $|B| - \varepsilon n$ vertices $b \in B \cap N(a)$ such that $c(ab) = c_a$, and at least $|A| - \varepsilon n$ vertices $a' \in A \cap N(a)$ such that $c_a \neq c(aa') \neq c_{a'}$;
- (E3) for each $b \in B$, there are at least $|A| - \varepsilon n$ vertices $a \in A \cap N(b)$ such that $c(ab) = c_a$.

Next we show that if G is (δ, ε) -extremal, then there exists $4\sqrt{\varepsilon}$ -extremal pair in G .

Lemma 3.1. *Let $0 < 1/n \ll \varepsilon \ll 1$ and let $\delta > 4\sqrt{\varepsilon}$. Let G be a critical edge-coloured graph on n vertices with $\delta^c(G) \geq \delta n$. Suppose that G is (δ, ε) -extremal. Then there exist disjoint $A, B \subseteq V(G)$ such that (A, B) is $4\sqrt{\varepsilon}$ -extremal, $|A| \geq (\delta - 4\sqrt{\varepsilon})n$, $|B| \geq (1 - \delta - \varepsilon)n$ and, for each $b \in B$, $d_G(b) \leq (\delta + \varepsilon)n$.*

Proof. Let $\varepsilon' := 4\sqrt{\varepsilon}$. Since G is (δ, ε) -extremal, there exist disjoint $A^*, B^* \subseteq V(G)$ satisfying (A1)–(A3).

Note that $|V(G) \setminus (A^* \cup B^*)| \leq 2\varepsilon n$. We say that an edge aa' in $G[A^*]$ is *good* if $c_a \neq c(aa') \neq c_{a'}$. We bound the number of good edges from below as follows. Define a directed graph D on A^* such that there is a directed edge from a to a' if and only if $c_a \neq c(aa')$. For each $a \in A^*$, a sends at most $1 + \varepsilon n + |V(G) \setminus (A^* \cup B^*)| \leq 3\varepsilon n + 1$ distinct colours (including the colour c_a) to $V(G) \setminus A^*$ by (A2). So the outdegree of a in D is at least $\delta n - 3\varepsilon n - 1 \geq |A^*| - 5\varepsilon n - 1$. Since the number of good edges equals the number of 2-cycles in D , the number of good edges is at least $(|A^*| - 5\varepsilon n - 1)|A^*| - \binom{|A^*|}{2} = |A^*|(|A^*| - 10\varepsilon n - 1)/2$. Let A' be the set of $a \in A^*$ that is incident with at most $|A^*| - \varepsilon' n$ good edges. Note that $|A'| \leq 3\sqrt{\varepsilon}n$.

Let $A := A^* \setminus A'$. Thus $|A| \geq |A^*| - 3\sqrt{\varepsilon}n \geq (\delta - \varepsilon')n$ by (A1). Moreover, every $a \in A$ is incident with at least $|A| - \varepsilon' n$ good edges in $G[A]$ implying (E2). Set $B := B^*$. So $|B| \geq (1 - \delta - \varepsilon)n$. Also, (A3) implies that (E3) holds and that, for each $b \in B$, $d_G(b) \leq (\delta + \varepsilon)n$. Therefore (A, B) is ε' -extremal. \square

In the next two lemma, we find properly coloured cycles spanning $A \cup B$, when (A, B) is ε -extremal.

Lemma 3.2. *Let $\varepsilon < 1/36$. Let G be an edge-coloured graph on $3m$ vertices. Suppose that there is a partition A, B of $V(G)$ such that (A, B) is ε -extremal, $|A| = 2m$ and $|B| = m$. Then G has a properly coloured Hamilton cycle.*

Proof. Partition A into X and Y each of size m . Let H_X be the subgraph of $G[X, B]$ induced by edges of colour in $\{c_a : a \in A\}$. By (E2) and (E3), H_X is a bipartite graph with $\delta(H_X) \geq m - 3\varepsilon m$. Hence by Hall's theorem, there exists a perfect matching M_X in H_X .

Similarly, let H_Y be the subgraph of $G[Y, B]$ induced by edges of colour in $\{c_a : a \in A\}$ and there exists a perfect matching M_Y in H_Y . Note that $M_X \cup M_Y$ is a union of m vertex-disjoint path of length 2 each with midpoint in B . By (E1), $M_X \cup M_Y$ is properly coloured. Let $M_X \cup M_Y = \{x_i b_i y_i : x_i \in X, b_i \in B, y_i \in Y \text{ and } i \leq m\}$.

Now define an oriented graph F on vertex set $Z = \{z_1, \dots, z_m\}$ such that there is a directed edge from z_i to z_j if and only if $y_i x_j$ is an edge (in G) with $c_{y_i} \neq c(y_i x_j) \neq c_{x_j}$. By (E2), each z_i has indegree and outdegree at least $m - 3\varepsilon m \geq m/2$. Therefore F contains a directed Hamilton cycle by a result of Ghouila-Houri [9], $z_1 z_2 \dots z_m z_1$ say. Then $x_1 b_1 y_1 x_2 b_2 y_2 \dots z_m x_1$ is a properly coloured Hamilton cycle in G as desired. \square

Lemma 3.3. *Let $\ell \in \mathbb{N}$ and $0 < 1/n \ll \varepsilon \ll \alpha < 1/3$ with $\ell < \alpha n$. Let G be a critical edge-coloured graph on n vertices. Suppose that (A, B) is ε -extremal such that $\alpha n + \ell + 1 \leq |B| \leq |A|/2 + \ell$. Suppose that \mathcal{P} is a union of ℓ vertex-disjoint properly coloured paths such that each path has both of its endpoints in B and $|(A \cup B) \cap V(\mathcal{P})| = 2\ell$. Then G contains a properly coloured cycle with vertex set $V(C) = A \cup B \cup V(\mathcal{P})$.*

Proof. First suppose that $|B| < |A|/2 + \ell$. Let $p := |A| - 2(|B| - \ell - 1)$, so $3 \leq p \leq |A| - 2\alpha n$. By (E2) and a greedy argument, G contains a properly colour path $ba_1 a_2 \dots a_p b'$ such that

$a_1, \dots, a_p \in A$ and $b, b' \in B \setminus V(\mathcal{P})$. We add the path $ba_1a_2 \dots a_pb'$ to \mathcal{P} and call the resulting set \mathcal{P}' . Let $A' = A \setminus \{a_1, \dots, a_p\}$, so $|A'| = |A| - p = 2(|B| - \ell - 1)$. Furthermore (A', B) is ε -extremal. Therefore by replacing A, B, \mathcal{P} with A', B, \mathcal{P}' , we may assume that without loss of generality that $|A| = 2m$ and $|B| = m + \ell$ for some integer $m \geq \alpha n$ with $\ell \leq m$.

Consider $G[A \cup B] \cup \mathcal{P}$. Suppose that P_1, \dots, P_ℓ are the paths of \mathcal{P} . We now contract each P_i as follows. Let b_i and b'_i be the end vertices of P_i , so $b_i, b'_i \in B$. Let N_i be the common neighbours $a \in A$ of b_i and b'_i such that $c(ab_i) = c(ab'_i) = c_a \notin C_{P_i}(b_i) \cup C_{P_i}(b'_i)$. Note that $|N_i| \geq |A| - 2\varepsilon n - 2 \geq 2m - 3\varepsilon\alpha^{-1}m \geq 2m - 3\sqrt{\varepsilon}m$ by (E3). We replace each $V(P_i)$ with a new vertex x_i and join x_i to each vertices $a \in N_i$ with colour c_a . Call the resulting graph H . So $A \subseteq H$ and $|H| = 3m$. Note that, for each $i \leq \ell$, $d_H(x_i, A) = |N_i| \geq 2m - 3\sqrt{\varepsilon}m$. Since $V(H) \setminus A = B \setminus V(\mathcal{P}) \cup \{x_1, \dots, x_\ell\}$, it is easy to see that $(A, V(H) \setminus A)$ is $\sqrt{\varepsilon}$ -extremal in H . Lemma 3.2 implies that H has a properly coloured Hamiltonian cycle C . By replacing each x_i in C with P_i we obtain a properly coloured cycle in G with vertex set $A \cup B \cup V(\mathcal{P})$ as required. \square

By Lemmas 3.1 and 3.3, to prove Lemma 2.1 it suffices to find a union of suitable properly coloured paths. We would need a finer partition $V(G) \setminus (A \cup B)$ into Y and Z as follows. Let Y be the set of $v \in V(G) \setminus (A \cup B)$ such that $d_G^c(v, B) \geq 10\varepsilon n$ or $|\{c(av) : a \in N_G(v) \cap A \text{ and } c(av) \neq c_a\}| \geq 10\varepsilon n$. Let $Z := V(G) \setminus (A \cup B \cup Y)$.

Proposition 3.4. *Let $\varepsilon, \delta > 0$. Let G be a critical edge-coloured graph on n vertices with $\delta^c(G) \geq \delta n$. Suppose that (A, B) is ε -extremal such that $|A| \geq (\delta - \varepsilon)n$ and $|B| \geq (1 - \delta - \varepsilon)n$. Let Y, Z be a partition of $V(G) \setminus (A \cup B)$ as above. For each $v \in Z$, there are at least $|A| - 24\varepsilon n$ vertices $a \in N_G(v) \cap A$ such that $c(av) = c_a$. Moreover, $(A, B \cup Z)$ is 24ε -extremal.*

Proof. Note that $|Y| + |Z| \leq 2\varepsilon n$. Consider any $v \in Z$. Since $d_G^c(v, B) < 10\varepsilon n$, we have

$$d_G^c(v, A) \geq d_G^c(v) - d_G^c(v, B) - |Y| - |Z| \geq (\delta - 12\varepsilon)n \geq |A| - 14\varepsilon n.$$

On the other hand, $|\{c(av) : a \in N_G(v) \cap A \text{ and } c(av) \neq c_a\}| < 10\varepsilon n$. Thus there are at least $|A| - 24\varepsilon n$ vertices $a \in N_G(v) \cap A$ such that $c(av) = c_a$. \square

Instead of finding a union of suitable properly coloured paths, the next lemma shows that finding a suitable matching is sufficient.

Lemma 3.5. *Let $0 < 1/n \ll \varepsilon \ll \alpha < 1/3$. Let G be a critical edge-coloured graph on n vertices. Suppose that (A, B) is ε -extremal such that $|A| \geq (2\alpha + 6\varepsilon)n + 2$ and $|B| \geq (\alpha + 4\varepsilon)n + 1$. Let Y be the set of $v \in V(G) \setminus (A \cup B)$ such that $d_G^c(v, B) \geq 10\varepsilon n$ or $|\{c(av) : a \in N_G(v) \cap A \text{ and } c(av) \neq c_a\}| \geq 10\varepsilon n$. Let $Z := V(G) \setminus (A \cup B \cup Y)$. Suppose that M and M' are vertex-disjoint matchings such that*

- (i) *there are at most $2\varepsilon n$ edges in $M \cup M'$;*
- (ii) *$M \subseteq G \setminus A$;*
- (iii) *$M' \subseteq G[A, B \cup Z]$ and for each edges $av \in M'$ with $a \in A$, $c(av) \neq c_a$.*

Then G contains a properly coloured cycle C such that

$$|C| \geq \min \left\{ n, \left\lfloor \frac{3|A|}{2} + |M| + \frac{|M'|}{2} + |Y| - \frac{|V(M) \cap Y|}{2} \right\rfloor \right\}.$$

Proof. Note that $(A, B \cup Z)$ is 24ε -extremal by Proposition 3.4. Our aim is to extend $M \cup M'$ into a suitable path system \mathcal{P} (see Claim 3.6 for the precise properties) such that we can apply Lemma 3.3. The key features of \mathcal{P} are that every path is properly coloured

with both endpoints in $B \cup Z$ and that \mathcal{P} covers Y . Here, we give a rough outline on how to construct \mathcal{P} from $M \cup M'$ (that is, the proof of Claim 3.6). For simplicity, we assume that $M \subseteq G[B \cup Z]$ (so the edges of M can be already viewed as paths with both endpoints in $B \cup Z$). For each edge $av \in M'$ with $a \in A$, we add the edge ab with $b \in B$ such that $c(ab) = c_a \neq c(av)$. In order to cover Y , consider any $y \in Y$. If $d_G^c(y, B) \geq 10\epsilon n$, then we extend y to a path byb' with $b, b' \in B$. Otherwise, we have $|\{c(av) : a \in N_G(v) \cap A \text{ and } c(av) \neq c_a\}| \geq 10\epsilon n$, so we construct the path $baya'b'$ with $a, a' \in A$ and $b, b' \in B$.

We now give the formal definition of \mathcal{P} in the following claim.

Claim 3.6. *Let $q := |V(M) \cap Y|$. There exists a properly coloured subgraph \mathcal{P} of G such that $M \cup M' \subseteq \mathcal{P}$ and*

- (i') \mathcal{P} is a union of ℓ^* vertex-disjoint paths such that each path has both endpoints in $B \cup Z$;
- (ii') $\ell^* = |M| + |M'| + |Y| - q \leq 4\epsilon n$;
- (iii') \mathcal{P} covers Y ;
- (iv') \mathcal{P} contains precisely $2\ell^*$ vertices in $B \cup Z$, that is, each vertex in $V(\mathcal{P}) \cap (B \cup Z)$ is an endpoint of some path in \mathcal{P} ;
- (v') \mathcal{P} contains at most $|M'| + 2|Y| - q$ vertices in A .

Proof of claim. We construct \mathcal{P}_0 as follows. Initially, we set $\mathcal{P}_0 := M \cup M'$. For each edge $av \in M'$ with $a \in A$, we add an edge ab to \mathcal{P}_0 such that $b \in B \setminus V(\mathcal{P})$ is distinct and $c(ab) = c_a \neq c(av)$ (which exists by (E2)). Thus \mathcal{P}_0 is a union of $|M| + |M'|$ vertex-disjoint paths such that each path has both endpoints in $V(G) \setminus A$,

$$|V(\mathcal{P}_0) \setminus A| = 2|M| + 2|M'|, \quad |V(\mathcal{P}_0) \cap Y| = q \quad \text{and} \quad |V(\mathcal{P}_0) \cap A| = |M'|.$$

Let $Y := \{y_1, \dots, y_{|Y|}\}$ be such that $V(\mathcal{P}_0) \cap Y = \{y_1, \dots, y_q\}$. Suppose that for some $i \leq |Y|$ we have already constructed $\mathcal{P}_0 \subseteq \dots \subseteq \mathcal{P}_{i-1}$ such that for all $j < i$

- (Q1) \mathcal{P}_j is a union of $|M| + |M'| + \max\{0, j - q\}$ vertex-disjoint properly coloured paths;
- (Q2) $|(B \cup Z) \cap V(\mathcal{P}_j)| = 2|M| + 2|M'| - q + j + \max\{0, j - r\}$ and $|A \cap V(\mathcal{P}_j)| \leq |M'| + j + \max\{0, j - q\}$;
- (Q3) every vertex in $V(\mathcal{P}_j) \cap (B \cup Z)$ is an endpoint of some paths in \mathcal{P}_j ;
- (Q4) for all $j' \leq j$, $d_{\mathcal{P}_j}(y_{j'}) = 2$ and for all $j' > j$, $d_{\mathcal{P}_j}(y_{j'}) = d_{\mathcal{P}_{j-1}}(y_{j'})$.

We now construct \mathcal{P}_i as follows. By (Q2), $|B \cap V(\mathcal{P}_{i-1})|, |A \cap V(\mathcal{P}_{i-1})| \leq 8\epsilon n$.

Note that by (Q4)

$$d_{\mathcal{P}_{i-1}}(y_i) = d_{\mathcal{P}_0}(y_i) = d_M(y_i) = \begin{cases} 1 & \text{if } i \leq q \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $i \leq q$. Let c' be the colour of the edge incident with y_i in \mathcal{P}_{i-1} . If $d_G^c(y_i, B) \geq 10\epsilon n$, then there exists an edge by_i such that $b \in B \setminus V(\mathcal{P}_{i-1})$ and $c(by_i) \neq c'$ and set $\mathcal{P}_i := \mathcal{P}_{i-1} \cup by_i$. Thus, we may assume that there exist at least $10\epsilon n$ vertices $a \in A \cap N_G(y_i)$ such that $c(ay_i) \neq c_a$ and these $c(ay_i)$ are distinct. So there exists a vertex $a \in (A \cap N_G(y_i)) \setminus V(\mathcal{P}_{i-1})$ such that $c_a \neq c(ay_i) \neq c'$. By (E2), there exists a vertex $b \in B \cap N_G(a) \setminus V(\mathcal{P}_{i-1})$ such that $c(ab) = c_a \neq c(ay_i)$. Set $\mathcal{P}_i := \mathcal{P}_{i-1} \cup \{ay_i, ab\}$. A similar argument also holds for the case when $i > q$, where we apply the previous argument twice. Finally, set $\mathcal{P} := \mathcal{P}_{|Y|}$. \square

Let $A^* := A \setminus V(\mathcal{P})$. Let B^* be a subset of $B \cup Z$ such that $V(\mathcal{P}) \cap (B \cup Z) \subseteq B^*$ and $|B^*| = \min\{|B| + |Z|, \lfloor |A^*|/2 \rfloor + \ell^*\}$.

Note that $|B| \geq (\alpha + 4\varepsilon n) + 1 \geq \alpha n + \ell^* + 1$, where the last inequality holds by Claim 3.6(ii'). Since $|Y| \leq 2\varepsilon n$, together with Claim 3.6(v') and (i), we have

$$|A^*| \geq |A| - (|M'| + 2|Y|) \geq |A| - 6\varepsilon n \geq 2\alpha n + 2.$$

Therefore, we deduce that $|B^*| \geq \alpha n + \ell^* + 1$.

Note that (A^*, B^*) is 24ε -extremal (as $(A, B \cup Z)$ is by Proposition 3.4). By Lemma 3.3, G contains a properly coloured cycle C with vertex set $A^* \cup B^* \cup V(\mathcal{P}) = A \cup B^* \cup Y$ by Claim 3.6(iii'). If $|B^*| = |B| + |Z|$, then C is a properly coloured Hamilton cycle of G . If $|B^*| = \lfloor |A^*|/2 \rfloor + \ell^*$, then

$$\begin{aligned} |C| &= |A| + |Y| + |B^*| = |A| + |Y| + \lfloor |A^*|/2 \rfloor + \ell^* \\ &= |A| + |Y| + \lfloor (|A| - |V(\mathcal{P}) \cap A|)/2 \rfloor + \ell^* \\ &\stackrel{(ii'), (v')}{\geq} |A| + \left\lfloor \frac{|A| - (|M'| + 2|Y| - q)}{2} \right\rfloor + |M| + |M'| + 2|Y| - q \\ &= \left\lfloor \frac{3|A|}{2} + |M| + \frac{|M'|}{2} + |Y| - \frac{q}{2} \right\rfloor \end{aligned}$$

as required. \square

We are ready to prove Lemma 2.1.

Proof of Lemma 2.1. Let $\varepsilon' := 4\sqrt{\varepsilon}$ and without loss of generality (by adjusting ε' slightly), we have $(\delta - \varepsilon')n \in \mathbb{Z}$. Let α such that $\varepsilon \ll \alpha \ll \delta$. Apply Lemma 3.1 and obtain an ε' -extremal pair (A, B) such that $|A| \geq (\delta - \varepsilon')n$,

$$|B| \geq (1 - \delta - \varepsilon')n \geq (\alpha + 8\varepsilon')n + 1.$$

and

$$d_G(b) \leq (\delta + \varepsilon)n \text{ for each } b \in B. \quad (3.1)$$

By removing vertices of A if necessary, we may assume that

$$|A| = (\delta - \varepsilon')n \geq (2\alpha + 12\varepsilon')n + 2. \quad (3.2)$$

Let Y be the set of $v \in V(G) \setminus (A \cup B)$ such that $d_G^c(v, B) \geq 10\varepsilon'n$ or $|\{c(av) : a \in N_G(v) \cap A \text{ and } c(av) \neq c_a\}| \geq 10\varepsilon'n$. Let $Z := V(G) \setminus (A \cup B \cup Y)$. Let $p := \max\{\varepsilon'n - |Y|, 0\}$, so

$$|Y| \geq \varepsilon'n - p. \quad (3.3)$$

Let $F := G \setminus A$. So $\delta(F) \geq \varepsilon'n$. Let R be the set of vertices $v \in V(F)$ such that $d_F(v) \leq 10\varepsilon'n$ and let $S := V(F) \setminus R$. Note that $|R| \geq (1 - \delta - \varepsilon')n$ as $B \subseteq R$ by (E3) and (3.1). Since $\Delta(F[R]) \leq 10\varepsilon'n$, Vizing's theorem implies that there exists a matching M_R in $F[R]$ such that $|M_R| \geq e(F[R])/(10\varepsilon'n + 1) \geq 8e(F[R])/|R|$. By summing the degrees $d_F(v)$ in $v \in R$, we have

$$\begin{aligned} |R|\varepsilon'n &\leq \sum_{v \in R} d_F(v) = 2e(F[R]) + e(F[R, S]) \leq |R||M_R|/4 + |R||S|, \\ \varepsilon'n &\leq |M_R|/4 + |S|. \end{aligned} \quad (3.4)$$

We now divide the proof into two different cases.

Case 1: $|M_R| + |S| \geq \varepsilon'n + p/2$. We claim that there exists a matching M in $F = G \setminus A$ such that $|M| = \lceil \varepsilon'n + p/2 \rceil$. Indeed, there is nothing to prove if $|M_R| \geq \varepsilon'n + p/2$. If $|M_R| < \varepsilon'n + p/2$, then we can extend M_R into a matching M of size $\lceil \varepsilon'n + p/2 \rceil$ by adding (appropriate) edges incident with S (as $d_F(s) \geq 10\varepsilon'n$ for all $s \in S$ and $p \leq \varepsilon'n$).

Note that $|M| = \lceil \varepsilon'n + p/2 \rceil \leq 2\varepsilon'n$ and

$$\begin{aligned} \left\lfloor \frac{3|A|}{2} + |M| + |Y| - \frac{|V(M_R) \cap Y|}{2} \right\rfloor &\geq \left\lfloor \frac{3|A|}{2} + |M| + \frac{|Y|}{2} \right\rfloor \\ &\stackrel{(3.2),(3.3)}{\geq} \left\lfloor \frac{3(\delta - \varepsilon')n}{2} + \varepsilon'n + \frac{p}{2} + \frac{\varepsilon'n - p}{2} \right\rfloor = \lfloor 3\delta n/2 \rfloor. \end{aligned}$$

By Lemma 3.5 (with $M, \emptyset, \varepsilon'$ playing the roles of M, M', ε), G contains a properly coloured cycle C such that $|C| \geq \min\{n, \lfloor 3\delta n/2 \rfloor\}$ as desired.

Case 2: $|M_R| + |S| < \varepsilon'n + p/2$. Together with (3.4) we have $|M_R| < 2p/3$ and $p > 0$. Thus $|Y| = \varepsilon'n - p$.

Case 2a: $|S \cap Y| \leq \varepsilon'n - 10p/3$. Note that by (3.3)

$$|Y \setminus (S \cup V(M_R))| \geq |Y| - |S \cap Y| - 2|M_R| \geq \varepsilon'n - p - (\varepsilon'n - 10p/3) - 4p/3 = p.$$

By (3.4), $|M_R| + |S| \geq \varepsilon'n$. We can extend M_R into a matching M in $F = G \setminus A$ such that $|M| = \lceil \varepsilon'n \rceil$ and $|Y \setminus V(M)| \geq p$. Indeed this is possible, by adding appropriate edges between S and $V(F) \setminus Y$ as $d_F(s) \geq 10\varepsilon'n \geq |Y| + 9\varepsilon'n$ for all $s \in S$. Hence

$$\begin{aligned} \left\lfloor \frac{3|A|}{2} + |M| + |Y| - \frac{|V(M) \cap Y|}{2} \right\rfloor &= \left\lfloor \frac{3|A|}{2} + |M| + \frac{|Y| + |Y \setminus V(M_R)|}{2} \right\rfloor \\ &\stackrel{(3.2),(3.3)}{\geq} \left\lfloor \frac{3(\delta - \varepsilon')n}{2} + \varepsilon'n + \frac{(\varepsilon'n - p) + p}{2} \right\rfloor = \left\lfloor \frac{3\delta n}{2} \right\rfloor. \end{aligned}$$

We are done by Lemma 3.5 (with $M, \emptyset, \varepsilon'$ playing the roles of M, M', ε).

Case 2b: $|S \cap Y| > \varepsilon'n - 10p/3$. Recall that $|M_R| < 2p/3$ and $|M_R| + |S| \leq \varepsilon'n + p/2$. So

$$\begin{aligned} |(S \cup V(M_R)) \cap (B \cup Z)| &= |(S \cup V(M_R)) \setminus Y| \leq |S| + 2|M_R| - |S \cap Y| \\ &\leq \varepsilon'n + p/2 + 2p/3 - (\varepsilon'n - 10p/3) = 9p/2. \end{aligned} \quad (3.5)$$

Let F' be the subgraph $G[A, B \cup Z]$ obtained by removing all edges uv with $c(uv) = c_a$ for some $a \in A$. Note that for each $a \in A$,

$$d_{F'}(a) \geq \delta^c(G) - (1 + |V(G) \setminus (B \cup Z)| - 1) = \delta n - |A| - |Y| = \varepsilon'n - |Y| = p.$$

Hence, $e(F') \geq p|A| \geq p(\delta - \varepsilon')n$ and $\Delta(F') \leq 24\varepsilon'n$ as $(A, B \cup Z)$ is $24\varepsilon'$ -extremal by Proposition 3.4. Since $\varepsilon' \ll \delta$, König's theorem implies that there is a matching

$$e(F')/\Delta(F') \geq 11p/2 \stackrel{(3.5)}{\geq} p + |(S \cup V(M_R)) \cap V(F')|.$$

Thus there is a matching M' in $F' \subseteq G[A, B \cup Z]$ such that $|M'| = p$ and $V(M') \cap (V(M_R) \cup S) = \emptyset$. By adding (appropriate) edges of F incident with S , we can extend M_R into a matching M in $F = G \setminus A$ satisfying $V(M) \cap V(M') = \emptyset$, $|M| = \lceil \varepsilon'n \rceil$. Note that $|M| + |M'| = p + \lceil \varepsilon'n \rceil \leq 2\varepsilon'n + 1$ and

$$\begin{aligned} \left\lfloor \frac{3|A|}{2} + |M| + \frac{|M'|}{2} + |Y| - \frac{|V(M) \cap Y|}{2} \right\rfloor &\geq \left\lfloor \frac{3|A|}{2} + |M| + \frac{|M'|}{2} + \frac{|Y|}{2} \right\rfloor \\ &\stackrel{(3.2),(3.3)}{\geq} \left\lfloor \frac{3(\delta - \varepsilon')n}{2} + \varepsilon'n + \frac{p}{2} + \frac{\varepsilon'n - p}{2} \right\rfloor = \left\lfloor \frac{3\delta n}{2} \right\rfloor. \end{aligned}$$

Again, we are done by Lemma 3.5 (with $M, M', 2\varepsilon'$ playing the roles of M, M', ε). \square

4. ABSORBING CYCLES

In this section, we prove Lemma 2.2. We need the following definitions. Given a vertex x , we say that a path P is an *absorbing path for x* if the following conditions hold:

- (i) $P = z_1 z_2 z_3 z_4$ is a properly coloured path of length 3;
- (ii) $x \notin V(P)$;
- (iii) $z_1 z_2 x z_3 z_4$ is a properly coloured path.

Next we define an absorbing path for two disjoint edges. Given two vertex-disjoint edges $x_1 x_2, y_1 y_2$, we say that a path P is an *absorbing path for $(x_1, x_2; y_1, y_2)$* if the following conditions hold:

- (i) $P = z_1 z_2 z_3 z_4$ is a properly coloured path of length 3;
- (ii) $V(P) \cap \{x_1, x_2, y_1, y_2\} = \emptyset$;
- (iii) both $z_1 z_2 x_1 x_2$ and $y_1 y_2 z_3 z_4$ are properly coloured paths of length 3.

Note that the ordering of $(x_1, x_2; y_1, y_2)$ is important. We would also need the following proposition from [14].

Proposition 4.1. *Let $P' = x_1 x_2 \dots x_{\ell-1} x_\ell$ be a properly coloured path with $\ell \geq 4$. Let $P = z_1 z_2 z_3 z_4$ be an absorbing path for $(x_1, x_2; x_{\ell-1}, x_\ell)$ with $V(P) \cap V(P') = \emptyset$. Then $z_1 z_2 x_1 x_2 \dots x_{\ell-1} x_\ell z_3 z_4$ is a properly coloured path.*

Given a vertex x , let $\mathcal{L}(x)$ be the set of absorbing paths for x . Similarly, given two vertex-disjoint edges $x_1 x_2, y_1 y_2$, let $\mathcal{L}(x_1, x_2; y_1, y_2)$ be the set of absorbing paths for $(x_1, x_2; y_1, y_2)$. The following lemma follows immediately from Lemmas 4.3 and 4.5 of [14].

Lemma 4.2. *Let $0 < 1/n \ll \gamma \ll \varepsilon < 1/2$. Let G be an edge-coloured graph on n vertices with $\delta^c(G) \geq (1/2 + \varepsilon)n$. Then there exists a family \mathcal{F} of vertex-disjoint properly coloured paths each of length 3, which satisfies the following properties:*

$$|\mathcal{F}| \leq \gamma^{1/2} n, \quad |\mathcal{L}(x) \cap \mathcal{F}| \geq \gamma n, \quad |\mathcal{L}(x_1, x_2; y_1, y_2) \cap \mathcal{F}| \geq \gamma n$$

for all $x \in V(G)$ and for all distinct vertices $x_1, x_2, y_1, y_2 \in V(G)$ with $x_1 x_2, y_1 y_2 \in E(G)$.

To prove Lemma 2.2, we aim to join the paths in \mathcal{F} given by Lemma 4.2 into a properly coloured cycle. First, we need the following definition, which are only used in this section.

Let G be an edge-coloured graph on n vertices. Let $x, y \in V(G)$ be distinct and let $\ell \in \mathbb{N}$. Define $\mathcal{P}_\ell^G(x; y)$ to be the set of properly coloured paths P of length ℓ from x to y . Define $\mu_\ell^G(x; y) := |\mathcal{P}_\ell^G(x; y)|/n^{\ell-1}$ and $\mu_{\leq \ell}^G(x; y) := \sum_{\ell' \leq \ell} \mu_{\ell'}^G(x; y)$. For a colour set C_y , let $\mathcal{P}_\ell^G(x; y, C_y)$ be the set of paths $P \in \mathcal{P}_\ell^G(x; y)$ such that $C_P(y) \in C_y$. Define $\mu_\ell^G(x; y, C_y)$ and $\mu_{\leq \ell}^G(x; y, C_y)$ analogously. For $\ell \in \mathbb{N}$ and $\eta > 0$, we say that y is $(\leq \ell, \eta)$ -reachable from x in G if $\mu_{\leq \ell}^G(x; y) \geq \eta$. We say that y is *strongly* $(\leq \ell, \eta)$ -reachable from x in G if for any colour c_0 , y is $(\leq \ell, \eta)$ -reachable from x in $G - \{yz \in E(G) : c(yz) = c_0\}$. Equivalently, y is strongly $(\leq \ell, \eta)$ -reachable from x in G if $\mu_{\leq \ell}^G(x; y, C(G) \setminus c_0) \geq \eta$ for all colours $c_0 \in C(G)$.

Proposition 4.3. *Let $\ell \in \mathbb{N}$ and let $\eta > 0$. Let G be an edge-coloured graph on n vertices. Let x, y, v be distinct vertices in $V(G)$.*

- (i) *If y is strongly $(\leq \ell, \eta)$ -reachable from x , then for any colour c_0 , we have $\mu_{\leq \ell}^{G \setminus v}(x; y, C(G) \setminus c_0) \geq \eta - \ell^2/n$.*

If y is not strongly $(\leq \ell, \eta)$ -reachable from x but is $(\leq \ell, 2\eta)$ -reachable, then

- (ii) *there exists a unique colour c_y such that $\mu_{\leq \ell}^G(x; y, c_y) \geq \eta$;*

$$(iii) \mu_{\leq \ell}^{G \setminus v}(x; y, c_y) \geq \eta - \ell^2/n.$$

Proof. For each $\ell' \in \mathbb{N}$, v is in at most $(\ell' - 1)n^{\ell' - 2}$ paths of length ℓ' from x to y . Hence for all $\ell' \leq \ell$,

$$\begin{aligned} \mu_{\ell'}^{G \setminus v}(x; y, C(G) \setminus c_0) &\geq \mu_{\ell'}^G(x; y, C(G) \setminus c_0) - (\ell' - 1)/n \\ &\geq \mu_{\ell'}^G(x; y, C(G) \setminus c_0) - \ell/n, \end{aligned}$$

so (i) holds. The definitions of $(\leq \ell, 2\eta)$ -reachable and strongly $(\leq \ell, \eta)$ -reachable implying (ii). The proof of (i) can be adapted to prove (iii). \square

Lemma 4.4. *Let $0 < 1/n \ll \varepsilon < 1/2$. Suppose that G is an edge-coloured graph on n vertices with $\delta^c(G) \geq (1/2 + \varepsilon)n + 2$. Let $x, y \in V(G)$ be distinct and let c_x, c_y be any two colours. Then there exists a properly coloured path P from x to y of length at most ε^{-2} such that $C_P(x) \neq \{c_x\}$ and $C_P(y) \neq \{c_y\}$.*

Proof. Let $\ell_0 := \lfloor \varepsilon^{-2} \rfloor$ and let η be such that $1/n \ll \eta \ll \varepsilon$. Let G, x, y, c_x, c_y be as defined in the lemma. Remove all edges at x with colour c_x and all edges at y with colour c_y . So $d(x), d(y) \geq (1/2 + \varepsilon)n$ and $d^c(v) \geq (1/2 + \varepsilon)n$ for all $v \in V(G) \setminus \{x, y\}$. Therefore to prove the lemma, it suffices to show that there exists a properly coloured path from x to y of length at most ℓ_0 . Note that for all $v \in V(G)$, all $\ell \leq \ell_0$ and all $P \in \mathcal{P}_\ell^G(x; v)$, we may assume that $y \notin V(P)$ or else the lemma holds.

For each $\ell \in \mathbb{N}$, let S_ℓ be the set of vertices $v \in V(G) \setminus x$ that are strongly $(\leq \ell, \eta)$ -reachable from x , and let T_ℓ be the set of vertices $v \in V(G) \setminus (S_\ell \cup x)$ that are $(\leq \ell, 2\eta^\ell)$ -reachable from x . Since a $(\leq \ell, 2\eta^\ell)$ -reachable vertex from x is also $(\leq \ell + 1, 2\eta^{\ell+1})$ -reachable from x and a similar statement for strongly reachable, we have

$$S_\ell \subseteq S_{\ell+1} \text{ and } S_\ell \cup T_\ell \subseteq S_{\ell+1} \cup T_{\ell+1} \text{ for all } \ell \in \mathbb{N}. \quad (4.1)$$

Also $S_1 = \emptyset$ and T_1 is the set of vertex $v \in N(x)$, so

$$|T_1| \geq (1/2 + \varepsilon)n. \quad (4.2)$$

Suppose that there exists $s \in S_\ell \cap N(y)$. Let $P \in \mathcal{P}_\ell^G(x; s)$ with $c(sy) \notin C_P(s)$ (which exists as s is strongly $(\leq \ell, \eta)$ -reachable from x). Note that Py is a properly coloured path from x to y of length at most $\ell + 1$. Thus we may assume that $|S_\ell| \leq (1/2 - \varepsilon)n$ for all $\ell < \ell_0$. If $2|S_{\ell+1}| + |T_{\ell+1}| \geq 2|S_\ell| + |T_\ell| + \varepsilon^2 n$ for all $1 \leq \ell < \ell_0 - 1$, then together with (4.2) we have $2|S_{\ell_0-1}| + |T_{\ell_0-1}| \geq 3n/2$. Hence $|S_{\ell_0-1}| \geq n/2$, a contradiction. Therefore, we may assume that for some $\ell < \ell_0 - 1$,

$$2|S_{\ell+1}| + |T_{\ell+1}| < 2|S_\ell| + |T_\ell| + \varepsilon^2 n. \quad (4.3)$$

By (4.1), we have

$$|(S_{\ell+1} \cup T_{\ell+1}) \setminus (S_\ell \cup T_\ell)| \leq \varepsilon^2 n. \quad (4.4)$$

Let $W := T_\ell \cap T_{\ell+1}$. Recall that $|S_\ell| \leq (1/2 - \varepsilon)n$. By (4.1) and (4.2), we have

$$|T_\ell| \geq |S_\ell \cup T_\ell| - |S_\ell| \geq |T_1| - (1/2 - \varepsilon)n \geq 2\varepsilon n.$$

Since $T_\ell \setminus W = T_\ell \setminus T_{\ell+1} \subseteq S_{\ell+1} \setminus S_\ell \subseteq (S_{\ell+1} \cup T_{\ell+1}) \setminus (S_\ell \cup T_\ell)$ by (4.1), (4.4) implies that

$$|T_\ell \setminus W| \leq \varepsilon^2 n \quad (4.5)$$

and so

$$|W| \geq |T_\ell| - |T_\ell \setminus W| \geq 2\varepsilon n - \varepsilon^2 n \geq \varepsilon n. \quad (4.6)$$

For each $w \in W \subseteq T_\ell$, Proposition 4.3(ii) implies that there exists a unique colour c_w such that $\mu_{\leq \ell}^G(x; w, c_w) \geq \eta^\ell$. Define an auxiliary digraph H with on $V(G) \setminus x$ and edge set $E(H) := \{wv : w \in W, v \in N_G(w) \setminus x \text{ and } c(wv) \neq c_w\}$. Note that for each $w \in W$, we have $d_H^+(w) \geq d_G^c(w) - 1 \geq (1 + \varepsilon)n/2$ and so

$$e(H) \geq (1 + \varepsilon)n|W|/2. \quad (4.7)$$

We now bound $e(H)$ from above (to obtain a contradiction) in the following claim.

Claim 4.5. *Let $e_H(X, Y)$ denote the number of edges from X to Y . Then*

- (i) $e_H(W, (S_{\ell+1} \cup T_{\ell+1}) \setminus (S_\ell \cup T_\ell)) < \varepsilon^2 n|W|$;
- (ii) $e_H(W, T_\ell \setminus W) < \varepsilon^2 n|W|$;
- (iii) $e_H(W, V(G) \setminus (S_{\ell+1} \cup T_{\ell+1} \cup x)) < 4\eta\varepsilon^{-1}n|W|$;
- (iv) $e_H(W, S_\ell) < 2\eta n|W|$;
- (v) $e_H(W, W) < (1/2 - \varepsilon + 2\eta)n|W|$.

Proof of claim. Note that (i) and (ii) follow from (4.4) and (4.5), respectively. To see (iii), note that if $wv \in E(H)$ with $w \in W$ and $v \in V(G) \setminus x$ and $P \in \mathcal{P}_{\ell'}^{G \setminus v}(x; w, c_w)$, then Pv is a properly coloured path of length $\ell' + 1$ from x to v . By Proposition 4.3(iii), for each $v \in V(G) \setminus x$,

$$\mu_{\leq \ell+1}^G(x, v) \geq \frac{1}{n} \sum_{w \in N_H(v)} \mu_{\leq \ell}^{G \setminus x}(x; w, c_w) \geq \eta^\ell e_H(W, v)/2n.$$

Therefore, for all $v \in V(G) \setminus (S_{\ell+1} \cup T_{\ell+1} \cup x)$, we have $e_H(W, v) < 4\eta n \leq 4\eta\varepsilon^{-1}|W|$, where the last inequality is due to (4.6). Thus (iii) holds.

Consider the edge $ws \in E(H)$ with $w \in W$ and $s \in S_\ell$. If $P \in \mathcal{P}_{\ell'}^{G \setminus v}(x; s, C(G) \setminus c(ws))$, then Pw is a properly coloured path of length $\ell' + 1$ from x to w with $C_P(w) \neq \{c_w\}$. We must have $e_H(w, S_\ell) < 2\eta n$ for all $w \in W$, which in turn implies (iv). Indeed, if $e_H(w, S_\ell) \geq 2\eta n$, then by Proposition 4.3(iii),

$$\begin{aligned} \mu_{\leq \ell+1}^G(x; w, C(G) \setminus c_w) &\geq \frac{1}{n} \sum_{s \in N_H(w) \cap S_\ell} \mu_{\leq \ell}^{G \setminus v}(x; s, C(G) \setminus c(ws)) \\ &\geq \frac{1}{n} e_H(w, S_\ell) (\eta^\ell - \ell^2/n) \geq \eta^{\ell+1} \end{aligned}$$

and so $w \in S_{\ell+1}$ (as $w \in W \subseteq T_{\ell+1}$ implying that $\mu_{\leq \ell+1}^G(x; w, c_w) \geq \eta^{\ell+1}$), a contradiction.

By a similar argument with $(T_\ell$ playing the role of $S_\ell)$, we deduce that every $w \in W \subseteq T_{\ell+1}$ has less than $2\eta n$ edges ww' in G such that $w' \in W \subseteq T_\ell$ and $c_w \neq c(ww') \neq c_{w'}$. This means that, in H , each $w \in W$ is contained less than $2\eta n$ 2-cycles. Since each $w \in W$ is incident to at most $(1/2 - \varepsilon)n$ edges of the same colour in G , we have $e_H(W, w) < (1/2 - \varepsilon)n + 2\eta n = (1/2 - \varepsilon + 2\eta)n$ implying (v). \square

By Claim 4.5, we deduce that

$$e(H) \leq (\varepsilon^2 + \varepsilon^2 + 4\varepsilon^{-1}\eta + 2\eta + 1/2 - \varepsilon + 2\eta) n|W| < (1 + \varepsilon)n|W|/2,$$

contradicting (4.7). This complete the proof of Lemma 4.4. \square

We now prove Lemma 2.2.

Proof of Lemma 2.2. Let ε_0 be such that $1/n \ll \varepsilon_0 \ll \varepsilon$. Apply Lemma 4.2 and obtain a family \mathcal{F} of vertex-disjoint properly coloured paths each of length 3 such that for all $x \in V(G)$ and for all distinct vertices $x_1, x_2, y_1, y_2 \in V(G)$ with $x_1x_2, y_1y_2 \in E(G)$,

$$|\mathcal{F}| \leq 3\gamma^{1/2}n, \quad |\mathcal{L}(x) \cap \mathcal{F}| \geq 3\gamma n, \quad |\mathcal{L}(x_1, x_2; y_1, y_2) \cap \mathcal{F}| \geq 3\gamma n.$$

Let $P_1, \dots, P_{|\mathcal{F}|}$ be paths in \mathcal{F} . Let x_i and y_i be endvertices of P_i for all $i \leq |\mathcal{F}|$. Suppose that for $j \leq |\mathcal{F}|$, we have already found Q_1, \dots, Q_{j-1} such that

- (a) for all $i < j$, Q_i is a path from y_i to x_{i+1} of length at most ε_0^{-2} ;
- (b) for all $i < j$, $P_iQ_iP_{i+1}$ is a properly coloured path;
- (c) $Q_1, \dots, Q_{j-1}, P_{j+1}, \dots, P_{|\mathcal{F}|}$ are disjoint.

We now find Q_j as follows. Let $C_{P_j}(y_j) = \{c_y\}$, let $C_{P_{j+1}}(x_{j+1}) = \{c_x\}$ and let $W := (\bigcup_{i \leq |\mathcal{F}|} V(P_i) \cup \bigcup_{i' < j} V(Q_{i'})) \setminus \{y_j, x_{j+1}\}$, where we take $P_{|\mathcal{F}|+1} = P_1$ and $x_{|\mathcal{F}|+1} = x_1$. Note that $|W| \leq 3\gamma^{1/2}n(4 + \varepsilon_0^{-2}) \leq \varepsilon n/2$. Let $G' = G \setminus W$. So $\delta^c(G') \geq (1/2 + \varepsilon/2)n \geq (1/2 + \varepsilon_0)|G'|$. Apply Lemma 4.4 and obtain a properly coloured path Q_j in G' from y_j to x_{j+1} of length at most ε_0^{-2} such that $C_{Q_j}(y_j) \neq \{c_y\}$ and $C_{Q_j}(x_{j+1}) \neq \{c_x\}$. Thus we have found $Q_1, \dots, Q_{|\mathcal{F}|}$.

Let $C := P_1Q_1P_2 \dots P_{|\mathcal{F}|}Q_{|\mathcal{F}|}$ be a properly coloured cycle in G . Note that $|C| \leq 3\gamma^{1/2}n(4 + \varepsilon_0^{-2}) \leq \varepsilon n/2$. Let \mathcal{P} be any set of k vertex-disjoint properly coloured paths in $G \setminus V(C)$ with $k \leq \gamma n$. Let \mathcal{P}' be the set of properly coloured paths obtained from \mathcal{P} by breaking up every path $P \in \mathcal{P}$ with $|P| \leq 3$ into isolated vertices. Thus $|\mathcal{P}'| \leq 3\gamma n$ and for each $P \in \mathcal{P}'$, $|P| = 1$ or $|P| \geq 4$. For each $P \in \mathcal{P}'$, there exists a distinct $P' \in \mathcal{F}$ such that $P' \in \mathcal{L}(V(P))$ if $|P'| = 1$, and $P' \in \mathcal{L}(u_1, u_2; u_\ell' u_{\ell-1})$ if $P = u_1 u_2 \dots u_\ell$. By Proposition 4.1 and the definition of an absorbing path for a vertex, there exists a properly coloured cycle C' with vertex set $V(C) \cup V(\bigcup \mathcal{P})$. \square

5. PROPERLY COLOURED 1-PATH-CYCLE

A *1-path-cycle* is a disjoint union of cycles and at most one path. In this section, we prove the following lemma, which immediately implies Lemma 2.3.

Lemma 5.1. *Let $0 < 1/n \ll \beta \ll \varepsilon \ll 1/2 < \delta$. Suppose that G is a critical edge-coloured graph on n vertices with $\delta^c(G) \geq \delta n + 1$. Then one of the following statements holds*

- (i) *G contains a properly coloured 1-path-cycle H such that $|H| \geq \min\{(3\delta + \beta)n/2, n\}$ and every cycle in H has length at least $\beta n/100$;*
- (ii) *G is (δ, ε) -extremal.*

To prove Lemma 5.1, we need the following terminology. Let $\mathbf{x} = (x, c_x)$ and $\mathbf{y} = (y, c_y)$ be pairs with vertices $x, y \in V(H)$ and colours c_x, c_y . For $\rho > 0$, we say that H is a *1-path-cycle with parameters ρ - $(\mathbf{x}; \mathbf{y})$* if H satisfies the following four properties:

- (a) H is a properly coloured 1-path-cycle;
- (b) every cycle in H has length at least ρn ;
- (c) the path component P in H has length at least ρn with endvertices x and y ;
- (d) $C_H(x) = \{c_x\}$ and $C_H(y) = \{c_y\}$.

Note that c_x and c_y are precisely the colours of the edges in P (and H) incident with x and y , respectively. The order of \mathbf{x} and \mathbf{y} is important. If ρ is known from the context, we simply write $(\mathbf{x}; \mathbf{y})$ instead of ρ - $(\mathbf{x}; \mathbf{y})$.

Orient the cycles of H into directed cycles arbitrarily and orient the path P into a directed path from x to y . For each $v \in V(H) \setminus y$, define $c_+(v)$ to be $c(vv_+)$, where v_+ is the successor of v , and for each $w \in V(H) \setminus x$, define $c_-(w)$ to be $c(ww_-)$, where w_- is

the ancestor of w . From now on every 1-path cycle is assumed to be oriented as above. For an oriented cycle C and $u, v \in V(C)$, we write uC^+v for the path $uu_+ \dots v_-v$ in C and uC^-v for the path $uu_- \dots v_+v$ in C .

Lemma 5.2. *Let $\rho > 0$. Let G be an edge-coloured graph on n vertices with $\delta^c(G) \geq \rho n + 1$. Suppose that H is a properly coloured 1-path-cycle in G of maximum order such that every cycle in H has length at least ρn , and that $|H| < n$. Then there exists a 1-path-cycle H' with parameters ρ - $(\mathbf{x}; \mathbf{y})$ such that $V(H') = V(H)$.*

Proof. If H contains no path component, then $H + w$ is a properly coloured 1-path-cycle such that every cycle has length at least ρn , where $w \in V(G) \setminus V(H)$. This contradicts the maximality of $|H|$. So we may assume that H contains a path component P .

Suppose that P has length less than ρn . Let x be an endvertex of P . Let $\mathbf{x} = (x, c_x)$ with $C_P(x) = \{c_x\}$ if $|V(P)| \geq 2$, and c_x is an arbitrary colour otherwise. Note that $|N(\mathbf{x})| \geq \delta^c(G) - 1 \geq \rho n \geq |V(P) \setminus x|$. So there exists $w \in N(\mathbf{x}) \setminus V(P)$. If $w \notin V(H)$, then we can extend P thus enlarging H , a contradiction. Hence $w \in V(H) \setminus V(P)$ and let C be the cycle in H containing w . Without loss of generality, we may assume that $c(xw) \neq c_-(w)$. Then $H' = H + xw - ww_-$ is a properly coloured 1-path-cycles on vertex set $V(H)$ such that every cycle in H has length at least ρn and the path component is $P' = w_+C^+wxP$ of length at least $|C| \geq \rho n$. Therefore H' is a 1-path-cycles with parameters $(\mathbf{w}_+; \mathbf{y})$, where $\mathbf{w}_+ = (w_+, c_+(w_+))$ and $\mathbf{y} = (y, c_y)$ such that y is the other endvertex of P' and $C_{P'}(y) = \{c_y\}$. \square

In the next proposition, we show how we can change from 1-path-cycle to another one by ‘switching edges’.

Proposition 5.3. *Let G be an edge-coloured graph. Let $\rho > 0$. Let H be a 1-path-cycle in G with parameters $(\mathbf{x}; \mathbf{y})$, where $\mathbf{x} = (x, c_x)$ and $\mathbf{y} = (y, c_y)$. Suppose that $w \in V(H) \cup N_G(\mathbf{x})$ such that $\text{dist}_H(w, x), \text{dist}_H(w, y) \geq \rho n + 1$. Then*

- (i) *if $c(xw) \neq c_-(w)$, then $H + xw - ww_+$ is a 1-path-cycle with parameters $((w_+, c_+(w_+)); \mathbf{y})$;*
- (ii) *if $c(xw) \neq c_+(w)$, then $H + xw - ww_-$ is a 1-path-cycle with parameters $((w_-, c_-(w_-)); \mathbf{y})$.*

A similar statement holds for $w \in V(H) \cup N_G(\mathbf{y})$ with $\text{dist}_H(w, x), \text{dist}_H(w, y) \geq \rho n + 1$.

Proof. Suppose that $c(xw) \neq c_-(w)$. If w is in the path component P of H , then $P + xw - ww_+$ is a properly coloured graph consisting of a cycle $xPwx$ and a path w_+Py (as $c(xw) \neq c_x$). Since $\text{dist}_H(w, x), \text{dist}_H(w, y) \geq \rho n + 1$, both of these components have size at least ρn . Thus $H + xw - ww_+$ is a 1-path-cycle with parameters $((w_+, c_+(w_+)); \mathbf{y})$. If C is the cycle in H containing w , then $P + C + xw - ww_+$ is a properly coloured path w_+C_+wxPy . Hence $H + xw - ww_+$ is a 1-path-cycle with parameters $((w_-, c_-(w_-)); \mathbf{y})$. Therefore (i) holds, and (ii) holds by a similar argument. \square

Let H be 1-path-cycle in G with parameters $(\mathbf{x}; \mathbf{y})$ and let H' be an 1-path-cycle with parameters $(\mathbf{z}; \mathbf{y})$ in G obtained from H by switching one edges. Note that we can deduce which edges were involved in the switching by analysing \mathbf{z} as follows. Let $\mathbf{z} = (z, c_z)$ be a pair with vertex $z \in V(H) \setminus \{x, y\}$ and colour $c_z \in C_H(z)$. Define the vertex

$$w_{\mathbf{z}} := \begin{cases} z_- & \text{if } c_z = c_+(z), \\ z_+ & \text{if } c_z = c_-(z). \end{cases}$$

Note that $H' = H + xw_{\mathbf{z}} - w_{\mathbf{z}}z$ by Proposition 5.3.

Let $X_1(H)$ be the set of pairs $\mathbf{z} = (z, c_z)$ with vertex $z \in V(H)$ and colour $c_z \in C_H(z)$ such that

- $H + xw_{\mathbf{z}} - w_{\mathbf{z}}z$ is a 1-path-cycle with parameters $(\mathbf{z}; \mathbf{y})$;
- $\text{dist}_H(w_{\mathbf{z}}, x), \text{dist}_H(w_{\mathbf{z}}, y) \geq 2\rho n$.

Note that $\{(\mathbf{z}; \mathbf{y}) : \mathbf{z} \in X_1(H)\}$ is a subset of possible parameters of the 1-path-cycle that can be obtained from H by switching one edge of H with an edge incident to x . We obtain the following properties of $X_1(H)$.

Proposition 5.4. *Let G be an edge-coloured graph on n vertices and let $\rho > 0$. Suppose that H is a properly coloured 1-path-cycle in G of maximum order, and that H has parameters $\rho - (\mathbf{x}; \mathbf{y})$. Let $z \in N_G(\mathbf{x})$ such that $\text{dist}_H(z, x), \text{dist}_H(z, y) \geq 2\rho n + 1$. Then the following statements hold*

- $N_G(\mathbf{x}) \subseteq V(H)$;
- if $c(xz) \neq c_-(z)$, then $(z_+, c_+(z_+)) \in X_1(H)$;
- if $c(xz) \neq c_+(z)$, then $(z_-, c_-(z_-)) \in X_1(H)$;
- for $\mathbf{z} \in X_1(H)$, $N_G(\mathbf{z}) \subseteq V(H)$.

Proof. If $z \in N_G(\mathbf{x}) \setminus V(H)$, then $H + xz$ is a 1-path-cycle with parameters $(z, c(xz); \mathbf{y})$ contradicting the maximality of H . Thus (a) holds, and (d) is proved similarly (by considering $H + xw_{\mathbf{z}} - w_{\mathbf{z}}z$ instead of H).

If $c(xz) \neq c_-(z)$, then $H + xz - zz_+$ is a 1-path-cycle with parameters $((z_+, c_+(z_+)); \mathbf{y})$ by Proposition 5.3(i). So $(z_+, c_+(z_+)) \in X_1(H)$ implying (b). A similar argument shows that (c) holds. \square

We would also need to consider the set of 1-path-cycles with parameters $(\mathbf{z}; \mathbf{y})$ that can be obtained from H by replacing two edges of H . We now define X_2 , which is the analogue of X_1 for replacing two edges of H (with some additional constraints). Let $X_2(H)$ be the set of pairs $\mathbf{z} = (z, c_z)$ with vertex $z \in V(H)$ and colour $c_z \in C_H(z)$ such that there exist at least $10\rho n$ pairs $\mathbf{z}' = (z', c_{z'}) \in X_1(H)$ satisfying

- $\text{dist}_H(z, x), \text{dist}_H(z, y), \text{dist}_H(z', z) \geq 2\rho n$ and
- $H + xw_{\mathbf{z}'} + z'w_{\mathbf{z}} - zw_{\mathbf{z}} - z'w_{\mathbf{z}'}$ is a 1-path-cycle with parameters $(\mathbf{z}; \mathbf{y})$.

In the next lemma, we show that if $|X_1(H) \cup X_2(H)|$ is bounded above, then there exist disjoint $W^*, Z^* \subseteq V(G)$ such that $G[W^* \cup Z^*]$ is extremal with partition W^*, Z^* . The proof relies on analysing the structure of $X_1(H)$, $X_2(H)$ and $N(\mathbf{z})$ for $\mathbf{z} \in X_1(H)$.

Lemma 5.5. *Let $0 < 1/n \ll \rho \leq \alpha/1000 < 1/1000$ and let $1/2 + 3\alpha < \delta \leq 2/3$. Let G be a critical edge-coloured graph on n vertices with $\delta^c(G) \geq \delta n + 1$. Suppose that H is a properly coloured 1-path-cycle in G of maximum order. Suppose that H has parameters $(\mathbf{x}; \mathbf{y})$, that $|X_1(H) \cup X_2(H)| \leq (\delta + \alpha)n$ and that $|H| < n$. Then there exist disjoint $W^*, Z^* \subseteq V(H)$ such that*

- $|W^*| \geq (\delta - 7\sqrt{\alpha})n$ and $|Z^*| \geq (2\delta - 1 - 3\alpha^{1/4})n$;
- for each $w \in W^*$, there exists a distinct colour c_w^* such that there are at least $|Z^*| - 3\sqrt{\alpha}n$ vertices $z \in Z^* \cap N_G(w)$ such that $c(zw) = c_w^*$;
- for each $z \in Z^*$, $d_G(z) \leq (\delta + 4\alpha^{1/4})n$ and there are at least $(\delta - 6\alpha^{1/4})n$ vertices $w \in W^* \cap N_G(z)$ and $c(zw) = c_w^*$.

Proof. Write X_1 for $X_1(H)$ and X_2 for $X_2(H)$. Let Z be the set of vertices $z \in V(H)$ such that $\text{dist}_H(z, x), \text{dist}_H(z, y) \geq 2\rho n$ and

- (*) there exists a colour $c_z \in C_H(z)$ such that $\mathbf{z} = (z, c_z) \in X_1$ with $c(zw_{\mathbf{z}}) = c(xw_{\mathbf{z}})$.

Let Z' be the set of vertices $z \in Z$ such that both colours $c_z \in C_H(z)$ satisfy (*). Clearly $Z' \subseteq Z$.

We now bound the sizes of Z and Z' from below.

Claim 5.6. $|Z| + |Z'| \geq (\delta - 2\alpha)n \geq n/2$.

Proof of claim. Let

$$N := \{u \in N_G(\mathbf{x}) : \text{dist}_H(u, x), \text{dist}_H(u, y) > 2\rho n\}, \quad N' := \{u \in N : c(xu) \in C_H(u)\}.$$

Thus $|N| \geq \delta^c(G) - 1 - 2 \cdot 2\rho n \geq (\delta - 4\rho)n$ and $N \subseteq V(H)$ by Proposition 5.4(a). By Proposition 5.4(b) and (c),

$$|X_1| \geq |N'| + 2|N \setminus N'| = |N| + |N \setminus N'| \geq (\delta - 4\rho)n + |N \setminus N'|.$$

Since $|X_1 \cup X_2| \leq (\delta + \alpha)n$, we have $|N \setminus N'| \leq (4\rho + \alpha)n$ and so

$$|N'| \geq |N| - |N \setminus N'| \geq (\delta - \alpha - 8\rho)n \geq (\delta - 2\alpha)n.$$

Let X'_1 be the subset of X_1 generated by the edges xv with $v \in N'$, that is, $X'_1 := \{(x', c_{x'}) \in X_1 : w_{(x', c_{x'})} \in N'\}$. So $|X'_1| \geq (\delta - 2\alpha)n$. Thus if $(z, c_z) \in X'_1$, then $w_{\mathbf{z}} \in N'$ and $c(zw_{\mathbf{z}}) = c(xw_{\mathbf{z}})$. Note that Z contains all vertices $z \in V(H)$ such that $(z, c_z) \in X'_1$ for some colour c_z . Similarly, Z' contains all vertices $z \in V(H)$ such that $(z, c_+(z)), (z, c_-(z)) \in X'_1$. Hence, $|Z| + |Z'| \geq |X'_1| \geq (\delta - 2\alpha)n \geq n/2$ as required. \square

Define a directed graph F on $V(H)$ such that there exists a directed edge from z to w if and only if

- $(z, c_z) \in X_1$ and $z \in Z \cap N_H(w)$ and $c(wz) \neq c_z$;
- $\text{dist}_H(w, x), \text{dist}_H(w, y), \text{dist}_H(w, z) \geq 2\rho n$.

We also colour the edges uv (in F) by $c(uv)$. We now establish some properties of F .

Claim 5.7.

- (a) $e(F) \geq e_F(Z, V(F)) \geq (\delta - 6\rho)n|Z| + \sum_{z \in Z'} (d_G(z) - \delta n)$.
- (b) If $w \in V(H)$ has $10\rho n$ edges zw in F with $c(zw) \neq c_+(w)$, then $(w_-, c_-(w_-)) \in X_2$.
- (c) If $w \in V(H)$ has $10\rho n$ edges zw in F with $c(zw) \neq c_-(w)$, then $(w_+, c_+(w_+)) \in X_2$.

Proof of claim. For $\mathbf{z} \in X_1$, $N_G(\mathbf{z}) \subseteq V(H)$ by Proposition 5.4(d). Hence, for each $z \in Z$, $d_F^+(z) \geq |N_G(\mathbf{z})| - 3 \cdot 2\rho n \geq (\delta - 6\rho)n$. A similar argument implies that, for each $z \in Z'$, $d_F^+(z') \geq d_G(z') - 6\rho n$. Hence (a) holds.

Suppose that zw is an edge in F with $c(zw) \neq c_+(w)$. Thus there is $\mathbf{z} = (z, c_z) \in X_1$ such that $c_z \neq c(zw)$. Note that by the definition of X_1 , $H' = H + xw_{\mathbf{z}} - w_{\mathbf{z}}z$ is a 1-path-cycle with parameters $(\mathbf{z}; \mathbf{y})$. Since $\text{dist}_H(w, x), \text{dist}_H(w, y), \text{dist}_H(w, z) \geq 2\rho n$, we have $\text{dist}_{H'}(w, z), \text{dist}_{H'}(w, y) \geq \rho n + 1$. Proposition 5.3(ii) implies that $H' + zw - ww_-$ is a 1-path-cycle with parameters $((w_-, c_-(w_-)); \mathbf{y})$. This implies (b), and (c) is proven similarly. \square

Let $W := \{w \in V(F) : d_F^-(w) \geq 20\rho n\}$ and $W' := \{w \in V(F) : d_F^-(w) \geq (1 - 2\sqrt{\alpha})|Z|\}$. Let W^* be the set of $w \in W'$ such that there exists a colour c_w^* and there are at most $10\rho n$ vertices $z \in N_G(w)$ with $c(zw) \neq c_w^*$.

Claim 5.8. $|W^*| \geq (\delta - 7\sqrt{\alpha})n$, $|W \setminus W^*| \leq 5\sqrt{\alpha}n$ and

$$\frac{1}{n} \sum_{z \in Z'} (d_G(z) - \delta n) + |W' \setminus W^*| \leq 4\sqrt{\alpha}n. \quad (5.1)$$

Proof of claim. If $|W \setminus W^*| > \sqrt{\alpha}n$, then Claim 5.7(a) implies that

$$\begin{aligned} (\delta - 6\rho)n|Z| &\leq e_F(Z, V(F)) \leq e_F(Z, W) + 20\rho n^2 \leq |Z||W| - 2\sqrt{\alpha}|Z||W \setminus W^*| + 20\rho n^2 \\ &\leq |Z||W| - 2\alpha|Z|n + 20\rho n^2 \leq (|W| - 2\alpha n + 80\rho n)|Z|, \end{aligned}$$

where the last inequality holds as $|Z| \geq n/4$ by Claim 5.6. This implies that $|W| > (\delta + \alpha)n$. By Claim 5.7(b) and (c), we have $|X_2| \geq |W|$, a contradiction. Hence,

$$|W \setminus W'| \leq \sqrt{\alpha n}.$$

Thus we have

$$e_F(Z, V(F)) \leq e_F(Z, W) + 20\rho n^2 \leq (|W'| + (\sqrt{\alpha} + 80\rho)n)|Z| \leq (|W'| + 2\sqrt{\alpha n})|Z|.$$

By Claim 5.7(a), we have

$$\begin{aligned} |W'| &\geq (\delta - 2\sqrt{\alpha} - 6\rho)n + \frac{1}{|Z|} \sum_{z \in Z'} (d_G(z) - \delta n) \\ &\geq (\delta - 3\sqrt{\alpha})n + \frac{1}{n} \sum_{z \in Z'} (d_G(z) - \delta n). \end{aligned} \quad (5.2)$$

Note that if $w \in W' \setminus W^*$, then $(w_-, c_-(w_-)), (w_+, c_+(w_+)) \in X_2$ by Claim 5.7(b) and (c). Thus $|X_2| \geq |W'| + |W' \setminus W^*|$. Since $|X_2| \leq (\delta + \alpha)n$, (5.2) implies that

$$\frac{1}{n} \sum_{z \in Z'} (d_G(z) - \delta n) + |W' \setminus W^*| \leq (\alpha + 3\sqrt{\alpha})n \leq 4\sqrt{\alpha n},$$

so (5.1) holds. Moreover, $|W' \setminus W^*| \leq 4\sqrt{\alpha n}$, so $|W \setminus W^*| \leq 5\sqrt{\alpha n}$. Together with (5.2), $|W^*| = |W'| - |W' \setminus W^*| \geq (\delta - 7\sqrt{\alpha})n$. \square

Recall that for each $w \in W^* \subseteq W'$, $d_F^-(w) \geq (1 - 2\sqrt{\alpha})|Z|$. So for each $w \in W^*$, the number of edges zw of colour c_w^* in G is at least

$$|\{z \in N_G(w) : c(zw) = c_w^*\}| \geq (1 - 2\sqrt{\alpha})|Z| - 10\rho n \geq |Z| - 3\sqrt{\alpha n}. \quad (5.3)$$

Since $\delta^e(G) \geq \delta n$, the left hand side of the inequality is bounded above by $(1 - \delta)n$. Thus $|Z| \leq (1 - \delta + 3\sqrt{\alpha})n$ and so Claim 5.6 implies that

$$|Z'| \geq (2\delta - 1 - 4\sqrt{\alpha})n. \quad (5.4)$$

Let Z^* be the set of vertices $z \in Z$ satisfying (iii). We now bound the size of Z^* from below.

Claim 5.9. $|Z^*| \geq (2\delta - 1 - 3\alpha^{1/4})n$.

Proof of claim. Let Z_1 be the set of $z \in Z'$ such that $d_G(z) \geq (\delta + 4\alpha^{1/4})n$. So (5.1) implies that

$$|Z_1| \leq \alpha^{1/4}n.$$

Let Z_2 be the set of $z \in Z$ such that $d_G(z, V(F) \setminus W) \geq 20\sqrt{\rho n}$. Note that

$$|Z_2| \leq e_F(Z, V(F) \setminus W) / 20\sqrt{\rho n} \leq \sqrt{\rho n}.$$

Let Z_3 be the set of $z \in Z$ such that there exist at least $4\alpha^{1/4}n$ vertices $w \in W^*$ with $c(zw) \neq c_w^*$. By (5.3), each $w \in W^*$ is incident with at most $3\sqrt{\alpha n}$ edges zw with $z \in Z$ and $c(zw) \neq c_w^*$. Hence

$$|Z_3| \leq 3\sqrt{\alpha n}^2 / (4\alpha^{1/4}n) < \alpha^{1/4}n.$$

For each $z \in Z \setminus (Z_2 \cup Z_3)$, the number of edges zw (in both G and F) such that $w \in W^*$ and $c(zw) = c_w^*$ is at least

$$d_G(z, W^*) - 4\alpha^{1/4}n \geq d_G(z) - 20\sqrt{\rho n} - |W \setminus W^*| - 4\alpha^{1/4}n \geq (\delta - 6\alpha^{1/4})n,$$

where the last inequality is due to Claim 5.8. Hence $Z^* \supseteq Z' \setminus (Z_1 \cup Z_2 \cup Z_3)$. Together with (5.4), we have $|Z^*| \geq (2\delta - 1 - 3\alpha^{1/4})n$. \square

Note that properties (i) and (ii) holds by Claims 5.8 and 5.9 and (5.3), and (iii) holds by our construction. To complete the proof, it suffices to show that W^* and Z^* are disjoint. For each $w \in W^*$, (ii) and (i) imply that

$$d_G(w) \geq d_G^c(w) - 1 + |Z^*| - 3\sqrt{\alpha}n \geq (3\delta - 1 - 3\alpha^{1/4} - 3\sqrt{\alpha})n > (\delta + 4\alpha^{1/4})n,$$

so $w \notin Z^*$ as required. \square

Let G be an edge-coloured graph and let H be 1-path-cycle with parameters $(\mathbf{x}; \mathbf{y})$ with path component P . Let H' be the 1-path-cycle with parameters $(\mathbf{y}; \mathbf{x})$ obtained from H by reversing the orientations of all edges. Let $Y_1(H) := X_1(H')$ and $Y_2(H) := X_2(H')$. We study the edges between $X_1(H) \cup X_2(H)$ and $Y_1(H) \cup Y_2(H)$ in the following lemma.

Lemma 5.10. *Let G be a critical edge-coloured graph on n vertices and let $\rho > 0$. Suppose that H is a properly coloured 1-path-cycle in G of maximum order. Suppose that H has parameters $(\mathbf{x}; \mathbf{y})$ and that $|H| < n$. Then for all $(x', c_{x'}) \in X_1(H) \cup X_2(H)$ and all $(y', c_{y'}) \in Y_1(H) \cup Y_2(H)$ such that $\text{dist}_H(x, y) \geq 2\rho n$, either $xy \notin E(G)$, $c(xy) = c_x$ or $c(xy) = c_y$.*

Proof. Consider any $\mathbf{x}' = (x', c_{x'}) \in X_1(H) \cup X_2(H)$ and any $\mathbf{y}' = (y', c_{y'}) \in Y_1(H) \cup Y_2(H)$ such that $\text{dist}_H(x, y) \geq 2\rho n$. To prove the lemma, it is sufficient to show that there exists a 1-path-cycle H_0 with $V(H_0) = V(H)$ and parameters $(\mathbf{x}'; \mathbf{y}')$. To see this suppose that $x'y' \in E(G)$ and $c_{x'} \neq c(xy) \neq c_{y'}$, then $H_0 + x'y'$ is a vertex-disjoint union of cycles each of length at least ρn . For $z \notin V(H)$, $(H_0 + x'y') \cup z$ is a 1-path-cycle contradicting the maximality of $|H|$.

We will only consider the case when $\mathbf{x}' \in X_2(H)$ and $\mathbf{y}' \in Y_2(H)$, since the other cases proved by similar (and simpler) arguments. Choose $\mathbf{z} = (z, c_z) \in X_1(H)$ and $\mathbf{v} = (v, c_v) \in Y_1(H)$ such that

- any pair of $\{x, y, x', y', z, v\}$ are distance at least $\rho n + 10$ apart in H ;
- $H' := H + xw_{\mathbf{z}} + zw_{\mathbf{x}'} - zw_{\mathbf{z}} - x'w_{\mathbf{x}'}$ is a 1-path-cycle with parameters $(\mathbf{x}'; \mathbf{y})$.
- $H + yw_{\mathbf{v}} + vw_{\mathbf{y}'} - vw_{\mathbf{v}} - y'w_{\mathbf{y}'}$ is a 1-path-cycle with parameters $(\mathbf{x}; \mathbf{y}')$.

Note that \mathbf{z} and \mathbf{v} exist since $\mathbf{x}' \in X_2(H)$ and $\mathbf{y}' \in Y_2(H)$. Since $\text{dist}_H(v, x), \text{dist}_H(v, y), \text{dist}_H(v, z) \geq \rho n + 10$, we have $\text{dist}_{H'}(v, x'), \text{dist}_{H'}(v, y) \geq \rho n + 1$. Proposition 5.3 implies that $H'' := H' + yw_{\mathbf{v}} - vw_{\mathbf{v}}$ is a 1-path-cycle with parameters $(\mathbf{x}'; \mathbf{v})$. By a similar argument, we deduce that $H'' + vw_{\mathbf{y}'} - y'w_{\mathbf{y}'}$ is a 1-path-cycle with parameters $(\mathbf{x}'; \mathbf{y}')$ as required. \square

The next lemma plays a key role in the proof of Lemma 5.1.

Lemma 5.11. *Let $\varepsilon, \rho, \alpha$ be such that $1/n \ll \alpha, \varepsilon \ll 1$. Let G be an edge-coloured graph on n vertices with $\delta^c(G) \geq \delta n + 1$. Then one of following statements holds*

- (a) G contains a properly coloured 1-path-cycle such that $|H| \geq \min\{n, (3\delta + \alpha/2)n/2\}$ and every cycle in H has length at least $\alpha n/100$;
- (b) there exist disjoint $W^*, Z^* \subseteq V(G)$ such that
 - (i) $|W^*| \geq (\delta - 7\sqrt{\alpha})n$ and $|Z^*| \geq (2\delta - 1 - 3\alpha^{1/4})n$;
 - (ii) for each $w \in W^*$, there exists a distinct colour c_w^* such that there are at least $|Z^*| - 3\sqrt{\alpha}n$ vertices $z \in Z^*$ such that $c(zw) = c_w^*$;
 - (iii) for each $z \in Z^*$, $d_G(z) \leq (\delta + 4\alpha^{1/4})n$ and there are at least $(\delta - 6\alpha^{1/4})n$ edges zw such that $w \in W^*$ and $c(zw) = c_w^*$.

Here we give a brief description of the proof. By Lemma 5.5, we may assume that $|X_1(H) \cup X_2(H)|$ is bounded below (or else (b) holds). Similarly $|Y_1(H) \cup Y_2(H)|$ is also bounded below. Using Lemma 5.10, we then show that $|H| \geq (3\delta + \alpha/2)n/2$ as desired.

Proof of Lemma 5.11. Let $\rho := \alpha/1000$. Let H be a properly coloured 1-path-cycle in G such that every cycle in H has length at least ρn . Suppose that $|H|$ is maximum. We may assume that $|H| < \min\{n, (3\delta + \alpha/2)n/2\}$ or else we are done. By Lemma 5.2, we further assume that H is a 1-path-cycle with parameters $\rho(\mathbf{x}; \mathbf{y})$.

Let $X := X_1(H) \cup X_2(H)$ and let $Y := Y_1(H) \cup Y_2(H)$. By Lemma 5.5, we may assume that $|X| \geq (\delta + \alpha)n$. Similarly, by reversing all orientation of H and Lemma 5.5, we may also assume that $|Y| \geq (\delta + \alpha)n$. Let S_X be the set of vertices $v \in V(H)$ such that $(v, c_+(v)), (v, c_-(v)) \in X$. Let $R_X := \{(x', c_{x'}) \in X : x' \notin S_X\}$. Note that

$$2|S_X| + |R_X| = |X| \geq (\delta + \alpha)n. \quad (5.5)$$

Consider any $\mathbf{y}' = (y', c_{y'}) \in Y$. Proposition 5.4 and Lemma 5.10 imply that

$$|N_G(\mathbf{y}')| \geq \delta n, \quad N_G(\mathbf{y}') \subseteq V(H), \quad |N_G(\mathbf{y}') \cap S_X| \leq 4\rho n. \quad (5.6)$$

If $R_X = \emptyset$, then

$$|H| \geq |N_G(\mathbf{y}')| + |S_X| - 4\rho n \geq \delta n + (\delta + \alpha)n/2 - 4\rho n \geq (3\delta + \alpha/2)n/2,$$

a contradiction. Thus $R_X \neq \emptyset$. Similarly, let S_Y be the set of vertices $v \in V(H)$ such that $(v, c_+(v)), (v, c_-(v)) \in Y$ and $R_Y := \{(y', c_{y'}) \in Y : y' \notin S_Y\}$.

Define F to be the auxiliary directed bipartite graph on vertex classes R_X and R_Y such that there exists a directed edge from $\mathbf{v} = (v, c_v)$ to $\mathbf{w} = (w, c_w)$ if and only if

- $\text{dist}_G(v, w) \geq 2\rho n$;
- vw is an edge in G with $c(vw) \neq c_v$.

By Lemma 5.10, F is an oriented graph, that is, F has no directed 2-cycle. Consider any $\mathbf{y}' = (y', c_{y'}) \in Y$. We have

$$\begin{aligned} d_F^+(\mathbf{y}') &\geq |N_G(\mathbf{y}') \cap R_X| - 4\rho n \geq |N_G(\mathbf{y}') \cap (R_X \cup S_X)| - 4\rho n - |N_G(\mathbf{y}') \cap S_X| \\ &\stackrel{(5.6)}{\geq} \delta n + |R_X| + |S_X| - |H| - 8\rho n \stackrel{(5.5)}{\geq} \frac{(3\delta + \alpha - 16\rho)n + |R_X|}{2} - |H|. \end{aligned}$$

Similarly, for any $\mathbf{x}' \in R_X$, $d_F^+(\mathbf{x}') \geq \frac{(3\delta + \alpha - 16\rho)n + |R_Y|}{2} - |H|$. Since F is an oriented graph, we have

$$\begin{aligned} |R_X||R_Y| &\geq e(F) \geq \sum_{\mathbf{x}' \in R_X} d_F^+(\mathbf{x}') + \sum_{\mathbf{y}' \in R_Y} d_F^+(\mathbf{y}') \\ &\geq |R_X| \left(\frac{(3\delta + \alpha - 16\rho)n + |R_Y|}{2} - |H| \right) + |R_Y| \left(\frac{(3\delta + \alpha - 16\rho)n + |R_X|}{2} - |H| \right), \\ &0 \geq (|R_X| + |R_Y|)((3\delta + \alpha - 16\rho)n/2 - |H|). \end{aligned}$$

This implies that $|H| \geq (3\delta + \alpha - 16\rho)n/2 \geq (3\delta + \alpha/2)n/2$ as $R_X \cup R_Y \neq \emptyset$, a contradiction. \square

When $\delta \geq 2/3$, Lemma 5.11 implies Lemma 5.1. For $1/2 < \delta < 2/3$, we present a rough sketch proof of Lemma 5.1 using Lemma 5.11. Suppose that Lemma 5.1 holds for any δ' with $\delta' > \delta$. Apply Lemma 5.11 and we may assume that Lemma 5.11(b) holds (or else we are done). Thus there exist disjoint $Z^*, W^* \subseteq V(G)$ satisfying Lemma 5.11(b). Let $\delta^* := (\delta - 4\alpha^{1/8})n/|G \setminus Z^*|$. So $\delta^* > \delta$. If $d^c(v, Z^*) \leq 4\alpha^{1/8}n$ for all vertices $v \notin Z^*$, then $\delta^c(G \setminus Z^*) \geq (\delta - 4\alpha^{1/8})n = \delta^*|G \setminus Z^*|$. Since $\delta^* > \delta$, we apply Lemma 5.1 to $G \setminus Z^*$. We

have either a large enough properly coloured 1-path-cycle or $G \setminus Z^*$ is $(\delta^*, \varepsilon^*)$ -extremal for some small ε^* or both. In the second case, we then show that G is (δ, ε) -extremal. This argument is formalised in the lemma below.

We would need the following notation. For $\phi \geq 0$, let $I_0(\phi) := [2/3 - \phi, 1)$. For $s \in \mathbb{N}$, let $I_s(\phi) := \{p \in [0, 1) \setminus \bigcup_{0 \leq i < s} I_i(\phi) : \frac{p-\phi}{3/2-p} \in I_{s-1}(\phi)\}$. Let $s_\phi(\delta)$ be the integer s such that $\delta \in I_s(\phi)$.

Lemma 5.12. *Let $0 < 1/n \ll \alpha_{s_\phi(\delta)} \ll \alpha_{s_\phi(\delta)-1} \ll \dots \ll \alpha_0 \ll \phi \ll \varepsilon \ll 1/2 \ll \delta \leq \delta^* < 1$. Suppose that $4^{s_\phi(\delta^*)}\varepsilon \ll \delta - 1/2$, and that G is a critical edge-coloured graph on $n^* \geq 2^{s_\phi(\delta^*)}n$ vertices with $\delta^c(G) \geq \delta^*n^* + 1$. Then one of the following statements holds:*

- (i*) G contains a properly coloured 1-path-cycle H such that $|H| \geq (3\delta^* + \alpha_{s_\phi(\delta^*)}/2)n^*/2$ and every cycle in H has length at least $\alpha_{s_\phi(\delta^*)}n^*/100$;
- (ii*) G is $(\delta^*, 4^{s_\phi(\delta^*)}\varepsilon)$ -extremal.

Proof. Fix δ^* and write s^* and α for $s_\phi(\delta^*)$ and $\alpha_{s_\phi(\delta^*)}$, respectively. Without loss of generality, $\delta^* \leq 2/3$. Suppose that G satisfies the hypothesis. Apply Lemma 5.11 to G with $\rho = \alpha_{s^*}/100$. We may assume that Lemma 5.11(b) holds or else we are done. Thus there exist disjoint $W^*, Z^* \subseteq V(G)$ such that

- (i') $|W^*| \geq (\delta^* - 7\sqrt{\alpha})n^*$ and $|Z^*| \geq (2\delta^* - 1 - 3\alpha^{1/4})n^*$;
- (ii') for each $w \in W^*$, there exists a distinct colour c_w^* such that there are at least $|Z^*| - 3\sqrt{\alpha}n^*$ vertices $z \in Z^* \cap N_G(w)$ such that $c(zw) = c_w^*$;
- (iii') for each $z \in Z^*$, $d_G(z) \leq (\delta^* + 4\alpha^{1/4})n^*$ and there are at least $(\delta^* - 6\alpha^{1/4})n^*$ edges zw such that $w \in W^* \cap N_G(z)$ and $c(zw) = c_w^*$.

First suppose that $s^* = 0$. Since $\delta^* \geq 2/3 - \phi$ and $\alpha, \phi \ll \varepsilon$, (i') implies that

$$|Z^*| \geq (2\delta^* - 1 - 3\alpha^{1/4})n^* = (1 - \delta^* + (3\delta^* - 2) - 3\alpha^{1/4})n^* \geq (1 - \delta^* - \varepsilon)n^*.$$

Thus G is (δ^*, ε) -extremal. So we may assume that $s \geq 1$ and the lemma holds for all $s' < s$.

Let F be the subgraph of G induced by edges zv such that $z \in Z^*$ and either $v \notin W^*$ or $v \in W^*$ with $c(zv) \neq c_v$. Note that by (iii'), $e(F) \leq 10\alpha^{1/4}n^*|Z^*|$. Let V_F be the set of vertices v such that $d_F(v) \geq 5\alpha^{1/8}n^*$. So $|V_F| \leq 5\alpha^{1/8}n^*$. For any $w \in W^*$, (i') and (ii') imply that

$$d_G(w) \geq (d_G^c(w) - 1) + |Z^*| - 3\sqrt{\alpha}n^* \geq (3\delta^* - 1 - 4\alpha^{1/4})n^*. \quad (5.7)$$

We split the proof into two cases depending on the value of δ^* .

Case 1: $\delta^* < \frac{3(1-15\alpha^{1/8})}{5(1-10\alpha^{1/8})}$. Let Z_1 be a subset of Z^* of size $|Z_1| = (\delta^* - 1/2)n^* - |V_F|$ and let $Z_2 := Z^* \setminus Z_1$. Note that by (i'),

$$|Z_2| \geq (\delta^* - 1/2 - 3\alpha^{1/4})n^*. \quad (5.8)$$

Let $G' := G \setminus (Z_1 \cup V_F)$. We claim that

$$\delta^c(G') \geq (\delta^* - 10\alpha^{1/8})n^* + 1 \quad (5.9)$$

If $v \in V \setminus W^*$, then $d_G^c(v, Z_1 \cup V_F) \leq d_G(v, Z^*) + |V_F| \leq d_F(v) + |V_F| \leq 10\alpha^{1/8}n^*$. If $w \in W^*$, then by (ii'), $d_G^c(w, Z_1 \cup V_F) \leq d_G^c(w, Z^*) + |V_F| \leq 1 + 3\sqrt{\alpha}n^* + |V_F| \leq 10\alpha^{1/8}n^*$. Hence (5.9) holds.

Let

$$n' := |G'| = (3/2 - \delta^*)n^* \quad \text{and} \quad \delta' := \frac{\delta^* - 10\alpha^{1/8}}{3/2 - \delta^*} \geq \frac{\delta^* - \phi}{3/2 - \delta^*}.$$

Note that $s_\phi(\delta') < s^*$, $\alpha n^* \ll \alpha_{s_\phi(\delta')} n'$ and $\delta^c(G') \geq \delta' n' + 1$. Also,

$$\frac{(3\delta' + \alpha_{s_\phi(\delta')}/2)n'}{2} = \frac{3(\delta^* - 10\alpha^{1/8})n^* + \alpha_{s_\phi(\delta')}n'/2}{2} > \frac{3(\delta^* + \alpha/2)n^*}{2}.$$

By our assumption on δ^* , we have $(3\delta' + \alpha/2)n'/2 < n'$. Clearly, $|G'| \geq n^*/2 \geq 2^{s_\phi(\delta')}n$. Let $\varepsilon' := 4^{s_\phi(\delta')}\varepsilon$. By induction hypothesis, we may assume that G' is (δ', ε') -extremal (or else we are done). Thus there exist disjoint $A', B' \subseteq V(G')$ such that

- (A1') $|A'| \geq (\delta' - \varepsilon')n'$ and $|B'| \geq (1 - \delta' - \varepsilon')n'$;
- (A2') for each $a \in A'$, there exists a distinct colour c'_a such that there are at least $|B'| - \varepsilon'n'$ vertices $b \in B'$ such that $c(ab) = c'_a$;
- (A3') for each $b \in B'$, $d_G(b) \leq (\delta' + \varepsilon')n'$ and b has at least $|A'| - \varepsilon'n'$ neighbours $a \in A'$ such that $c(ab) = c'_a$.

Let $U' := V(G') \setminus (A' \cup B')$, so $|U'| \leq 2\varepsilon'n'$. Recall that $W^* \subseteq V(G')$ and that $\varepsilon', \alpha \ll \delta^* - 1/2$. For any $w \in W^*$,

$$\begin{aligned} d_{G'}(w) &\geq d_G(w) - |Z_1 \cup V_F| \stackrel{(5.7)}{\geq} (3\delta^* - 1 - 4\alpha^{1/4})n^* - (\delta^* - 1/2)n^* \\ &= (2\delta^* - 1/2 - 4\alpha^{1/4})n^* \geq (\delta^* + \varepsilon')n^* \geq (\delta' + \varepsilon')n'. \end{aligned}$$

Therefore $W^* \cap B' = \emptyset$ by (A3'). Let $A := W^* \cap A'$. So

$$|A| \geq |W^*| - |U'| \stackrel{(i')}{\geq} (\delta^* - 7\sqrt{\alpha})n^* - 2\varepsilon'n' \geq (\delta^* - 4^{s^*}\varepsilon)n^* \quad (5.10)$$

and $|A' \setminus A| \leq (\delta' + \varepsilon')n' - |A| \leq 2 \cdot 4^{s^*}\varepsilon n^*$. Since $Z_2 \cap W^* = \emptyset$, we have $Z_2 \cap A' \subseteq A \cap A'$. Hence

$$|Z_2 \cap B'| \geq |Z_2| - |Z_2 \cap A'| - |Z_2 \setminus (A' \cup B')| \geq |Z_2| - |A \cap A'| - |U'| \stackrel{(5.8)}{>} 3\sqrt{\alpha}n^* + \varepsilon'n'.$$

Consider any $a \in A$. By (ii') and (A2'), there exists vertex $z \in Z_2 \cap B'$ such that $c_a^* = c(az) = c'_a$. Therefore we have $c_a^* = c'_a$ for all $a \in A$.

Let $B := B' \cup Z_1$. Note that

$$|B| = |V(G) \setminus (A' \cup U' \cup V(F))| \geq n^* - |A'| - |U'| - |V_F| \geq (1 - \delta - 4^{s^*}\varepsilon)n. \quad (5.11)$$

We now claim that G is $(\delta, 4^{s^*}\varepsilon)$ -extremal with partition (A, B) . Note that (A1) holds by (5.10) and (5.11). Statements (ii') and (A2') imply (A2). Similarly, statements (iii') and (A3') imply (A3).

Case 2: $\delta^* \geq \frac{3(1-15\alpha^{1/8})}{5(1-10\alpha^{1/8})}$. Note that $s^* = 1$. Case 2 is proved via a similar argument used in Case 1, where we let Z_1 be a subset of Z^* of size $|Z_1| = (1 - (3\delta^* + \alpha/2)/2)n^* - |V_F|$. \square

We now prove Lemma 5.1 by choosing $\phi, \alpha_0, \alpha_1, \dots, \alpha_{s_\phi(\delta)}$ appropriately.

Proof of Lemma 5.1. Let $s_0 := s_0(\delta)$ and let $\varepsilon' := 4^{-2s_0}\varepsilon$. Choose $\beta \ll \phi \ll \varepsilon', \delta - 1/2$ such that $s_\phi(\delta) \leq 2s_0$. So $4^{s_\phi(\delta)}\varepsilon' \leq \varepsilon$. Next choose $\beta < \alpha_{s_\phi(\delta)} \ll \alpha_{s_\phi(\delta)-1} \ll \dots \ll \alpha_0 \ll \phi$. Therefore, Lemma 5.12 with ε' playing the role of ε implies Lemma 5.1. \square

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