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Evseev, Anton; Kleshchev, Alexander

DOI:
10.4007/annals.2018.188.2.2

Document Version
Peer reviewed version

Citation for published version (Harvard):
Evseev, A & Kleshchev, A 2018, 'Blocks of symmetric groups, semicuspidal KLR algebras and zigzag Schur-Weyl duality', *Annals of Mathematics*, vol. 188, no. 2, pp. 453-512. https://doi.org/10.4007/annals.2018.188.2.2

BLOCKS OF SYMMETRIC GROUPS, SEMICUSPIDAL KLR ALGEBRAS AND ZIGZAG SCHUR-WEYL DUALITY

ANTON EVSEEV AND ALEXANDER KLESHCHEV

We record with deep sadness the passing of Anton Evseev on February 21, 2017.

Abstract. We prove Turner's conjecture, which describes the blocks of the Hecke algebras of the symmetric groups up to derived equivalence as certain explicit Turner double algebras. Turner doubles are Schur-algebra-like 'local' objects, which replace wreath products of Brauer tree algebras in the context of the Broué abelian defect group conjecture for blocks of symmetric groups with non-abelian defect groups. The main tools used in the proof are generalized Schur algebras corresponding to wreath products of zigzag algebras and imaginary semicuspidal quotients of affine KLR algebras.

1. Introduction

Let $H_n(q)$ be the Hecke algebra of the symmetric groups $\mathfrak{S}_n$ over a field $\mathbb{F}$ with parameter $q \in \mathbb{F}^\times$. An important special case is $q = 1$, when $H_n(q) = \mathbb{F}\mathfrak{S}_n$. Let e be the quantum characteristic of q. In this paper we assume that $e > 0$, i.e. there exists $k \in \mathbb{Z}_{>0}$ such that $1 + q + \cdots + q^{k-1} = 0$, and e is the smallest such k.

Representations of $H_n(q)$ for all $n \geq 0$ categorify the basic module $V(\Lambda_0)$ with highest weight Λ_0 of the affine Kac-Moody Lie algebra $\mathfrak{g} = \widehat{\mathfrak{sl}}_e(\mathbb{C})$, see for example [$\mathbf{A}_1$,$\mathbf{A}_2$,$\mathbf{Gro}$,$\mathbf{K}_1$]. In particular, each weight space $V(\Lambda_0)_{\Lambda_0-\alpha}$ for α in the positive part of the root lattice is identified with the complexified Grothendieck group of the corresponding block $H_\alpha(q)$ of some $H_n(q)$.

The Weyl group W of $\mathfrak{g}$ acts on the weights of $V(\Lambda_0)$, and the orbits are precisely $\mathcal{O}_d := W(\Lambda_0 - d\delta) = W\Lambda_0 - d\delta$ for all $d \in \mathbb{Z}_{\geq 0}$, where δ is the null-root. Chuang and Rouquier [**CR**] lift this action of W on the weights to derived equivalences between the corresponding blocks. Therefore, all blocks $H_\alpha(q)$ with $\Lambda_0 - \alpha \in \mathcal{O}_d$ for a fixed d are derived equivalent.

Moreover, for every $d \in \mathbb{Z}_{\geq 0}$, Rouquier [$\mathbf{R}_1$] and Chuang and Kessar [**CK**] identify special representatives $\Lambda_0 - \alpha \in \mathcal{O}_d$ for which the corresponding blocks $H_\alpha(q)$ have a particularly nice structure. These blocks are known as *RoCK blocks*. Thus for any n, every block of $H_n(q)$ is derived equivalent to a RoCK block.

Let $H_\alpha(q)$ be a RoCK block. Turner [$\mathbf{Tu}_1, \mathbf{Tu}_2, \mathbf{Tu}_3$] developed a theory of double algebras and conjectured that $H_\alpha(q)$ is Morita equivalent to an appropriate double [$\mathbf{Tu}_1$, Conjecture 165]. The aim of this paper is to prove Turner's

2010 *Mathematics Subject Classification.* 20C08, 20C30, 20G43.

The first author was supported by the EPSRC grant EP/L027283/1 and thanks the Max-Planck-Institut for hospitality. The second author was supported by the NSF grant DMS-1161094, the Max-Planck-Institut and the Fulbright Foundation.

Conjecture. In fact, we prove a slightly more general result stated in terms of cyclotomic KLR algebras over $\mathbb{Z}$.

To state the result precisely, we now recap Turner's theory as developed in [**EK**]. Let Q be a type A_{e-1} quiver, and let P_Q be the quotient of the path algebra $\mathbb{Z}Q$ by all paths of length 2. As a $\mathbb{Z}$-module, P_Q has an obvious basis whose elements are identified with the vertices and the edges of Q. We view P_Q as a $\mathbb{Z}$-superalgebra with $P_{Q,\bar{0}}$ being the span of the vertices and $P_{Q,\bar{1}}$ being the span of the edges. We denote by $\bar{x} \in \{\bar{0}, \bar{1}\}$ the degree of a homogeneous element x of any superalgebra. Let $n \in \mathbb{Z}_{>0}$, and consider the matrix superalgebra $X := M_n(P_Q)$.

For every $d \in \mathbb{Z}_{\geq 0}$ we have a superalgebra structure on $X^{\otimes d}$ induced by that on X. So $\bigoplus_{d\geq 0} X^{\otimes d}$ is a superalgebra, with the product on each summand $X^{\otimes d}$ being as above, and $xy = 0$ for $x \in X^{\otimes d}$ and $y \in X^{\otimes f}$ with $d \neq f$. In fact, $\bigoplus_{d\geq 0} X^{\otimes d}$ is even a *superbialgebra* with the coproduct

$$\Delta\colon X^{\otimes d} \to \bigoplus_{0\leq f\leq d} X^{\otimes f} \otimes X^{\otimes (d-f)},$$
$$\xi_1 \otimes \cdots \otimes \xi_d \mapsto \sum_{0\leq f\leq d} (\xi_1 \otimes \cdots \otimes \xi_f) \otimes (\xi_{f+1} \otimes \cdots \otimes \xi_d).$$

The symmetric group $\mathfrak{S}_d$ acts on $X^{\otimes d}$ by signed place permutations with superalgebra automorphisms, so the set of fixed points $\mathtt{Inv}^d X := \left(X^{\otimes d}\right)^{\mathfrak{S}_d}$ is a subsuperalgebra of $X^{\otimes d}$, and $\mathtt{Inv}\, X := \bigoplus_{d\geq 0} \mathtt{Inv}^d X$ is a subsuperbialgebra of $\bigoplus_{d\geq 0} X^{\otimes d}$.

There is a superbialgebra structure on $(\mathtt{Inv}\, X)^* := \bigoplus_{d\geq 0} (\mathtt{Inv}^d X)^*$ which is dual to that on $\mathtt{Inv}\, X$. We also have an $\mathtt{Inv}\, X$-bimodule structure on $(\mathtt{Inv}\, X)^*$ defined by

$$(x \cdot \xi)(\eta) = x(\xi\eta),\ (\xi \cdot x)(\eta) = x(\eta\xi) \qquad (\xi, \eta \in \mathtt{Inv}\, X,\ x \in (\mathtt{Inv}\, X)^*).$$

The *Turner double* is the superalgebra $DX := \mathtt{Inv}\, X \otimes (\mathtt{Inv}\, X)^*$, with the product defined, using Sweedler's notation for the coproduct Δ, by

$$(\xi \otimes x)(\eta \otimes y) = \sum (-1)^{\bar{\xi}_{(1)}(\bar{\xi}_{(2)}+\bar{\eta}+\bar{x})+\bar{\eta}_{(1)}\bar{x}} \xi_{(2)}\eta_{(1)} \otimes (x \cdot \eta_{(2)})(\xi_{(1)} \cdot y)$$

for all homogeneous $\xi, \eta \in \mathtt{Inv}\, X$ and $x, y \in (\mathtt{Inv}\, X)^*$. We have a decomposition $DX = \bigoplus_{d\geq 0} D^d X$ as a direct sum of superalgebras, where

$$D^d X := \bigoplus_{0\leq f\leq d} \mathtt{Inv}^f X \otimes (\mathtt{Inv}^{d-f} X)^*$$

is a direct sum of $\mathbb{Z}$-supermodules. Each superalgebra $D^d X$ is symmetric.

The superalgebra P_Q is $\mathbb{Z}$-graded with all vertices of Q being in degree 0 and all edges in degree 1. This induces gradings on $M_n(P_Q)$ and $\mathtt{Inv}\, X = \mathtt{Inv}\, M_n(P_Q)$ in the natural way. We view each $(\mathtt{Inv}^d X)^*$ as a graded $\mathbb{Z}$-module, with the grading induced from $\mathtt{Inv}^d X$ shifted by $2d$, i.e. for $x \in (\mathtt{Inv}^d X)^*$ we have $\deg x = m$ if and only if x is zero on all graded components of $\mathtt{Inv}^d X$ other than the $(2d-m)$th component. Then $D^d X$ is a $\mathbb{Z}$-graded superalgebra concentrated in degrees $0, 1, \ldots, 2d$. In fact, this grading is a refinement of the superstructure on

D^dX, in the sense that $(D^dX)_{\bar{0}}$ is the sum of even graded components of D^dX and $(D^dX)_{\bar{1}}$ is the sum of odd graded components. From now on, we forget the superstructure on D^dX and view

$$D_Q(n,d) := D^dX$$

simply as a graded $\mathbb{Z}$-algebra.

As before, let $H_\alpha(q)$ be a RoCK block, with $\Lambda_0 - \alpha \in \mathcal{O}_d$. The precise conditions that α must satisfy in order for this to be the case are stated in §5.4. Let $R_\alpha^{\Lambda_0}$ be the corresponding cyclotomic KLR algebra, which has a natural grading, see §4.2.

Theorem A. *If $n \geq d$, then the $\mathbb{Z}$-algebras $R_\alpha^{\Lambda_0}$ and $D_Q(n,d)$ are graded Morita equivalent.*

For any graded $\mathbb{Z}$-algebra A, define $A_{\mathbb{F}} := A \otimes_{\mathbb{Z}} \mathbb{F}$. The $\mathbb{F}$-algebra $R_{\alpha,\mathbb{F}}^{\Lambda_0}$ is isomorphic to the RoCK block $H_\alpha(q)$ of a Hecke algebra, see [$\mathbf{BK}_1$,$\mathbf{R}_2$]. Applying this result and the aforementioned theorem of Chuang-Rouquier, one deduces the following from Theorem A:

Corollary B. *If $n \geq d$, then:*

(i) *The RoCK block $H_\alpha(q)$ is Morita equivalent to $D_Q(n,d)_{\mathbb{F}}$.*
(ii) *For every β with $\Lambda_0 - \beta \in \mathcal{O}_d$, the algebra $H_\beta(q)$ is derived equivalent to $D_Q(n,d)_{\mathbb{F}}$.*

Alperin's Weight Conjecture [**Al**] predicts an equality between the number of simple modules of an arbitrary block of a finite group and the number of 'weights' defined in terms of normalisers of local p-subgroups. In the case of blocks with abelian defect group, the conjecture of Broué [**Br**] lifts Alperin's Weight Conjecture to the categorical level, but no such categorical conjecture is known for blocks of arbitrary finite groups with non-abelian defect groups.

An important step in the proof of Broué's conjecture for symmetric groups is the theorem [**CK**] asserting that, if $q = 1$ and $d < \operatorname{char} \mathbb{F}$, then there is a Morita equivalence between the RoCK block $H_\alpha(1)$ and the wreath product algebra $H_\delta(1)^{\otimes d} \rtimes \mathbb{F}\mathfrak{S}_d$. Corollary B shows that, for an arbitrary block of a symmetric group, the corresponding double $D_Q(n,d)_{\mathbb{F}}$ is a 'local' object that can replace $H_\delta(1)^{\otimes d} \rtimes \mathbb{F}\mathfrak{S}_d$ in the context of Broué's conjecture.

In fact, the wreath product $H_\delta(q)^{\otimes d} \rtimes \mathbb{F}\mathfrak{S}_d$ has a $\mathbb{Z}$-form $(R_\delta^{\Lambda_0})^{\otimes d} \rtimes \mathbb{Z}\mathfrak{S}_d$ that is closely related to $D_Q(n,d)$. Denote by Z the zigzag algebra of type A_{e-1} over $\mathbb{Z}$, and consider the wreath product $W_d := \mathsf{Z}^{\otimes d} \rtimes \mathbb{Z}\mathfrak{S}_d$, see §3.1. Then Z is Morita equivalent to $R_\delta^{\Lambda_0}$, and more generally W_d is (graded) Morita equivalent to $(R_\delta^{\Lambda_0})^{\otimes d} \rtimes \mathbb{F}\mathfrak{S}_d$, see the proof of Proposition 8.29. On the other hand, the double $D_Q(n,d)$ can be identified with a subalgebra of a *generalized Schur algebra* $S^{\mathsf{Z}}(n,d)$, and there is a Schur-Weyl duality between $S^{\mathsf{Z}}(n,d)$ and W_d, see §3.2. In particular, for a certain explicit idempotent $\xi_\omega \in D_Q(n,d)$, we have

$$\xi_\omega D_Q(n,d)\xi_\omega = \xi_\omega S^{\mathsf{Z}}(n,d)\xi_\omega \cong W_d.$$

Thus, the idempotent truncation $\xi_\omega D_Q(n,d)\xi_\omega$ is Morita equivalent to $R_\delta^{\Lambda_0} \rtimes \mathbb{F}\mathfrak{S}_d$.

If $d < \operatorname{char} \mathbb{F}$ or $\operatorname{char} \mathbb{F} = 0$, the double $D_Q(n,d)_{\mathbb{F}}$ is Morita equivalent to

$$\xi_\omega D_Q(n,d)_{\mathbb{F}}\xi_\omega \cong (R_{\delta,\mathbb{F}}^{\Lambda_0})^{\otimes d} \rtimes \mathbb{F}\mathfrak{S}_d \cong H_\delta(q)^{\otimes d} \rtimes \mathbb{F}\mathfrak{S}_d,$$

see Proposition 8.29. When $d \geq \operatorname{char} \mathbb{F} > 0$, the algebra $D_Q(n,d)_{\mathbb{F}}$ has more isomorphism classes of simple modules than $\xi_\omega D_Q(n,d)_{\mathbb{F}} \xi_\omega$. As was predicted in [$\mathbf{Tu}_1$, Conjecture 82] and proved in [**Ev**], a certain explicit idempotent truncation of the RoCK block $H_\alpha(q)$ is Morita equivalent to $H_\delta(q)^{\otimes d} \rtimes \mathbb{F}\mathfrak{S}_d$ in all cases. Corollary B(i) strengthens this result, replacing the idempotent truncation by the *whole* RoCK block $H_\alpha(q)$.

We now outline the proof of Theorem A and the contents of the paper. Section 2 contains some general definitions and notation. In Section 3, we review necessary definitions and results from [**EK**]. In particular, we introduce the zigzag algebra Z and the wreath product W_d. An important role is played by the (right) *colored permutation W_d-modules* $M_{\lambda,\boldsymbol{c}}$, which are parametrized by *colored compositions* $(\lambda, \boldsymbol{c})$. Here, λ is a composition of d and $\boldsymbol{c}$ is a tuple consisting of elements of $\{1, \dots, e-1\}$ that assigns a 'color' to each part of λ. In fact, the proof of Theorem A uses only colored compositions with $n(e-1)$ parts of the form $(\lambda, \boldsymbol{c}^0)$, where $\boldsymbol{c}^0$ is given by (3.13), but it is more natural to work with more general colored compositions. We define the generalized Schur algebra $S^{\mathsf{Z}}(n,d)$ as the endomorphism algebra of the direct sum of the appropriate W_d-modules $M_{\lambda,\boldsymbol{c}^0}$ and review results identifying the Turner double $D_Q(n,d)$ as an explicit $\mathbb{Z}$-subalgebra of $S^{\mathsf{Z}}(n,d)$.

Section 4 begins with the definition and standard properties of the KLR algebras R_θ and their cyclotomic quotients $R_\theta^{\Lambda_0}$. In §4.5 we define the *semicuspidal algebra* $\hat{C}_{d\delta}$ as an explicit quotient of $R_{d\delta}$. In 4.6, we consider parabolic subalgebras of $\hat{C}_{d\delta}$.

In §5.4, we recall the definition of a RoCK block $R_\alpha^{\Lambda_0}$ and construct a natural homomorphism $\Omega\colon \hat{C}_{d\delta} \to R_\alpha^{\Lambda_0}$. The quotient $C_{\rho,d} := \hat{C}_{d\delta}/\ker\Omega$ is isomorphic to an idempotent truncation of $R_\alpha^{\Lambda_0}$, which is later shown to be Morita equivalent to $R_\alpha^{\Lambda_0}$. We note that $C_{\rho,d}$ is finitely generated as a $\mathbb{Z}$-module, but $\hat{C}_{d\delta}$ is not. The arguments of §5.4 rely on results connecting cyclotomic KLR algebras with the combinatorics of standard tableaux and abaci, which are reviewed and developed in §§5.1–5.3.

In §6.1 we define the *Gelfand-Graev idempotent* $\gamma^{\lambda,\boldsymbol{c}} \in R_{d\delta}$ for every colored composition $(\lambda, \boldsymbol{c})$ of d and an 'uncolored' idempotent $\gamma^\omega \in R_{d\delta}$. The latter may be viewed as a KLR counterpart of $\xi_\omega \in S^{\mathsf{Z}}(n,d)$. The following two results are key to the proof of Theorem A:

(i) There is an explicit algebra isomorphism $W_d \xrightarrow{\sim} \gamma^\omega C_{\rho,d} \gamma^\omega$ (see Theorem 8.9).
(ii) If $\gamma^\omega C_{\rho,d}\gamma^\omega$ is identified with W_d via the isomorphism in (i), then there is an explicit isomorphism $M_{\lambda,\boldsymbol{c}} \xrightarrow{\sim} \gamma^{\lambda,\boldsymbol{c}} C_{\rho,d}\gamma^\omega$ of right W_d-modules (see Theorem 8.15).

The isomorphism (i) is a slight generalization of the main result of [**Ev**] and is constructed in §§7.1,8.1 using a homomorphism [$\mathbf{KM}_3$] from W_d to $\gamma^\omega \hat{C}_{d\delta}\gamma^\omega$. In order to prove (ii), we first show that $\gamma^{\lambda,\boldsymbol{c}} C_{\rho,d}\gamma^\omega$ and $M_{\lambda,\boldsymbol{c}}$ have the same rank as free $\mathbb{Z}$-modules, see Corollary 6.31. This relies on combinatorial results about RoCK blocks proved in §§6.2–6.4. Secondly, in §§7.3,7.4, we prove several results on the structure of $\gamma^{\lambda,\boldsymbol{c}} \hat{C}_{d\delta}\gamma^\omega$. In particular, we find an explicit element that

generates $\gamma^{\lambda,\boldsymbol{c}}\hat{C}_{d\delta}\gamma^{\omega}$ as a right $\gamma^{\omega}\hat{C}_{d\delta}\gamma^{\omega}$-module, see Corollary 7.32. We use this element to construct a homomorphism from $M_{\lambda,\boldsymbol{c}}$ to $\gamma^{\lambda,\boldsymbol{c}}\hat{C}_{d\delta}\gamma^{\omega}$ and ultimately to prove (ii).

In §8.3, we define the algebra $E(n,d)$ as the endomorphism algebra of the direct sum of (graded shifts of) certain projective left $C_{\rho,d}$-modules $C_{\rho,d}\gamma^{\lambda,\boldsymbol{c}}$. Using the right W_d-modules $\gamma^{\lambda,\boldsymbol{c}}C_{\rho,d}\gamma^{\omega}$ and the isomorphism (ii), we construct a natural homomorphism $\Phi\colon E(n,d)\to S^{\mathsf{Z}}(n,d)$. Finally, using the identification of the Turner double $D_Q(n,d)$ as a subalgebra of $S^{\mathsf{Z}}(n,d)$ stated in Section 3 as well as results about the semicuspidal algebra proved in Section 7, we show that Φ is injective with image exactly $D_Q(n,d)$, so that $E(n,d)\cong D_Q(n,d)$ (see Theorem 8.23).

A priori, it follows from our constructions that $E(n,d)$ is Morita equivalent to an idempotent truncation of the RoCK block $R_\alpha^{\Lambda_0}$. In §8.4, we prove that $E(n,d)\cong D_Q(n,d)$ is (graded) Morita equivalent to $R_\alpha^{\Lambda_0}$ by showing that the scalar extensions of $D_Q(n,d)$ and $R_\alpha^{\Lambda_0}$ to any algebraically closed field have the same number of simple modules.

2. Preliminaries

For any $m,n\in\mathbb{Z}$, we define the (possibly empty) *segment*

$$[m,n]:=\{l\in\mathbb{Z}\mid m\le l\le n\}.$$

Let $l,m,n\in\mathbb{Z}_{\ge0}$ and I be a set. For any $i\in I$ and tuples $\boldsymbol{i}=(i_1,\dots,i_l)\in I^l$, $\boldsymbol{j}=(j_1,\dots,j_m)\in I^m$, we set

$$i^n:=\underbrace{(i,\dots,i)}_{n}\in I^n,\quad \boldsymbol{ij}:=(i_1,\dots,i_l,j_1,\dots,j_m)\in I^{l+m},\quad \boldsymbol{i}^n:=\underbrace{\boldsymbol{i}\dots\boldsymbol{i}}_{n}\in I^{ln}.$$

We write $i_1\dots i_l$ instead of $(i_1,\dots,i_l)$ when there is no possibility of confusion.

2.1. **Partitions and compositions.** Fix $n\in\mathbb{Z}_{>0}$ and $d\in\mathbb{Z}_{\ge0}$. We denote by $\Lambda(n)$ the set of compositions $\lambda=(\lambda_1,\dots,\lambda_n)$ with $\lambda_1,\dots,\lambda_n\in\mathbb{Z}_{\ge0}$. For $\lambda\in\Lambda(n)$ we write $|\lambda|:=\lambda_1+\dots+\lambda_n$, and set

$$\Lambda(n,d):=\{\lambda\in\Lambda(n)\mid|\lambda|=d\}.$$

If $m\in\mathbb{Z}_{\ge0}$, we define $m\lambda:=(m\lambda_1,\dots,m\lambda_n)\in\Lambda(n)$.

Let S be an arbitrary finite set. We define $\Lambda^S(n,d)$ to be the set of tuples $\boldsymbol{\lambda}=(\lambda^{(i)})_{i\in S}$ of compositions in $\Lambda(n)$ such that $\sum_{i\in S}|\lambda^{(i)}|=d$.

We denote by $\mathscr{P}$ the set of all partitions. For $\lambda=(\lambda_1,\dots,\lambda_m)\in\mathscr{P}$ we write $\ell(\lambda):=\max\{k\mid\lambda_k>0\}$ and $|\lambda|:=\lambda_1+\dots+\lambda_m$. We set

$$\mathscr{P}(d):=\{\lambda\in\mathscr{P}\mid|\lambda|=d\}.$$

We do not assume that the parts λ_k of the partition λ are positive, and we identify a partition $(\lambda_1,\dots,\lambda_m)$ with any partition $(\lambda_1,\dots,\lambda_m,0,\dots,0)$.

We define the set of *S-multipartitions* $\mathscr{P}^S$ as the set of tuples $\boldsymbol{\lambda}=(\lambda^{(i)})_{i\in S}$ of partitions. For $\boldsymbol{\lambda}\in\mathscr{P}^S$, we set $|\boldsymbol{\lambda}|:=\sum_{i\in S}|\lambda^{(i)}|$ and $\mathscr{P}^S(d):=\{\boldsymbol{\lambda}\in\mathscr{P}^S\mid|\boldsymbol{\lambda}|=d\}$. The only multipartition in $\mathscr{P}^S(0)$ is denoted by $\varnothing$.

We set $\mathsf{N}^S := \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \times S$ and refer to the elements of N^S as *nodes*. When S has one element, we identify N^S with $\mathsf{N} := \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$. If $\boldsymbol{\lambda} = (\lambda^{(i)})_{i \in S} \in \mathscr{P}^S$ is an S-multipartition, its *Young diagram*, which we often identify with $\boldsymbol{\lambda}$, is

$$[\![\boldsymbol{\lambda}]\!] := \{(r, s, i) \in \mathsf{N}^S \mid s \leq \lambda^{(i)}_r\}.$$

If $(r, s, i) \in \mathsf{N}^S$, we say that $(r, s+1, i)$ is *the right neighbor* of (r, s, i) and $(r+1, s, i)$ is the *bottom neighbor* of (r, s, i). Define a partial order $<$ on N^S as follows: $(r, s, i) \leq (r', s', i')$ if and only if $i = i'$, $r \leq r'$ and $s \leq s'$. Given a multipartition $\boldsymbol{\lambda} \in \mathscr{P}^S$, a function $\mathtt{T}\colon [\![\boldsymbol{\lambda}]\!] \to \mathbb{Z}_{>0}$ is said to be *weakly increasing* if whenever $u \leq v$ are in $[\![\boldsymbol{\lambda}]\!]$ we have $\mathtt{T}(u) \leq \mathtt{T}(v)$. If $u, v \in \mathsf{N}^S$ and neither $u \leq v$ nor $v \leq u$, then we say that u and v are *independent*. Two subsets $U, V \subseteq \mathsf{N}^S$ are said to be *independent* if every element of U is independent from every element of V. We say that a subset $U \subseteq \mathsf{N}^S$ is *convex* if whenever $u \leq v \leq w$ are in N^S and $u, w \in U$, we have $v \in U$.

A *skew partition* is a pair (λ, μ) of partitions such that $[\![\mu]\!] \subseteq [\![\lambda]\!]$. We denote it by $\lambda \setminus \mu$ and set $|\lambda \setminus \mu| := |\lambda| - |\mu|$. We identify $\lambda \setminus \varnothing$ with λ.

2.2. **Symmetric groups and parabolic subgroups.** Let $d \in \mathbb{Z}_{\geq 0}$. We denote by $\mathfrak{S}_d$ the symmetric group on $\{1, \dots, d\}$ and set $s_r := (r, r+1) \in \mathfrak{S}_d$ for $r = 1, \dots, d-1$ to be the elementary transpositions. For every $n \in \mathbb{Z}_{>0}$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda(n, d)$, we have the standard parabolic subgroup

$$\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_n} \leq \mathfrak{S}_d.$$

Moreover, for an ordered set $S = \{1, \dots, l\}$ and $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(l)}) \in \Lambda^S(n, d)$, we define the parabolic subgroup

$$\mathfrak{S}_{\boldsymbol{\lambda}} := \mathfrak{S}_{\lambda^{(1)}} \times \cdots \times \mathfrak{S}_{\lambda^{(l)}} \leq \mathfrak{S}_d.$$

If $g \in \mathfrak{S}_d$ and $g = s_{r_1} \dots s_{r_l}$ is a *reduced decomposition* of g, i.e. a decomposition as a product of elementary transpositions with l smallest possible, then we define $\ell(g) := l$ and refer to l as the *length* of g. For any $\lambda, \mu \in \Lambda(n, d)$, we denote by $\mathscr{D}^\lambda$ the set of the minimal length coset representatives for $\mathfrak{S}_d / \mathfrak{S}_\lambda$, by ${}^\mu\mathscr{D}$ the set of the minimal length coset representatives for $\mathfrak{S}_\mu \backslash \mathfrak{S}_d$ and by ${}^\mu\mathscr{D}^\lambda$ the set of the minimal length coset representatives for $\mathfrak{S}_\mu \backslash \mathfrak{S}_d / \mathfrak{S}_\lambda$.

2.3. **Algebras and modules.** In this paper we mostly work over the ground ring $\mathbb{Z}$. Occasionally, we use the prime fields $\mathbb{F}_p$ and their algebraic closures $\overline{\mathbb{F}}_p$.

All gradings in this paper are $\mathbb{Z}$-gradings. Let q be an indeterminate. Given a graded free $\mathbb{Z}$-module $V \cong \bigoplus_{n=1}^k \mathbb{Z} v_k$ with homogeneous generators v_k, we write $\dim_q V$ for the graded rank of V, i.e. $\dim_q V := \sum_{n=1}^k q^{\deg(v_n)} \in \mathbb{Z}[q, q^{-1}]$ and $\dim V := k$. Throughout, V^n denotes the nth graded component of V for any $n \in \mathbb{Z}$. Given $m \in \mathbb{Z}$, let $q^m V$ denote the module obtained by shifting the grading on V up by m, i.e. $(q^m V)^n := V^{n-m}$. We use the notation $V^{>m} := \bigoplus_{n>m} V^n$. For any $m \in \mathbb{Z}$, we set $[m] := (q^m - q^{-m})/(q - q^{-1}) \in \mathbb{Z}[q, q^{-1}]$. If $m \in \mathbb{Z}_{\geq 0}$, we define $[m]^! := \prod_{k=1}^m [k]$.

Let A be a ($\mathbb{Z}$-)graded algebra. All A-modules are assumed to be graded. Let A-mod denote the category of all *finitely generated* (graded) A-modules, with morphisms being *degree-preserving* module homomorphisms. Given A-modules V and W, we denote by $\hom_A(V, W)$ the space of morphisms in A-mod. For any $m \in$

$\mathbb{Z}$, define $\mathrm{Hom}_A(V,W)^m := \hom_H(q^m V, W)$. This is the space of homomorphisms that are homogeneous of degree m. Set

$$\mathrm{Hom}_A(V,W) := \bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}_A(V,W)^m.$$

In particular, $\mathrm{End}_A(V) := \mathrm{Hom}_A(V,V)$ is a graded algebra. All homomorphisms between graded algebras are assumed to be degree-preserving. We have the grading shift functor $q\colon A\text{-mod} \to A\text{-mod}$, $V \mapsto qV$.

Given an A-module V and a commutative ring $\Bbbk$, we denote by $A_\Bbbk := A \otimes_{\mathbb{Z}} \Bbbk$ the (graded) algebra obtained by scalar extension, and by $V_\Bbbk := V \otimes_{\mathbb{Z}} \Bbbk$ the corresponding $A_\Bbbk$-module. If $B = A/K$ is the quotient of A by an ideal B and $x \in A$, we denote an element $x + K$ of B simply by x when there is no possibility of confusion.

If $\Bbbk$ is a field and A is a finite-dimensional graded $\Bbbk$-algebra, we denote by $\ell(A)$ the number of irreducible graded A-modules up to isomorphism and degree shift.

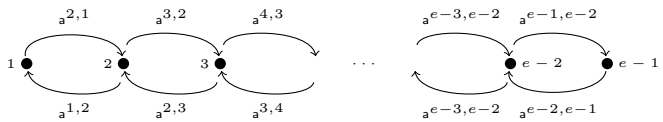
3. Zigzag algebras, wreath products and Turner doubles

Throughout the paper, we fix $e \in \mathbb{Z}_{\geq 2}$.

3.1. **Zigzag algebras and wreath products.** Let Q be a type A_{e-1} quiver with vertex set

$$J := \{1, \dots, e-1\}. \tag{3.1}$$

We will use the *zigzag algebra* Z *of type* A_{e-1}, defined in [**HK**] as follows. First assume that $e > 2$. Let $\hat{Q}$ be the quiver with vertex set J and an arrow $\mathsf{a}^{k,j}$ from j to k for all ordered pairs $(k,j) \in J^2$ such that $|k-j| = 1$:



Then Z is the path algebra $\mathbb{Z}\hat{Q}$, generated by length 0 paths e_j for $j \in J$ and length 1 paths $\mathsf{a}^{k,j}$, subject to the following relations:

(i) All paths of length three or greater are zero.
(ii) All paths of length two that are not cycles are zero.
(iii) All cycles of length 2 based at the same vertex are equal.

The algebra Z inherits the path length grading from $\mathbb{Z}\hat{Q}$. If $e = 2$, we define $\mathsf{Z} := \mathbb{Z}[\mathsf{c}]/(\mathsf{c}^2)$, where c is an indeterminate in degree 2.

If $k, j \in J$, we say that k and j are *neighbors* if $|k-j| = 1$. If $e > 2$, for every vertex $j \in J$ pick its neighbor k and denote $\mathsf{c}^{(j)} := \mathsf{a}^{j,k}\mathsf{a}^{k,j}$. The relations in Z imply that $\mathsf{c}^{(j)}$ is independent of choice of k. Define $\mathsf{c} := \sum_{j\in J} \mathsf{c}^{(j)}$. Then in all cases Z has a basis

$$B_{\mathsf{Z}} := \{\mathsf{a}^{k,j} \mid k \in J,\ j \text{ is a neighbor of } k\} \cup \{\mathsf{c}^m \mathsf{e}_j \mid j \in J,\ m \in \{0,1\}\}, \tag{3.2}$$

and

$$\dim_q \mathsf{Z} = (e-1)(1+q^2) + 2(e-2)q. \tag{3.3}$$

Moreover, using (3.2), we see that for any $j \in J$

$$\dim \mathsf{e}_j \mathsf{Z} = \begin{cases} 4 & \text{if } 1 < j < e-1, \\ 3 & \text{if } j \in \{1, e-1\} \text{ and } e > 2, \\ 2 & \text{if } j = 1 \text{ and } e = 2. \end{cases} \tag{3.4}$$

We will also consider the graded *wreath products*

$$W_d := \mathsf{Z}^{\otimes d} \rtimes \mathbb{Z}\mathfrak{S}_d, \tag{3.5}$$

with $\mathbb{Z}\mathfrak{S}_d$ concentrated in degree 0. (Note that, unlike [**EK**], we do not consider any superstructures here.) As usual, we identify $\mathsf{Z}^{\otimes d}$ and $\mathbb{Z}\mathfrak{S}_d$ with the subalgebras $\mathsf{Z}^{\otimes d} \otimes 1_{\mathfrak{S}_d}$ and $1_{\mathsf{Z}}^{\otimes d} \otimes \mathbb{Z}\mathfrak{S}_d$ of W_d, respectively. The multiplication in W_d is then uniquely determined by the additional requirement that

$$g^{-1}(x_1 \otimes \cdots \otimes x_d)g = x_{g1} \otimes \cdots \otimes x_{gd} \tag{3.6}$$

for $g \in \mathfrak{S}_d$ and $x_1, \dots, x_d \in \mathsf{Z}$. Given $x \in \mathsf{Z}$ and $1 \le a \le d$, we denote

$$x[a] := 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1 \in \mathsf{Z}^{\otimes d},$$

with x in the ath position. We have the idempotents

$$\mathsf{e}_{\boldsymbol{j}} := \mathsf{e}_{j_1} \otimes \cdots \otimes \mathsf{e}_{j_d} \in \mathsf{Z}^{\otimes d} \subseteq W_d \qquad (\boldsymbol{j} \in J^d).$$

Fix $n \in \mathbb{Z}_{>0}$. We define the set of *colored compositions*

$$\Lambda^{\mathrm{col}}(n,d) := \Lambda(n,d) \times J^n. \tag{3.7}$$

Let $(\lambda, \boldsymbol{c}) \in \Lambda^{\mathrm{col}}(n,d)$ with $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\boldsymbol{c} = (c_1, \dots, c_n)$. We define the idempotent

$$\mathsf{e}_{\lambda,\boldsymbol{c}} := \mathsf{e}_{c_1}^{\otimes \lambda_1} \otimes \cdots \otimes \mathsf{e}_{c_n}^{\otimes \lambda_n} \in \mathsf{Z}^{\otimes d} \tag{3.8}$$

and the *parabolic subalgebra*

$$W_{\lambda,\boldsymbol{c}} = \mathsf{e}_{\lambda,\boldsymbol{c}} \otimes \mathbb{Z}\mathfrak{S}_\lambda \subseteq \mathsf{e}_{\lambda,\boldsymbol{c}} W_d \mathsf{e}_{\lambda,\boldsymbol{c}}.$$

Note that $\mathsf{e}_{\lambda,\boldsymbol{c}}$ is the identity element of $W_{\lambda,\boldsymbol{c}}$, so $W_{\lambda,\boldsymbol{c}}$ is a (usually *non-unital*) subalgebra of W_d, isomorphic to the group algebra $\mathbb{Z}\mathfrak{S}_\lambda$.

We assign signs ζ_j to the elements $j \in J$ according to the following rule:

$$\zeta_j = \begin{cases} +1 & \text{if } j \text{ is odd}, \\ -1 & \text{if } j \text{ is even}. \end{cases} \tag{3.9}$$

Consider the function $\varepsilon_{\lambda,\boldsymbol{c}} \colon \mathfrak{S}_\lambda \to \{\pm 1\} \subseteq \mathbb{Z}$ defined by

$$\varepsilon_{\lambda,\boldsymbol{c}}(g_1, \dots, g_n) := \zeta_{c_1}^{\ell(g_1)} \cdots \zeta_{c_n}^{\ell(g_n)} \tag{3.10}$$

for all $(g_1, \dots, g_n) \in \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_n} = \mathfrak{S}_\lambda$. We define the $\boldsymbol{c}$-*alternating* right module $\mathtt{alt}_{\lambda,\boldsymbol{c}} = \mathbb{Z} \cdot 1_{\lambda,\boldsymbol{c}}$ over $W_{\lambda,\boldsymbol{c}}$ with the action on the basis element $1_{\lambda,\boldsymbol{c}}$ given by

$$1_{\lambda,\boldsymbol{c}} \cdot (\mathsf{e}_{\lambda,\boldsymbol{c}} \otimes g) = \varepsilon_{\lambda,\boldsymbol{c}}(g) 1_{\lambda,\boldsymbol{c}} \qquad (g \in \mathfrak{S}_\lambda).$$

We have identified $\mathsf{Z}^{\otimes d}$ and $\mathbb{Z}\mathfrak{S}_d$ as subalgebras of W_d, so we can also view $\mathsf{e}_{\lambda,\boldsymbol{c}}$ as an element of W_d. Then $W_{\lambda,\boldsymbol{c}} = \mathsf{e}_{\lambda,\mathrm{c}}(\mathbb{Z}\mathfrak{S}_\lambda)\mathsf{e}_{\lambda,\boldsymbol{c}}$ and $\mathsf{e}_{\lambda,\boldsymbol{c}} W_d$ is naturally a left $W_{\lambda,\boldsymbol{c}}$-module. We now define the *colored permutation module*

$$M_{\lambda,\boldsymbol{c}} := \mathtt{alt}_{\lambda,\boldsymbol{c}} \otimes_{W_{\lambda,\mathrm{c}}} \mathsf{e}_{\lambda,\boldsymbol{c}} W_d. \tag{3.11}$$

This is a right W_d-module with generator $m_{\lambda,\boldsymbol{c}} := 1_{\lambda,\boldsymbol{c}} \otimes \mathsf{e}_{\lambda,\boldsymbol{c}}$.

Lemma 3.12. *For each $j \in J$, set $d_j := \sum_{1\le r\le n,\, c_r=j} \lambda_r$. Then the module $M_{\lambda,\boldsymbol{c}}$ is $\mathbb{Z}$-free, with*

$$\dim M_{\lambda,\boldsymbol{c}} = \begin{cases} |\mathfrak{S}_d : \mathfrak{S}_\lambda| 3^{d_1+d_{e-1}} 4^{\sum_{j=2}^{e-2} d_j} & \text{if } e > 2, \\ |\mathfrak{S}_d : \mathfrak{S}_\lambda| 2^{d_1} & \text{if } e = 2. \end{cases}$$

Proof. This follows from (3.4) and [**EK**, Lemma 5.21]. □

3.2. **Turner doubles and generalized Schur algebras.** Let $n \in \mathbb{Z}_{>0}$ and $d \in \mathbb{Z}_{\ge 0}$. Set

$$\boldsymbol{c}^0 := (1,\dots,e-1)^n = (1,\dots,e-1,1,\dots,e-1,\dots,1,\dots,e-1) \in J^{n(e-1)}. \tag{3.13}$$

We have a bijection

$$\Lambda^J(n,d) \xrightarrow{\sim} \Lambda(n(e-1),d),$$
$$\boldsymbol{\lambda} = (\lambda^{(1)},\dots,\lambda^{(e-1)}) \mapsto (\lambda_1^{(1)},\dots,\lambda_1^{(e-1)},\dots,\lambda_n^{(1)},\dots,\lambda_n^{(e-1)}).$$

In this subsection, we use this bijection to translate the results of [**EK**, §7.2] into the present notation.

For any $\lambda \in \Lambda(n(e-1),d)$, we define

$$M^\lambda := M_{\lambda,\boldsymbol{c}^0}.$$

Let

$$M(n,d) := \bigoplus_{\lambda\in\Lambda(n(e-1),d)} M^\lambda. \tag{3.14}$$

Following [**EK**], we consider the *generalized Schur algebra*

$$S^{\mathsf{Z}}(n,d) := \mathrm{End}_{W_d}\left(M(n,d)\right).$$

Since the algebra W_d is non-negatively graded, so are the modules M^λ. Since M^λ has the degree zero generator

$$m^\lambda := m_{\lambda,\boldsymbol{c}^0}$$

as a W_d-module, it follows that the algebra $S^{\mathsf{Z}}(n,d)$ is non-negatively graded.

For $\lambda \in \Lambda(n(e-1),d)$, let $\xi_\lambda \in S^{\mathsf{Z}}(n,d)$ be the projection onto the direct summand M^λ of $M(n,d)$ along the decomposition (3.14). We always identify $\xi_\mu S^{\mathsf{Z}}(n,d)\xi_\lambda$ with $\mathrm{Hom}_{W_d}(M^\lambda, M^\mu)$ in the obvious way.

Let $\lambda \in \Lambda((n-1)(e-1), d-1)$. For $j \in J$, we define

$$\hat\lambda^j := (\underbrace{0,\dots,0,1,0,\dots,0}_{e-1 \text{ entries}},\lambda_1,\dots,\lambda_{(n-1)(e-1)}) \in \Lambda(n(e-1),d),$$

where 1 is in the jth position. Let $z \in \mathsf{e}_j\mathsf{Z}\mathsf{e}_k$ for some $j,k \in J$. By [**EK**, Lemma 7.5], there exists a unique endomorphism $\mathtt{i}^\lambda(z) \in S^{\mathsf{Z}}(n,d)$ with

$$\mathtt{i}^\lambda(z)\colon m^\mu \mapsto \begin{cases} m^{\hat\lambda^j} z[1] & \text{if } \mu = \hat\lambda^k, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, by [**EK**, Lemma 7.6], we have a (non-unital) injective algebra homomorphism

$$\mathtt{i}^\lambda\colon \mathsf{Z} \to S^{\mathsf{Z}}(n,d), z \mapsto \sum_{j,k\in J} \mathtt{i}^\lambda(\mathsf{e}_j z \mathsf{e}_k). \tag{3.15}$$

Define $T^{\mathsf{Z}}(n,d)$ to be the subalgebra of $S^{\mathsf{Z}}(n,d)$ generated by the set

$$S^{\mathsf{Z}}(n,d)^0 \cup \bigcup_{\lambda\in\Lambda((n-1)(e-1),d-1)} \mathtt{i}^{\lambda}(\mathsf{Z}).$$

Theorem 3.16. [**EK**, Theorem 7.7] *Suppose that $n \geq d$. There is a graded algebra isomorphism $D_Q(n,d) \xrightarrow{\sim} T^{\mathsf{Z}}(n,d)$.*

Theorem 3.17. [**EK**, Theorem 6.6]*, Suppose that $n \geq d$. If A is a subalgebra of $S^{\mathsf{Z}}(n,d)$ such that $T^{\mathsf{Z}}(n,d) \subseteq A \subseteq S^{\mathsf{Z}}(n,d)$ and $A_{\mathbb{F}_p}$ is a symmetric $\mathbb{F}_p$-algebra for every prime p, then $A = T^{\mathsf{Z}}(n,d)$.*

4. KLR algebras

4.1. **Lie-theoretic notation.** Let

$$I := \mathbb{Z}/e\mathbb{Z} = \{0,\dots,e-1\}.$$

We consider the quiver of type $A^{(1)}_{e-1}$ with vertex set I and a directed edge $i \to j$ whenever $j = i+1$. The corresponding *Cartan matrix* $(\mathtt{c}_{ij})_{i,j\in I}$ is defined by

$$\mathtt{c}_{ij} := \begin{cases} 2 & \text{if } i = j,\\ 0 & \text{if } j \neq i, i\pm 1,\\ -1 & \text{if } i \to j \text{ or } i \leftarrow j,\\ -2 & \text{if } i \rightleftarrows j. \end{cases}$$

Following [**Ka**], we fix a realization of the Cartan matrix $(\mathtt{c}_{ij})_{i,j\in I}$ with the *simple roots* $\{\alpha_i \mid i \in I\}$, the *fundamental dominant weights* $\{\Lambda_i \mid i \in I\}$, the *normalized invariant form* $(\cdot,\cdot)$ such that

$$(\alpha_i,\alpha_j) = \mathtt{c}_{ij}, \quad (\Lambda_i,\alpha_j) = \delta_{ij} \qquad (i,j \in I),$$

the *root system* Φ, the set of *positive roots* Φ_+, and the *null-root*

$$\delta := \alpha_0 + \alpha_1 + \dots + \alpha_{e-1} \in \Phi_+. \tag{4.1}$$

Let $Q_+ := \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$. For $\theta \in Q_+$ let $\mathrm{ht}(\theta)$ be the *height of* θ, i.e. $\mathrm{ht}(\theta)$ is the sum of the coefficients when θ is expanded in terms of the simple roots α_i. For any $m \in \mathbb{Z}_{\geq 0}$, the symmetric group $\mathfrak{S}_m$ acts from the left on the set I^m by place permutations. If $\boldsymbol{i} = i_1 \dots i_m \in I^m$ then its *weight* is $|\boldsymbol{i}| := \alpha_{i_1} + \dots + \alpha_{i_m} \in Q_+$. Then the $\mathfrak{S}_m$-orbits on I^m are the sets

$$I^\theta := \{\boldsymbol{i} \in I^m \mid |\boldsymbol{i}| = \theta\}$$

parametrized by all $\theta \in Q_+$ of height m.

We always identify $J = \{1,\dots,e-1\}$ with the subset $I \setminus \{0\}$ of I, cf. (3.1). Let $\mathtt{C}'$ be the type A_{e-1} Cartan matrix corresponding to J, and let $\Phi'_+ \subset \Phi_+$ be the corresponding positive part of the finite root system. We define

$$\Phi_+^{\prec\delta} := \{-\beta + n\delta \mid \beta \in \Phi'_+,\ n \in \mathbb{Z}_{>0}\} \text{ and } \Phi_+^{\succ\delta} := \{\beta + n\delta \mid \beta \in \Phi'_+,\ n \in \mathbb{Z}_{\geq 0}\}.$$

Set $\Phi_+^{\preceq\delta} := \Phi_+^{\prec\delta} \sqcup \{\delta\}$ and $\Phi_+^{\succeq\delta} := \Phi_+^{\succ\delta} \sqcup \{\delta\}$. Note that $\Phi_+ = \Phi_+^{\mathrm{im}} \sqcup \Phi_+^{\mathrm{re}}$, where $\Phi_+^{\mathrm{im}} = \{n\delta \mid n \in \mathbb{Z}_{>0}\}$ and $\Phi_+^{\mathrm{re}} = \Phi_+^{\prec\delta} \sqcup \Phi_+^{\succ\delta}$.

4.2. **Basics on KLR algebras.** Let $\theta \in Q_+$ be of height m. Following [**KL**,**R**$_2$], the *KLR algebra* (of type $A^{(1)}_{e-1}$) is the unital $\mathbb{Z}$-algebra R_θ generated by the elements $\{1_{\boldsymbol{i}} | \boldsymbol{i} \in I^\theta\} \cup \{y_1, \dots, y_m\} \cup \{\psi_1, \dots, \psi_{m-1}\}$, subject only to the following relations:

$$1_{\boldsymbol{i}} 1_{\boldsymbol{j}} = \delta_{\boldsymbol{i},\boldsymbol{j}} 1_{\boldsymbol{i}}; \qquad \textstyle\sum_{\boldsymbol{i} \in I^\theta} 1_{\boldsymbol{i}} = 1; \tag{4.2}$$

$$y_r 1_{\boldsymbol{i}} = 1_{\boldsymbol{i}} y_r; \qquad \psi_r 1_{\boldsymbol{i}} = 1_{s_r \boldsymbol{i}} \psi_r; \tag{4.3}$$

$$y_r y_s = y_s y_r; \tag{4.4}$$

$$\psi_r y_s = y_s \psi_r \qquad \text{if } s \neq r, r+1; \tag{4.5}$$

$$\psi_r \psi_s = \psi_s \psi_r \qquad \text{if } |r-s| > 1; \tag{4.6}$$

$$\psi_r y_{r+1} 1_{\boldsymbol{i}} = (y_r \psi_r + \delta_{i_r, i_{r+1}}) 1_{\boldsymbol{i}}; \tag{4.7}$$

$$y_{r+1} \psi_r 1_{\boldsymbol{i}} = (\psi_r y_r + \delta_{i_r, i_{r+1}}) 1_{\boldsymbol{i}}; \tag{4.8}$$

$$\psi_r^2 1_{\boldsymbol{i}} = \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\ 1_{\boldsymbol{i}} & \text{if } i_{r+1} \neq i_r, i_r \pm 1, \\ (y_{r+1} - y_r) 1_{\boldsymbol{i}} & \text{if } i_r \to i_{r+1}, \\ (y_r - y_{r+1}) 1_{\boldsymbol{i}} & \text{if } i_r \leftarrow i_{r+1}, \\ (y_{r+1} - y_r)(y_r - y_{r+1}) 1_{\boldsymbol{i}} & \text{if } i_r \rightleftarrows i_{r+1}; \end{cases} \tag{4.9}$$

$$\psi_r \psi_{r+1} \psi_r 1_{\boldsymbol{i}} = \begin{cases} (\psi_{r+1} \psi_r \psi_{r+1} + 1) 1_{\boldsymbol{i}} & \text{if } i_{r+2} = i_r \to i_{r+1}, \\ (\psi_{r+1} \psi_r \psi_{r+1} - 1) 1_{\boldsymbol{i}} & \text{if } i_{r+2} = i_r \leftarrow i_{r+1}, \\ \big(\psi_{r+1} \psi_r \psi_{r+1} - 2y_{r+1} & \\ \qquad + y_r + y_{r+2}\big) 1_{\boldsymbol{i}} & \text{if } i_{r+2} = i_r \rightleftarrows i_{r+1}, \\ \psi_{r+1} \psi_r \psi_{r+1} 1_{\boldsymbol{i}} & \text{otherwise.} \end{cases} \tag{4.10}$$

The *cyclotomic KLR algebra* $R^{\Lambda_0}_\theta$ is the quotient of R_θ by the two-sided ideal $I^{\Lambda_0}_\theta$ generated by the elements $y_1^{\delta_{i_1,0}} 1_{\boldsymbol{i}}$ for all $\boldsymbol{i} = (i_1, \dots, i_d) \in I^\theta$. We have the natural projection map

$$\pi_\theta \colon R_\theta \twoheadrightarrow R^{\Lambda_0}_\theta = R_\theta / I^{\Lambda_0}_\theta. \tag{4.11}$$

The algebras R_θ and $R^{\Lambda_0}_\theta$ have $\mathbb{Z}$-gradings determined by setting $1_{\boldsymbol{i}}$ to be of degree 0, y_r of degree 2, and $\psi_r 1_{\boldsymbol{i}}$ of degree $-\mathtt{c}_{i_r, i_{r+1}}$ for all admissible r and $\boldsymbol{i}$.

For $\kappa \in I = \mathbb{Z}/e\mathbb{Z}$ and $\boldsymbol{i} = (i_1, \dots, i_n) \in I^n$, we set $\boldsymbol{i}^{+\kappa} := (i_1 + \kappa, \dots, i_n + \kappa) \in I^n$. Then for any $d \in \mathbb{Z}_{>0}$, there is an automorphism

$$\mathtt{rot}_\kappa \colon R_{d\delta} \to R_{d\delta}, 1_{\boldsymbol{i}} \mapsto 1_{\boldsymbol{i}^{+\kappa}},\ y_r \mapsto y_r,\ \psi_s \mapsto \psi_s \tag{4.12}$$

for all admissible $\boldsymbol{i}, r, s$.

Fixing a *preferred reduced decomposition* $w = s_{r_1} \dots s_{r_l}$ for each element $w \in \mathfrak{S}_m$, we define the elements $\psi_w := \psi_{r_1} \dots \psi_{r_l} \in R_\theta$. In general, ψ_w depends on the choice of a preferred reduced decomposition of w.

Theorem 4.13. [**KL**, Theorem 2.5], [**R**$_2$, Theorem 3.7] *Let $\theta \in Q_+$ and $m = \mathrm{ht}(\theta)$. Then*

$$\{\psi_w y_1^{k_1} \dots y_m^{k_m} 1_{\boldsymbol{i}} \mid w \in \mathfrak{S}_m,\ k_1, \dots, k_m \in \mathbb{Z}_{\geq 0},\ \boldsymbol{i} \in I^\theta\}$$

is a $\mathbb{Z}$-basis of R_θ.

As a special case of [**KK**, Remark 4.20], we have

Theorem 4.14. *Let $\theta \in Q_+$. Then the $\mathbb{Z}$-module $R_\theta^{\Lambda_0}$ is free of finite rank.*

By [**SVV**, Proposition 3.10] (see also [**We**, Remark 3.19]), we have

Theorem 4.15. *Let $\theta \in Q_+$. Then for any field $\Bbbk$, the algebra $R_{\theta,\Bbbk}^{\Lambda_0}$ is symmetric. More precisely, $R_{\theta,\Bbbk}^{\Lambda_0}$ admits a symmetrizing form of degree $(\Lambda_0 - \theta, \Lambda_0 - \theta)$.*

4.3. **Parabolic subalgebras.** For $\theta_1, \dots, \theta_r \in Q_+$ and $\theta = \theta_1 + \dots + \theta_r$, we have the idempotent

$$1_{\theta_1,\dots,\theta_r} = \sum_{\boldsymbol{i}^{(1)} \in I^{\theta_1}, \dots, \boldsymbol{i}^{(r)} \in I^{\theta_r}} 1_{\boldsymbol{i}^{(1)} \dots \boldsymbol{i}^{(r)}}.$$

and an algebra embedding

$$\iota_{\theta_1,\dots,\theta_r} \colon R_{\theta_1} \otimes \dots \otimes R_{\theta_r} \to 1_{\theta_1,\dots,\theta_r} R_\theta 1_{\theta_1,\dots,\theta_r} \tag{4.16}$$

whose image is the *parabolic subalgebra*

$$R_{\theta_1,\dots,\theta_r} \subseteq 1_{\theta_1,\dots,\theta_r} R_\theta 1_{\theta_1,\dots,\theta_r} \subseteq R_\theta.$$

Denoting by 1_θ the identity element in R_θ, we have

$$1_{\theta_1,\dots,\theta_r} := \iota_{\theta_1,\dots,\theta_r}(1_{\theta_1} \otimes \dots \otimes 1_{\theta_r}). \tag{4.17}$$

Note that we always identify $R_{\theta_1} \otimes \dots \otimes R_{\theta_r}$ with $R_{\theta_1,\dots,\theta_r}$ via $\iota_{\theta_1,\dots,\theta_r}$.

We have the corresponding induction and restriction functors

$$\mathrm{Ind}_{\theta_1,\dots,\theta_r} \colon R_{\theta_1,\dots,\theta_r}\text{-mod} \to R_\theta\text{-mod},\ W \mapsto R_\theta 1_{\theta_1,\dots,\theta_r} \otimes_{R_{\theta_1,\dots,\theta_r}} W,$$
$$\mathrm{Res}_{\theta_1,\dots,\theta_r} \colon R_\theta\text{-mod} \to R_{\theta_1,\dots,\theta_r}\text{-mod},\ U \mapsto 1_{\theta_1,\dots,\theta_r} U.$$

Let $W_1 \in R_{\theta_1}\text{-mod}, \dots, W_r \in R_{\theta_r}\text{-mod}$. We define

$$W_1 \circ \dots \circ W_r := \mathrm{Ind}_{\theta_1,\dots,\theta_r} W_1 \boxtimes \dots \boxtimes W_r.$$

We refer to the elements of I^θ as *words*. Given $W \in R_\theta$-mod and $\boldsymbol{i} \in I^\theta$, we say that $\boldsymbol{i}$ is a *word of* W if $1_{\boldsymbol{i}} W \neq 0$. If every $1_{\boldsymbol{i}} W$ is free of finite rank as a $\mathbb{Z}$-module, we define the *formal character* of W as $\mathrm{ch}_q W = \sum_{\boldsymbol{i} \in I^\theta} (\dim_q 1_{\boldsymbol{i}} W) \boldsymbol{i} \in \mathbb{Z}[q, q^{-1}] \cdot I^\theta$.

Given a composition $\lambda \in \Lambda(r, m)$ and words $\boldsymbol{i}^{(1)} \in I^{\lambda_1}, \dots, \boldsymbol{i}^{(r)} \in I^{\lambda_r}$, a word $\boldsymbol{i} \in I^m$ is called a *shuffle* of $\boldsymbol{i}^{(1)}, \dots, \boldsymbol{i}^{(r)}$ if $\boldsymbol{i} = g \cdot (\boldsymbol{i}^{(1)} \dots \boldsymbol{i}^{(r)})$ for some $g \in \mathscr{D}^\lambda$. By [**KL**, Lemma 2.20], an element $\boldsymbol{i} \in I^m$ is a word of $W_1 \circ \dots \circ W_r$ if and only if $\boldsymbol{i}$ is a shuffle of words $\boldsymbol{i}^{(1)}, \dots, \boldsymbol{i}^{(r)}$ where $\boldsymbol{i}^{(s)}$ is a word of W_s for $s = 1, \dots, r$.

We will need the following weak version of the Mackey Theorem for KLR algebras, see [**Ev**, Proposition 3.7] or the proof of [**KL**, Proposition 2.18]:

Lemma 4.18. *Let $\theta_1, \dots, \theta_r, \theta'_1, \dots, \theta'_t \in Q_+$ satisfy $\theta_1 + \dots + \theta_r = \theta'_1 + \dots + \theta'_t =: \theta$. Define $m := \mathrm{ht}(\theta)$, $\lambda := (\mathrm{ht}(\theta_1), \dots, \mathrm{ht}(\theta_r)) \in \Lambda(r, m)$, and $\lambda' := (\mathrm{ht}(\theta'_1), \dots, \mathrm{ht}(\theta'_t)) \in \Lambda(t, m)$. Then*

$$1_{\theta_1,\dots,\theta_r} R_\theta 1_{\theta'_1,\dots,\theta'_t} = \sum_{w \in {}^\lambda \mathscr{D}^{\lambda'}} R_{\theta_1,\dots,\theta_r} \psi_w R_{\theta'_1,\dots,\theta'_t}.$$

With the notation as in the beginning of the subsection, we have the *parabolic subalgebra*

$$R^{\Lambda_0}_{\theta_1,\dots,\theta_r} := \pi_\theta(R_{\theta_1,\dots,\theta_r}) \subseteq R_\theta^{\Lambda_0}.$$

Let $\theta, \eta \in Q_+$. We have a natural embedding $\zeta_{\theta,\eta}\colon R_\theta \to R_{\theta,\eta},\ x \mapsto \iota_{\theta,\eta}(x \otimes 1_\eta)$. The map $\pi_{\theta+\eta} \circ \zeta_{\theta,\eta}$ factors through the quotient $R_\theta^{\Lambda_0}$ to give the natural *unital* algebra homomorphism

$$\zeta_{\theta,\eta}\colon R_\theta^{\Lambda_0} \to R_{\theta,\eta}^{\Lambda_0}. \tag{4.19}$$

4.4. **Divided power idempotents.** Fix $i \in I$. Let $m \in \mathbb{Z}_{\geq 0}$ and denote by w_0 the longest element of $\mathfrak{S}_m$. The algebra $R_{m\alpha_i}$ is known to be the nil-Hecke algebra and has an idempotent $1_{i^{(m)}} := \psi_{w_0} \prod_{s=1}^m y_s^{s-1}$, cf. [**KL**]. The fact that $1_{i^{(m)}}$ is an idempotent follows immediately from the equality

$$1_{i^{(m)}} \psi_{w_0} = \psi_{w_0} \tag{4.20}$$

noted in [**KL**, §2.2].

Lemma 4.21. *For any $x \in R_{m\alpha_i}$ there exists $y \in \mathbb{Z}[y_1, \dots, y_m]$ such that $1_{i^{(m)}} x = \psi_{w_0} y$.*

Proof. By Theorem 4.13, we can write $(\prod_{s=1}^m y_s^{s-1})x = \sum_{w \in \mathfrak{S}_m} \psi_w y(w)$ for some $y(w) \in \mathbb{Z}[y_1, \dots, y_m]$. So $1_{i^{(m)}} x = \psi_{w_0}(\prod_{s=1}^m y_s^{s-1})x = \sum_{w \in \mathfrak{S}_m} \psi_{w_0}\psi_w y(w) = \psi_{w_0} y(1)$. □

Let $\theta \in Q_+$. We define I^θ_{div} to be the set of all expressions of the form $(i_1^{(m_1)}, \dots, i_r^{(m_r)})$ with $m_1, \dots, m_r \in \mathbb{Z}_{\geq 0}$, $i_1, \dots, i_r \in I$ and $m_1\alpha_{i_1} + \dots + m_r\alpha_{i_r} = \theta$. We refer to such expressions as *divided power words*. Analogously to the words, for $\kappa \in I = \mathbb{Z}/e\mathbb{Z}$ and a divided power word $\boldsymbol{i} = (i_1^{(m_1)}, \dots, i_r^{(m_r)})$, we define the divided power word

$$\boldsymbol{i}^{+\kappa} := ((i_1 + \kappa)^{(m_1)}, \dots, (i_r + \kappa)^{(m_r)}).$$

We identify I^θ with the subset of I^θ_{div} which consists of all expressions as above with all $m_k = 1$. We use the same notation for concatenation of divided power words as for concatenation of words.

Fix $\boldsymbol{i} = (i^{(m_1)}, \dots, i^{(m_r)}) \in I^\theta_{\mathrm{div}}$. We have the *divided power idempotent*

$$1_{\boldsymbol{i}} := \iota_{m_1\alpha_{i_1}, \dots, m_r\alpha_{i_r}}(1_{i_1^{(m_1)}} \otimes \dots \otimes 1_{i_r^{(m_r)}}) \in R_\theta.$$

Define $\boldsymbol{i}! := [m_1]^! \cdots [m_r]^!$ and

$$\langle \boldsymbol{i} \rangle := \sum_{k=1}^r m_k(m_k - 1)/2. \tag{4.22}$$

Set

$$\hat{\boldsymbol{i}} := (i_1, \dots, i_1, \dots, i_r, \dots, i_r) \in I^\theta, \tag{4.23}$$

with i_k repeated m_k times. Note that $1_{\boldsymbol{i}} 1_{\hat{\boldsymbol{i}}} = 1_{\hat{\boldsymbol{i}}} 1_{\boldsymbol{i}} = 1_{\boldsymbol{i}}$.

Lemma 4.24. [**KL**, §2.5] *Let U (resp. W) be a left (resp. right) R_θ-module, free of finite rank as a $\mathbb{Z}$-module. For $\boldsymbol{i} \in I^\theta_{\mathrm{div}}$, we have*

$$\dim_q(1_{\hat{\boldsymbol{i}}} U) = \boldsymbol{i}! q^{\langle \boldsymbol{i} \rangle} \dim_q(1_{\boldsymbol{i}} U) \quad \text{and} \quad \dim_q(W 1_{\hat{\boldsymbol{i}}}) = \boldsymbol{i}! q^{-\langle \boldsymbol{i} \rangle} \dim_q(W 1_{\boldsymbol{i}}).$$

4.5. **Semicuspidal modules.** Let $d, f \in \mathbb{Z}_{\geq 0}$. A word $\boldsymbol{i} \in I^{d\delta}$ is called *separated* if whenever $\boldsymbol{i} = \boldsymbol{jk}$ for $\boldsymbol{j} \in I^{\theta}$ and $\boldsymbol{k} \in I^{\eta}$, it follows that θ is a sum of positive roots in $\Phi_+^{\prec\delta}$ and η is a sum of positive roots in $\Phi_+^{\succ\delta}$. We denote by $I^{d\delta}_{\mathrm{sep}}$ the set of all separated words in $I^{d\delta}$. An $R_{d\delta}$-module is *(imaginary) semicuspidal* if all of its words are separated. Note that a shuffle of separated words is separated, so:

Lemma 4.25. *If $U \in R_{d\delta}$-mod and $W \in R_{f\delta}$-mod are semicuspidal modules, then $U \circ W \in R_{(d+f)\delta}$-mod is semicuspidal.*

Set $1_{\mathrm{nsep}} := \sum_{\boldsymbol{i} \in I^{d\delta} \setminus I^{d\delta}_{\mathrm{sep}}} 1_{\boldsymbol{i}}$. The *(imaginary) semicuspidal algebra* is defined as

$$\hat{C}_{d\delta} := R_{d\delta} / R_{d\delta} 1_{\mathrm{nsep}} R_{d\delta}. \tag{4.26}$$

The category of finitely generated semicuspidal $R_{d\delta}$-modules is equivalent to the category $\hat{C}_{d\delta}$-mod. A word $\boldsymbol{i} \in I^{d\delta}$ is called *semicuspidal* if the idempotent $1_{\boldsymbol{i}}$ is non-zero in $\hat{C}_{d\delta}$. Denote by $I^{d\delta}_{\mathrm{sc}}$ the set of all semicuspidal words. Then, setting $1_{\mathrm{nsc}} := \sum_{\boldsymbol{i} \in I^{d\delta} \setminus I^{d\delta}_{\mathrm{sc}}} 1_{\boldsymbol{i}}$, we have $\hat{C}_{d\delta} \cong R_{d\delta} / R_{d\delta} 1_{\mathrm{nsc}} R_{d\delta}$. By definition, we always have $I^{d\delta}_{\mathrm{sc}} \subseteq I^{d\delta}_{\mathrm{sep}}$, but this containment may be strict, see Example 4.30 below.

Everything in this subsection so far makes sense over any ground ring. In particular the notion of a semicuspidal module over $R_{d\delta,\mathbb{F}}$ is defined for any field $\mathbb{F}$. We now explain the classification of the semicuspidal irreducible $R_{d\delta,\mathbb{F}}$-modules for an arbitrary field $\mathbb{F}$.

We begin with the case $d = 1$, in which case the semicuspidal irreducible $R_{d\delta,\mathbb{F}}$-modules are parametrized by the elements of $J = \{1, \dots, e-1\} = I \setminus \{0\}$. More precisely, let $j \in J$. We denote by $I^{\delta,j}$ the set of all words in I^{δ} of the form $0\boldsymbol{k}j$ where $\boldsymbol{k}$ is an arbitrary shuffle of the words $(1, 2, \dots, j-1)$ and $(e-1, e-2, \dots, j+1)$. Let $L_{\delta,j}$ be the graded $\mathbb{Z}$-module with basis $\{v_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^{\delta,j}\}$ where all basis elements have degree 0. By [**KR**, Theorem 3.4], there is a unique structure of a graded R_{δ}-module on $L_{\delta,j}$ such that

$$1_{\boldsymbol{j}} v_{\boldsymbol{i}} = \delta_{\boldsymbol{i},\boldsymbol{j}} v_{\boldsymbol{i}},\ y_r v_{\boldsymbol{i}} = 0,\ \psi_r v_{\boldsymbol{i}} = \begin{cases} v_{s_r \boldsymbol{i}} & \text{if } s_r \boldsymbol{i} \in I^{\delta,j}, \\ 0 & \text{if } s_r \boldsymbol{i} \notin I^{\delta,j} \end{cases} \tag{4.27}$$

for all admissible $\boldsymbol{i}, \boldsymbol{j}, r$. All the words in $I^{\delta,j}$ are separated, so the module $L_{\delta,j}$ is semicuspidal, which implies that $I^{\delta,j} \subseteq I^{\delta}_{\mathrm{sc}}$.

For example,

$$I^{\delta,1} = \{(0, e-1, e-2, \dots, 1)\} \quad \text{and} \quad I^{\delta,e-1} = \{(0, 1, \dots, e-1)\},$$

so $L_{\delta,1}$ and $L_{\delta,e-1}$ have $\mathbb{Z}$-rank 1. On the other hand, for $e \geq 3$, the module $L_{\delta,e-2}$ has $\mathbb{Z}$-rank $e-2$, since

$$I^{\delta,e-2} = \{(0, 1, \dots, r, e-1, r+1, r+2, \dots, e-2) \mid 0 \leq r < e-2\}.$$

For a composition $\boldsymbol{d} = (d_1, \dots, d_{e-1}) \in \Lambda(e-1, d)$, consider the semicuspidal $R_{d\delta}$-module $V^{\boldsymbol{d}} := L_{\delta,1}^{\circ d_1} \circ \cdots \circ L_{\delta,e-1}^{\circ d_{e-1}}$.

Theorem 4.28. *Let $\mathbb{F}$ be an arbitrary field. There is an assignment $\boldsymbol{\lambda} \mapsto L(\boldsymbol{\lambda})$ which maps every element $\boldsymbol{\lambda} \in \mathscr{P}^J(d)$ to a semicuspidal irreducible $R_{d\delta,\mathbb{F}}$-module $L(\boldsymbol{\lambda})$ such that*

(i) *$\{L(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \mathscr{P}^J(d)\}$ is a complete and irredundant set of irreducible semicuspidal $R_{d\delta,\mathbb{F}}$-modules;*

(ii) *Let* $\boldsymbol{d} = (d_1, \dots, d_{e-1}) \in \Lambda(e-1, d)$ *and*

$$\mathscr{P}^J(\boldsymbol{d}) = \{\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(e-1)}) \in \mathscr{P}^J(d) \mid |\lambda^{(j)}| = d_j \text{ for all } j \in J\}.$$

Then $\{L(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \mathscr{P}^J(\boldsymbol{d})\}$ *is the set of composition factors of* $V_{\mathbb{F}}^{\boldsymbol{d}}$.

Proof. This is essentially contained in [$\mathbf{K}_2$] and [$\mathbf{KM}_1$], but we provide some details for the reader's convenience. In this proof, we drop the subscript $\mathbb{F}$ from our notation. Fix $n \in \mathbb{Z}_{\geq d}$. Let $j \in J$, $m \in \mathbb{Z}_{\geq 0}$, and $\nu \in \Lambda(n, m)$. In [$\mathbf{KM}_1$, §1.4], certain submodules $Z_j^\nu \subseteq L_{\delta,j}^{\circ m}$ are constructed. Let $Z_j := \bigoplus_{\nu \in \Lambda(n,m)} Z_j^\nu$ and $\mathscr{S}_{m,j} := R_{d\delta}/\mathrm{Ann}_{R_{d\delta}}(Z_j)$. In [$\mathbf{KM}_1$, Theorems 4 and 6] a complete and irredundant family $\{L_j(\lambda) \mid \lambda \in \mathscr{P}(m)\}$ of irreducible $\mathscr{S}_{m,j}$-modules is constructed and it is proved that Z_j is a projective generator for $\mathscr{S}_{m,j}$, hence every $L_j(\lambda)$ appears as a composition factor of Z_j. But Z_j is a direct sum of submodules of $L_{\delta,j}^{\circ m}$ and one of the summands is $L_{\delta,j}^{\circ m}$ itself. So every $L_j(\lambda)$ appears as a composition factor of $L_{\delta,j}^{\circ m}$.

Now, for $\boldsymbol{\lambda} \in \mathscr{P}^J(d)$, we consider the $R_{d\delta}$-module $L(\boldsymbol{\lambda}) := L_1(\lambda^{(1)}) \circ \cdots \circ L_{e-1}(\lambda^{(e-1)})$. By [$\mathbf{K}_2$, Theorem 5.10], this module is semicuspidal and irreducible, and $\{L(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \mathscr{P}^J(d)\}$ is a complete and irredundant set of irreducible semicuspidal $R_{d\delta,\mathbb{F}}$-modules, proving (i). Now (ii) follows from the description of the composition factors of each $L_{\delta,j}^{\circ d_j}$ in the previous paragraph. □

Corollary 4.29. *The set* $I_{\mathrm{sc}}^{d\delta}$ *is exactly the set of all shuffles of words* $\boldsymbol{i}^{(1)}, \dots, \boldsymbol{i}^{(d)}$ *such that each* $\boldsymbol{i}^{(a)} \in \bigsqcup_{j \in J} I^{\delta,j}$.

Proof. If $\boldsymbol{i}$ is a shuffle of words $\boldsymbol{i}^{(1)}, \dots, \boldsymbol{i}^{(d)}$ such that $\boldsymbol{i}^{(a)} \in I^{\delta,j_a}$ for $a = 1, \dots, d$, then $\boldsymbol{i}$ is a word of the semicuspidal module $L_{\delta,j_1} \circ \cdots \circ L_{\delta,j_d}$, so $\boldsymbol{i} \in I_{\mathrm{sc}}^{d\delta}$. Conversely, let $\boldsymbol{i} \in I_{\mathrm{sc}}^{d\delta}$. By definition, $1_{\boldsymbol{i}}$ is non-zero in $\hat{C}_{d\delta}$. Since $1_{\boldsymbol{i}}$ is an idempotent, it follows that $1_{\boldsymbol{i},\mathbb{F}} := 1_{\boldsymbol{i}} \otimes 1_{\mathbb{F}}$ is non-zero in $\hat{C}_{d\delta,\mathbb{F}}$ for some field $\mathbb{F}$. Hence there is an irreducible semicuspidal $R_{d\delta,\mathbb{F}}$-module L such that $1_{\boldsymbol{i},\mathbb{F}} L \neq 0$. By Theorem 4.28, the word $\boldsymbol{i}$ is a shuffle of words $\boldsymbol{i}^{(1)}, \dots, \boldsymbol{i}^{(d)}$ such that each $\boldsymbol{i}^{(a)} \in \bigsqcup_{j \in J} I^{\delta,j}$. □

Example 4.30. Let $e = 5$ and $d = 2$. Then the word 0012342341 is in $I_{\mathrm{sep}}^{d\delta}$, but is not in $I_{\mathrm{sc}}^{d\delta}$ by Corollary 4.29.

4.6. **Induction and restriction of semicuspidal modules.** Throughout the subsection we fix $d \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{>0}$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda(n, d)$. Denote

$$R_{\lambda\delta} := R_{\lambda_1\delta,\dots,\lambda_n\delta} \subseteq R_{d\delta}.$$

Let $1_{\lambda\delta}$ denote the identity element of $R_{\lambda\delta}$. Define the semicuspidal parabolic subalgebra

$$\hat{C}_{\lambda\delta} \subseteq 1_{\lambda\delta} \hat{C}_{d\delta} 1_{\lambda\delta}$$

to be the image of $R_{\lambda\delta}$ under the natural projection $R_{d\delta} \twoheadrightarrow \hat{C}_{d\delta}$. Whereas the parabolic subalgebra $R_{\lambda\delta}$ has been identified with $R_{\lambda_1\delta} \otimes \cdots \otimes R_{\lambda_n\delta}$ via the embedding $\iota_{\lambda_1\delta,\dots,\lambda_n\delta}$, it is not clear a priori that $\hat{C}_{\lambda\delta} \cong \hat{C}_{\lambda_1\delta} \otimes \cdots \otimes \hat{C}_{\lambda_n\delta}$. This will be proved in Lemma 4.33.

We call an $R_{\lambda_1\delta} \otimes \cdots \otimes R_{\lambda_n\delta}$-module W semicuspidal if $(1_{\boldsymbol{i}^{(1)}} \otimes \cdots \otimes 1_{\boldsymbol{i}^{(n)}})W = 0$ whenever $\boldsymbol{i}^{(1)}, \dots, \boldsymbol{i}^{(n)}$ are not all separated. This is equivalent to the property that W factors through the natural quotient $\hat{C}_{\lambda_1\delta} \otimes \cdots \otimes \hat{C}_{\lambda_n\delta}$ of $R_{\lambda_1\delta} \otimes \cdots \otimes R_{\lambda_n\delta}$.

Lemma 4.31. *We have:*

(i) *If W is a semicuspidal $R_{d\delta}$-module, then $\mathrm{Res}_{\lambda_1\delta,\dots,\lambda_n\delta}W$ is a semicuspidal $R_{\lambda_1\delta}\otimes\dots\otimes R_{\lambda_n\delta}$-module.*

(ii) *If $\boldsymbol{i}^{(1)}\in I^{\lambda_1\delta},\dots,\boldsymbol{i}^{(n)}\in I^{\lambda_n\delta}$ and $\boldsymbol{i}^{(1)}\dots\boldsymbol{i}^{(n)}\in I^{d\delta}_{\mathrm{sc}}$, then we have that $\boldsymbol{i}^{(1)}\in I^{\lambda_1\delta}_{\mathrm{sc}},\dots,\boldsymbol{i}^{(n)}\in I^{\lambda_n\delta}_{\mathrm{sc}}$.*

Proof. This is known and can be proved combinatorially using Corollary 4.29. We sketch a representation-theoretic proof for the reader's convenience. Clearly (i) and (ii) are equivalent, and hence it suffices to prove (i) with scalars extended to $\mathbb{C}$ in the case where W is irreducible. This follows for example from [**McN**, Theorem 14.6]. □

Lemma 4.32. *If $\boldsymbol{i}^{(1)}\in I^{\lambda_1\delta}_{\mathrm{sc}},\dots,\boldsymbol{i}^{(n)}\in I^{\lambda_n\delta}_{\mathrm{sc}}$, then there is an isomorphism of $R_{d\delta}$-modules*

$$\begin{aligned}\hat{C}_{d\delta}1_{\boldsymbol{i}^{(1)}\dots\boldsymbol{i}^{(n)}}&\xrightarrow{\sim}\hat{C}_{\lambda_1\delta}1_{\boldsymbol{i}^{(1)}}\circ\dots\circ\hat{C}_{\lambda_n\delta}1_{\boldsymbol{i}^{(n)}},\\ 1_{\boldsymbol{i}^{(1)}\dots\boldsymbol{i}^{(n)}}&\mapsto 1_{\lambda_1\delta,\dots,\lambda_n\delta}\otimes 1_{\boldsymbol{i}^{(1)}}\otimes\dots\otimes 1_{\boldsymbol{i}^{(n)}}.\end{aligned}$$

Proof. Since $\hat{C}_{\lambda_1\delta}1_{\boldsymbol{i}^{(1)}}\circ\dots\circ\hat{C}_{\lambda_n\delta}1_{\boldsymbol{i}^{(n)}}$ is semicuspidal, we can consider it as a $\hat{C}_{d\delta}$-module. So there exists a homomorphism as in the lemma. To construct the inverse homomorphism, use adjointness of induction and restriction together with Lemma 4.31(i). □

Lemma 4.33. *The natural map $R_{\lambda_1\delta}\otimes\dots\otimes R_{\lambda_n\delta}\hookrightarrow R_{d\delta}\twoheadrightarrow\hat{C}_{d\delta}$ factors through $\hat{C}_{\lambda_1\delta}\otimes\dots\otimes\hat{C}_{\lambda_n\delta}$ and induces an isomorphism $\hat{C}_{\lambda_1\delta}\otimes\dots\otimes\hat{C}_{\lambda_n\delta}\xrightarrow{\sim}\hat{C}_{\lambda\delta}$. Moreover, $\hat{C}_{d\delta}1_{\lambda\delta}$ is a free right $\hat{C}_{\lambda\delta}$-module with basis $\{\psi_w\mid w\in\mathscr{D}^{e\lambda}\}$.*

Proof. That the map factors through $\hat{C}_{\lambda_1\delta}\otimes\dots\otimes\hat{C}_{\lambda_n\delta}$ follows from Lemma 4.31. For the remaining claims, let us consider the $R_{d\delta}$-module $W:=\hat{C}_{\lambda_1\delta}\circ\dots\circ\hat{C}_{\lambda_n\delta}$. By Lemma 4.25, the module W factors through $\hat{C}_{d\delta}$. On the other hand by the Basis Theorem 4.13 for $R_{d\delta}$, we can decompose $W=\bigoplus_{w\in\mathscr{D}^{e\lambda}}\psi_w1_{\lambda\delta}\otimes\hat{C}_{\lambda_1\delta}\otimes\dots\otimes\hat{C}_{\lambda_n\delta}$ as a $\mathbb{Z}$-module, with each summand being naturally isomorphic to $\hat{C}_{\lambda_1\delta}\otimes\dots\otimes\hat{C}_{\lambda_n\delta}$ as a $\mathbb{Z}$-module. The lemma follows. □

In view of the lemma we identify $\hat{C}_{\lambda_1\delta}\otimes\dots\otimes\hat{C}_{\lambda_n\delta}$ with $\hat{C}_{\lambda\delta}$. Then:

Corollary 4.34. *Suppose that for each $r=1,\dots,n$ we have a $\hat{C}_{\lambda_r\delta}$-module W_r. Then there is a natural isomorphism of semicuspidal $R_{d\delta}$-modules*

$$\begin{aligned}W_1\circ\dots\circ W_n&\xrightarrow{\sim}\hat{C}_{d\delta}1_{\lambda\delta}\otimes_{\hat{C}_{\lambda\delta}}(W_1\boxtimes\dots\boxtimes W_n),\\ u1_{\lambda\delta}\otimes w_1\otimes\dots\otimes w_n&\mapsto\bar{u}1_{\lambda\delta}\otimes w_1\otimes\dots\otimes w_n,\end{aligned}$$

where $\bar{u}\in\hat{C}_{d\delta}$ is the image of $u\in R_{d\delta}$ under the natural projection $R_{d\delta}\twoheadrightarrow\hat{C}_{d\delta}$.

From now on we identify the induced modules as in the corollary.

5. Abaci, tableaux and RoCK blocks

5.1. **Abaci.** We will use the abacus notation for partitions, see [**JK**, Section 2.7]. Recall that we have fixed a number $e\in\mathbb{Z}_{\geq 2}$ and $I=\mathbb{Z}/e\mathbb{Z}$. When convenient we identify I with the subset $\{0,1,\dots,e-1\}\subset\mathbb{Z}$. We define the *abacus* $\mathsf{A}^e:=$

$\mathbb{Z}_{\geq 0} \times I$. For $i \in I$, the subset $\mathsf{R}_i := \mathbb{Z}_{\geq 0} \times \{i\} \subset \mathsf{A}^e$ is referred to as the (ith) *runner* of A^e.

Let λ be a partition, and fix an integer $N \geq \ell(\lambda)$, so that we can write $\lambda = (\lambda_1, \dots, \lambda_N)$. Let

$$\mathsf{A}_N(\lambda) := \{\lambda_k + N - k \mid k = 1, \dots, N\} \subset \mathbb{Z}_{\geq 0}. \tag{5.1}$$

The *abacus display of* λ is

$$\mathsf{A}^e_N(\lambda) := \{(t, i) \in \mathsf{A}^e \mid et + i \in \mathsf{A}_N(\lambda)\}.$$

The elements of $\mathsf{A}^e_N(\lambda)$ are called the *beads* of $\mathsf{A}^e_N(\lambda)$, and the elements of $\mathsf{A}^e \setminus \mathsf{A}^e_N(\lambda)$ are called the *non-beads* of $\mathsf{A}^e_N(\lambda)$.

We have the total order $<$ on A^e defined by the condition that $(t, i) < (q, j)$ if and only if $et + i < eq + j$. If $(t, i) < (q, j)$, we say that (t, i) *precedes* (q, j) and (q, j) *succeeds* (t, i). For any $r \in \mathbb{Z}_{>0}$, we say that a bead (t, i) of $\mathsf{A}^e_N(\lambda)$ is the *bead with number* r in $\mathsf{A}^e_N(\lambda)$ if exactly $r - 1$ beads of $\mathsf{A}^e_N(\lambda)$ succeed (t, i), and we say that a non-bead (t, i) of $\mathsf{A}^e_N(\lambda)$ is the *non-bead with number* r in $\mathsf{A}^e_N(\lambda)$ if exactly $r - 1$ non-beads of $\mathsf{A}^e_N(\lambda)$ precede (t, i).

It is easy to see that the bead (t, i) with number r of $\mathsf{A}^e_N(\lambda)$ satisfies $et + i = N + \lambda_r - r$. Moreover, if $(\lambda'_1, \lambda'_2, \dots)$ is the conjugate partition to λ, then the non-bead (t, i) with number s of $\mathsf{A}^e_N(\lambda)$ satisfies $et + i = N - \lambda'_s + s - 1$. Using these observations, it is easy to prove the following well-known fact:

Lemma 5.2. *Let $\lambda \in \mathscr{P}$ and $(r, s) \in \mathsf{N}$. Then $(r, s) \in [\![\lambda]\!]$ if and only if the bead with number r succeeds the non-bead with number s in $\mathsf{A}^e_N(\lambda)$.*

For $\lambda \in \mathscr{P}$, we write $b_i(\lambda) := |\mathsf{A}^e_N(\lambda) \cap \mathsf{R}_i|$ for $i \in I$. The *e-core* of λ is the partition $\mathrm{core}(\lambda)$ defined by

$$\mathsf{A}^e_N(\mathrm{core}(\lambda)) = \{(t, i) \in \mathsf{A}^e \mid i \in I,\ 0 \leq t < b_i(\lambda)\}.$$

Recall the notation (5.1). The *e-quotient* of λ is defined as the multipartition $\mathrm{quot}_N(\lambda) = (\lambda^{(i)})_{i \in I} \in \mathscr{P}^I$ such that for every $i \in I$, the partition $\lambda^{(i)}$ is determined from $\mathsf{A}_{b_i(\lambda)}(\lambda^{(i)}) = \mathsf{A}^e_N(\lambda) \cap \mathsf{R}_i$, where we have identified R_i with $\mathbb{Z}_{\geq 0}$. The e-quotient of λ depends on the residue of N modulo e and changes by a 'cyclic permutation' of the components $\lambda^{(i)}$ when this residue changes. So the *e-weight* of λ, defined as $\mathrm{wt}(\lambda) := |\mathrm{quot}_N(\lambda)|$, does not depend on N.

Note that $\lambda = \mathrm{core}(\lambda)$ if and only $\mathrm{quot}_N(\lambda) = \varnothing$, in which case λ is said to be an *e-core*. For any e-core ρ and $d \in \mathbb{Z}_{\geq 0}$, we set

$$\mathscr{P}_\rho := \{\lambda \in \mathscr{P} \mid \mathrm{core}(\lambda) = \rho\}, \quad \mathscr{P}_{\rho,d} := \{\lambda \in \mathscr{P}_\rho \mid \mathrm{wt}(\lambda) = d\}.$$

The following is easy to check and well known:

Lemma 5.3. *The map $\lambda \mapsto \mathrm{quot}(\lambda)$ is a bijection from $\mathscr{P}_{\rho,d}$ to $\mathscr{P}^I(d)$.*

The *(e-)residue* of a node $(r, s) \in \mathsf{N}$ is $\mathrm{res}(r, s) := s - r + e\mathbb{Z} \in I = \mathbb{Z}/e\mathbb{Z}$. For $i \in I$, we say that (r, s) is an *i-node* if its residue is i. For $\lambda \in \mathscr{P}$, we define

$$\mathrm{cont}(\lambda) := \sum_{u \in [\![\lambda]\!]} \alpha_{\mathrm{res}(u)} \in Q_+.$$

Lemma 5.4. [**JK**, Theorem 2.7.41] *Let ρ be an e-core, $d \in \mathbb{Z}_{\geq 0}$, and $\lambda \in \mathscr{P}$. Then $\mathrm{cont}(\lambda) = \mathrm{cont}(\rho) + d\delta$ if and only if $\lambda \in \mathscr{P}_{\rho,d}$.*

5.2. **Tableaux.** Let ν be a partition. A node $u \in \mathsf{N}$ is called an *addable* node for ν if $u \notin [\![\nu]\!]$ and $[\![\nu]\!] \cup \{u\}$ is the Young diagram of a partition, and u is called a *removable* node of ν if $u \in [\![\nu]\!]$ and $[\![\nu]\!] \setminus \{u\}$ is the Young diagram of a partition. For $i \in I$, we denote by $\mathsf{Add}(\nu, i)$ (resp. $\mathsf{Rem}(\nu, i)$) the set of all addable (resp. removable) i-nodes for ν. We say that a node (r, s) is *above* a node (r', s') if $r < r'$. Given a node $v \in \mathsf{N}$ and a finite subset $U \subset \mathsf{N}$, denote by $a(v, U)$ the number of elements of U which are above v.

Let $i \in I$ and U be a set of removable i-nodes of ν. Define

$$d_U(\nu) = \sum_{v \in \mathsf{Add}(\nu,i)} a(v, U) - \sum_{v \in \mathsf{Rem}(\nu,i) \setminus U} a(v, U).$$

Let $\lambda \setminus \mu$ be a skew partition, and $\theta = \mathrm{cont}(\lambda \setminus \mu) := \sum_{u \in [\![\lambda]\!] \setminus [\![\mu]\!]} \alpha_{\mathrm{res}(u)} \in Q_+$. Fix $\boldsymbol{i} = (i_1^{(m_1)}, \dots, i_r^{(m_r)}) \in I^\theta_{\mathrm{div}}$. An *$\boldsymbol{i}$-standard tableau* of shape $\lambda \setminus \mu$ is a map $\mathtt{t} \colon [\![\lambda]\!] \setminus [\![\mu]\!] \to \{1, \dots, r\}$ such that

(i) $\mathtt{t}(u) < \mathtt{t}(u')$ whenever $u, u' \in [\![\lambda]\!] \setminus [\![\mu]\!]$ and $u < u'$;
(ii) for all $k = 1, \dots, r$ and $u \in \mathtt{t}^{-1}(k)$, we have $\mathrm{res}\, u = i_k$;
(iii) for all $k = 1, \dots, r$, we have $|\mathtt{t}^{-1}(k)| = m_k$.

We denote the set of all $\boldsymbol{i}$-standard tableaux of shape $\lambda \setminus \mu$ by $\mathsf{Std}(\lambda \setminus \mu, \boldsymbol{i})$. If $\mathtt{t} \in \mathsf{Std}(\lambda \setminus \mu, \boldsymbol{i})$, we define

$$\deg(\mathtt{t}) = \sum_{k=1}^{r} d_{\mathtt{t}^{-1}(k)}(\mathtt{t}^{-1}([1, k]) \cup [\![\mu]\!]).$$

Note that $\deg(\mathtt{t})$ depends on λ and μ, not just on the set $[\![\lambda]\!] \setminus [\![\mu]\!]$. If $\boldsymbol{i} \in I^\theta$ and $\mu = \varnothing$, then the notion of an $\boldsymbol{i}$-standard tableau is the same as the usual notion of a standard tableau with residue sequence $\boldsymbol{i}$ as in [**BKW**, §3.2], and the notion of the degree agrees with the one from [**BKW**, §3.5]. If $\boldsymbol{i} \in I^\eta_{\mathrm{div}}$ for some $\eta \neq \theta$, then we set $\mathsf{Std}(\lambda \setminus \mu, \boldsymbol{i}) := \varnothing$. We denote

$$\mathsf{Std}(\lambda \setminus \mu) := \bigsqcup_{\boldsymbol{i} \in I^{\mathrm{cont}(\lambda \setminus \mu)}} \mathsf{Std}(\lambda \setminus \mu, \boldsymbol{i}).$$

Let $\boldsymbol{i} = (i_1^{(m_1)}, \dots, i_r^{(m_r)}) \in I^\theta_{\mathrm{div}}$ and $\hat{\boldsymbol{i}} \in I^\theta$ be as in (4.23). Given $\mathtt{t} \in \mathsf{Std}(\lambda \setminus \mu, \boldsymbol{i})$, a tableau $\mathtt{s} \in \mathsf{Std}(\lambda \setminus \mu, \hat{\boldsymbol{i}})$ is called a refinement of $\mathtt{t}$ if

$$\mathtt{t}^{-1}(k) = \mathtt{s}^{-1}\big([m_1 + \dots + m_{k-1} + 1, m_1 + \dots + m_k]\big)$$

for all $k = 1, \dots, r$. Let $\hat{\mathtt{t}} \subseteq \mathsf{Std}(\lambda \setminus \mu, \hat{\boldsymbol{i}})$ denote the set of all refinements of $\mathtt{t}$.

Lemma 5.5. *For any $\mathtt{t} \in \mathsf{Std}(\lambda \setminus \mu, \boldsymbol{i})$, we have $\sum_{\mathtt{s} \in \hat{\mathtt{t}}} q^{\deg(\mathtt{s})} = \boldsymbol{i}! q^{\deg(\mathtt{t})}$.*

Proof. The lemma is easily reduced to the case $r = 1$. In that case, let $\mathtt{s} \in \hat{t}$ be the tableau such that for $u, v \in [\![\lambda]\!] \setminus [\![\mu]\!]$ the node u is above v if and only if $\mathtt{s}(u) < \mathtt{s}(v)$, in other words we assign the numbers $1, \dots, m := m_1$ to the nodes of $\lambda \setminus \mu$ from top to bottom. Then $\deg(\mathtt{s}) = \deg(\mathtt{t}) + m(m-1)/2$. We have $\hat{t} = \{w\mathtt{s} \mid w \in \mathfrak{S}_m\}$, where $w\mathtt{s}$ is the tableau defined by $(w\mathtt{s})(u) = w(\mathtt{s}(u))$. In view of [**BKW**, Proposition 3.13], we have $\deg(w\mathtt{s}) = \deg(\mathtt{s}) - 2\ell(w)$, where $\ell(w)$ is the length of $w \in \mathfrak{S}_d$. So

$$\sum_{\mathtt{s} \in \hat{\mathtt{t}}} q^{\deg(\mathtt{s})} = q^{\deg(\mathtt{t}) + m(m-1)/2} \sum_{w \in \mathfrak{S}_m} q^{-2\ell(w)} = [m]^! q^{\deg(\mathtt{t})},$$

where the last equality comes from the well-known formula for the Poincaré polynomial of the symmetric group [**Hu**, §3.15]. □

5.3. **Dimensions and core algebras.** Recall the notation (4.22). The following is a variation of a known result:

Theorem 5.6. *For any $\theta \in Q_+$ and $\boldsymbol{i}, \boldsymbol{j} \in I^\theta_{\mathrm{div}}$, the $\mathbb{Z}$-module $1_{\boldsymbol{i}} R^{\Lambda_0}_\theta 1_{\boldsymbol{j}}$ is free of graded rank*

$$\dim_q \left(1_{\boldsymbol{i}} R^{\Lambda_0}_\theta 1_{\boldsymbol{j}}\right) = q^{\langle \boldsymbol{j}\rangle - \langle \boldsymbol{i}\rangle} \sum_{\substack{\mu \in \mathscr{P} \\ \mathtt{s} \in \mathsf{Std}(\mu, \boldsymbol{i}) \\ \mathtt{t} \in \mathsf{Std}(\mu, \boldsymbol{j})}} q^{\deg(\mathtt{s}) + \deg(\mathtt{t})}.$$

In particular, the idempotent $1_{\boldsymbol{i}}$ is non-zero in $R^{\Lambda_0}_\theta$ if and only if $\mathsf{Std}(\mu, \boldsymbol{i}) \neq \varnothing$ for some $\mu \in \mathscr{P}$.

Proof. The freeness statement follows from Theorem 4.14. Extending scalars to $\mathbb{C}$ and using [$\mathbf{BK}_2$, Theorem 4.20] yields the graded rank formula in the case when $\boldsymbol{i}, \boldsymbol{j} \in I^\theta$, and the general case then follows from Lemmas 4.24 and 5.5. □

Recall the notation $\ell(A)$ for an algebra A from §2.3.

Theorem 5.7. *Let $\Bbbk$ be a field, ρ be an e-core and $d \in \mathbb{Z}_{\geq 0}$. Then*

$$\ell(R^{\Lambda_0}_{\mathrm{cont}(\rho)+d\delta, \Bbbk}) = |\mathscr{P}^J(d)|.$$

Proof. By [**KK**, Theorem 6.2] or [**LV**, Theorem 7.5], the number $\ell(R^{\Lambda_0}_{\mathrm{cont}(\rho)+d\delta,\Bbbk})$ is equal to the dimension of the weight space $V(\Lambda_0)_{\Lambda_0 - \mathrm{cont}(\rho) - d\delta}$ for the integrable highest weight module $V(\Lambda_0)$ over the Kac-Moody algebra $\mathfrak{g}$ of type $A^{(1)}_{e-1}$. It is well known that this dimension is equal to $|\mathscr{P}^J(d)|$, see e.g. [**Ka**, (13.11.5)] or [**LLT**, Sections 4,5]. □

Let ρ be an e-core. We pick an extremal word $(i_1^{a_1}, \dots, i_r^{a_r}) \in I^{\mathrm{cont}(\rho)}$ for the left regular module $R^{\Lambda_0}_{\mathrm{cont}(\rho)}$, see [$\mathbf{K}_2$, §2.8]. In particular, $i_k \neq i_{k+1}$ for $1 \leq k < r$. Let $\boldsymbol{i} = (i_1^{(a_1)}, \dots, i_r^{(a_r)}) \in I^{\mathrm{cont}(\rho)}_{\mathrm{div}}$.

Lemma 5.8. *Let ρ be an e-core and $\boldsymbol{i} \in I^{\mathrm{cont}(\rho)}_{\mathrm{div}}$ be chosen as above. Then there is an isomorphism of graded $\mathbb{Z}$-algebras $R^{\Lambda_0}_{\mathrm{cont}(\rho)} \xrightarrow{\sim} \mathrm{End}_{\mathbb{Z}}(R^{\Lambda_0}_{\mathrm{cont}(\rho)} 1_{\boldsymbol{i}})$, where $x \in R^{\Lambda_0}_{\mathrm{cont}(\rho)}$ gets mapped to the left multiplication by x.*

Proof. We clearly have a homomorphism $\varphi\colon R^{\Lambda_0}_{\mathrm{cont}(\rho)} \to \mathrm{End}_{\mathbb{Z}}(R^{\Lambda_0}_{\mathrm{cont}(\rho)} 1_{\boldsymbol{i}})$ as in the statement. In view of Theorem 4.14, to prove that φ is an isomorphism, it suffices to prove its scalar extension $\varphi_{\Bbbk}$ is an isomorphism for any algebraically closed field $\Bbbk$. By Theorem 5.7, the algebra $R^{\Lambda_0}_{\mathrm{cont}(\rho),\Bbbk}$ has only one irreducible module L up to isomorphism and degree shift. Considering the composition series of the left regular module over $R^{\Lambda_0}_{\mathrm{cont}(\rho),\Bbbk}$, we see that $\boldsymbol{i}$ is an extremal weight for L, hence by [$\mathbf{K}_2$, Lemma 2.8], the space $1_{\boldsymbol{i}} L$ is 1-dimensional. It follows that $\mathrm{Hom}_{R^{\Lambda_0}_{\mathrm{cont}(\rho),\Bbbk}}(R^{\Lambda_0}_{\mathrm{cont}(\rho),\Bbbk} 1_{\boldsymbol{i}}, L) \cong 1_{\boldsymbol{i}} L$ is 1-dimensional, so $R^{\Lambda_0}_{\mathrm{cont}(\rho),\Bbbk} 1_{\boldsymbol{i}}$ is the projective cover of L. We claim that in fact $R^{\Lambda_0}_{\mathrm{cont}(\rho),\Bbbk} 1_{\boldsymbol{i}} \cong L$. This

is known for $\Bbbk = \mathbb{C}$ since $R^{\Lambda_0}_{\mathrm{cont}(\rho),\mathbb{C}}$ is a simple algebra: indeed, by [$\mathbf{BK}_1$] it is a defect zero block of an Iwahori-Hecke algebra at an eth root of unity. Hence $\mathrm{Hom}_{R^{\Lambda_0}_{\mathrm{cont}(\rho),\mathbb{C}}}(R^{\Lambda_0}_{\mathrm{cont}(\rho),\mathbb{C}}1_{\boldsymbol{i}}, R^{\Lambda_0}_{\mathrm{cont}(\rho),\mathbb{C}}1_{\boldsymbol{i}}) \cong 1_{\boldsymbol{i}}R^{\Lambda_0}_{\mathrm{cont}(\rho),\mathbb{C}}1_{\boldsymbol{i}}$ is 1-dimensional. This proves that $1_{\boldsymbol{i}}R^{\Lambda_0}_{\mathrm{cont}(\rho)}1_{\boldsymbol{i}}$ has rank 1 as a $\mathbb{Z}$-module, whence $1_{\boldsymbol{i}}R^{\Lambda_0}_{\mathrm{cont}(\rho),\Bbbk}1_{\boldsymbol{i}} \cong \mathrm{Hom}_{R^{\Lambda_0}_{\mathrm{cont}(\rho),\Bbbk}}(R^{\Lambda_0}_{\mathrm{cont}(\rho),\Bbbk}1_{\boldsymbol{i}}, R^{\Lambda_0}_{\mathrm{cont}(\rho),\Bbbk}1_{\boldsymbol{i}})$ has dimension 1. Hence, $R^{\Lambda_0}_{\mathrm{cont}(\rho),\Bbbk}1_{\boldsymbol{i}} \cong L$. We deduce that $R^{\Lambda_0}_{\mathrm{cont}(\rho),\Bbbk}$ is a simple algebra and $\varphi_{\Bbbk}$ is an isomorphism. □

Recall the map $\zeta_{\theta,\eta}$ from (4.19).

Lemma 5.9. *If ρ is an e-core and $d \in \mathbb{Z}_{\geq 0}$, then the map*

$$\zeta_{\mathrm{cont}(\rho),d\delta} \colon R^{\Lambda_0}_{\mathrm{cont}(\rho)} \to 1_{\mathrm{cont}(\rho),d\delta}R^{\Lambda_0}_{\mathrm{cont}(\rho)+d\delta}1_{\mathrm{cont}(\rho),d\delta}$$

is injective.

Proof. By Theorem 4.14, it suffices to prove that the scalar extension of the map to $\mathbb{C}$ is injective. By Lemma 5.8, $R^{\Lambda_0}_{\mathrm{cont}(\rho),\mathbb{C}}$ is a simple algebra, so it is enough to show that $1_{\mathrm{cont}(\rho),d\delta}R^{\Lambda_0}_{\mathrm{cont}(\rho)+d\delta}1_{\mathrm{cont}(\rho),d\delta} \neq 0$. The last fact follows easily from Theorem 5.6. □

5.4. **RoCK blocks.** Let ρ be an e-core and $d \in \mathbb{Z}_{\geq 1}$. Following [$\mathbf{Tu}_1$, Definition 52], we say that ρ is a *d-Rouquier core* if there exists an integer $N \geq \ell(\rho)$ such that for all $i = 0, \dots, e-2$, the abacus display $\mathsf{A}^e_N(\rho)$ has at least $d-1$ more beads on runner $i+1$ than on runner i. In this case,

$$\kappa := -N + e\mathbb{Z} \in \mathbb{Z}/e\mathbb{Z}$$

is well-defined and is called the *residue* of ρ.

If ρ is a d-Rouquier core, we refer to the cyclotomic KLR algebra $R^{\Lambda_0}_{\mathrm{cont}(\rho)+d\delta}$ as a *RoCK block*.

Remark 5.10. The term *RoCK* comes from the names of Rouquier [$\mathbf{R}_1$], Chuang and Kessar [**CK**]. We refer to the algebra $R^{\Lambda_0}_{\mathrm{cont}(\rho)+d\delta}$ as a *block* since, with notation as in Section 1, the block $H_{\mathrm{cont}(\rho)+d\delta}(q)$ of an Iwahori–Hecke algebra is isomorphic to the $\mathbb{F}$-algebra $R^{\Lambda_0}_{\mathrm{cont}(\rho)+d\delta,\mathbb{F}}$, see [$\mathbf{BK}_1, \mathbf{R}_2$]. Note however that the analogous isomorphism in general does not make sense over $\mathbb{Z}$. Moreover, if $\Bbbk$ is a field such that $e = m \operatorname{char} k$ for some $m \in \mathbb{Z}_{>1}$, the algebra $R^{\Lambda_0}_{\theta,\Bbbk}$ is not in general isomorphic to a block of a Hecke algebra.

We now review and develop some results from [**Ev**, Section 4]. Throughout the subsection, we fix $d \in \mathbb{Z}_{>0}$ and a d-Rouquier core ρ of residue κ. We then set

$$\alpha := \mathrm{cont}(\rho) + d\delta \in Q_+.$$

Let

$$\Omega \colon R_{d\delta} \to R^{\Lambda_0}_{\mathrm{cont}(\rho),d\delta},\ x \mapsto \pi_\alpha\big(\iota_{\mathrm{cont}(\rho),d\delta}(1_{\mathrm{cont}(\rho)} \otimes \mathtt{rot}_\kappa(x))\big),$$

cf. (4.11), (4.12) and (4.16). Note that Ω is an algebra homomorphism.

Lemma 5.11. *Let $\boldsymbol{i} \in I^{d\delta}$, and $\boldsymbol{j} \in I^\rho$ be such that $\mathsf{Std}(\rho, \boldsymbol{j}) \neq \varnothing$. If $1_{\boldsymbol{j}(\boldsymbol{i}^{+\kappa})}$ is non-zero in $R^{\Lambda_0}_{\mathrm{cont}(\rho)+d\delta}$, then $\boldsymbol{i} \in I^{d\delta}_{\mathrm{sc}}$. In particular, Ω factors through $\hat{C}_{d\delta}$.*

Proof. This follows from [**Ev**, Lemma 4.6] thanks to Theorem 5.6 and Corollary 4.29. □

In view of the lemma, from now on, we will consider Ω as a homomorphism

$$\Omega : \hat{C}_{d\delta} \to R^{\Lambda_0}_{\mathrm{cont}(\rho),d\delta}. \tag{5.12}$$

Lemma 5.13. *Let σ be a partition such that $[\![\sigma]\!] \subsetneq [\![\rho]\!]$. Then the number of nodes of residue κ in $[\![\rho]\!] \setminus [\![\sigma]\!]$ is less than $(|\rho| - |\sigma|)/e$.*

Proof. In this proof we use abacus displays with N beads, where N is greater than the number of parts in all the partitions involved and $N + e\mathbb{Z} = -\kappa$. Recall from §5.1 that for $\tau \in \mathscr{P}$, we denote $b_i(\tau) := |\,\mathsf{A}^e_N(\tau) \cap \mathsf{R}_i|$. For $0 \leq l < e$, we denote $b_{\geq l}(\tau) := \sum_{i=l}^{e-1} b_i(\tau)$. Recall also the fundamental dominant weights Λ_i from §4.1. Let $0 \leq m < e$ be the integer such that $m + e\mathbb{Z} = -\kappa$.

For any $\tau \in \mathscr{P}$, we claim that

$$e(\Lambda_\kappa, \mathrm{cont}(\tau)) - |\tau| = \frac{(e-1)N - (e-m)m}{2} - \sum_{l=1}^{e-1} b_{\geq l}(\tau). \tag{5.14}$$

Indeed, it is straightforward to check that both sides are 0 when $\tau = \varnothing$, since $b_0(\varnothing) = \cdots = b_{m-1}(\varnothing) = b_m(\varnothing) + 1 = \cdots = b_{e-1}(\varnothing) + 1$. Furthermore, adding a box of residue $i \in I$ to τ changes both sides by $e - 1$ if $i = \kappa$ and by -1 if $i \neq \kappa$ (for the right-hand side, consult [**Ev**, Lemma 4.2]). The claim is proved.

Let $l \in \{0, \dots, e-1\}$ and $b = b_l(\rho)$. Suppose for a contradiction that $b_{\geq l}(\sigma) > b_{\geq l}(\rho)$. As ρ is a Rouquier core, $\mathsf{A}^e_N(\rho)$ contains the rectangle $[0, b-1] \times [l, e-1]$, whence

$$|\,\mathsf{A}^e_N(\sigma) \cap (\mathbb{Z}_{\geq b} \times [l, e-1])| > |\,\mathsf{A}^e_N(\rho) \cap (\mathbb{Z}_{\geq b} \times [l, e-1])|,$$

and it follows that $|\,\mathsf{A}_N(\sigma) \cap \mathbb{Z}_{\geq be}| > |\,\mathsf{A}_N(\rho) \cap \mathbb{Z}_{\geq be}|$. This is a contradiction to the hypothesis $[\![\sigma]\!] \subseteq [\![\rho]\!]$. Hence, $b_{\geq l}(\sigma) \leq b_{\geq l}(\rho)$ for all $l \in \{0, \dots, e-1\}$. Moreover, the inequality must be strict for at least one $l \in \{1, \dots, e-1\}$, for otherwise we have $b_i(\sigma) = b_i(\rho)$ for all $i \in I$, and so $\rho = \mathrm{core}(\sigma)$, contradicting the hypothesis $[\![\sigma]\!] \subsetneq [\![\rho]\!]$. Hence, using (5.14), we deduce that $e(\Lambda_\kappa, \mathrm{cont}(\sigma)) - |\sigma| > e(\Lambda_\kappa, \mathrm{cont}(\rho)) - |\rho|$, which implies the lemma. □

Recall that throughout the subsection $\alpha = \mathrm{cont}(\rho) + d\delta$ is such that $R^{\Lambda_0}_\alpha$ is a RoCK block.

Lemma 5.15. *We have $1_{\mathrm{cont}(\rho),d\delta} R^{\Lambda_0}_\alpha 1_{\mathrm{cont}(\rho),d\delta} = R^{\Lambda_0}_{\mathrm{cont}(\rho),d\delta}$.*

Proof. By Lemma 4.18, $1_{\mathrm{cont}(\rho),d\delta} R^{\Lambda_0}_\alpha 1_{\mathrm{cont}(\rho),d\delta}$ is generated by $R^{\Lambda_0}_{\mathrm{cont}(\rho),d\delta}$ together with the elements ψ_w for $w \in {}^{(|\rho|,de)}\mathscr{D}^{(|\rho|,de|)} \setminus \{1\}$. Thus, it will suffice to show that $1_{\mathrm{cont}(\rho),d\delta} \psi_w 1_{\mathrm{cont}(\rho),d\delta} = 0$ in $R^{\Lambda_0}_\alpha$ for each such w. If not, then $1_{\boldsymbol{j}'((\boldsymbol{i}')^{+\kappa})} \psi_w 1_{\boldsymbol{j}(\boldsymbol{i}^{+\kappa})} \neq 0$ for some $\boldsymbol{j}, \boldsymbol{j}' \in I^{\mathrm{cont}(\rho)}$ such that $\mathsf{Std}(\rho, \boldsymbol{j}), \mathsf{Std}(\rho, \boldsymbol{j}')$ are non-empty, and some $\boldsymbol{i}, \boldsymbol{i}' \in I^{d\delta}_{\mathrm{sc}}$, see Theorem 5.6 and Lemma 5.11. In this case $w(\boldsymbol{j}(\boldsymbol{i}^{+\kappa})) = \boldsymbol{j}'((\boldsymbol{i}')^{+\kappa})$. Moreover, $w = \prod_{t=1}^{m} (|\rho| - m + t, |\rho| + t)$ for some $m > 0$, and therefore the last m entries of $\boldsymbol{j}'$ are $i_1 + \kappa, \dots, i_m + \kappa$. Since $\boldsymbol{i}$ is semicuspidal, the number of entries κ in the tuple $(i_1 + \kappa, \dots, i_m + \kappa)$ is at least m/e. But by Lemma 5.13, this means that $\mathsf{Std}(\rho, \boldsymbol{j}') = \varnothing$, a contradiction. □

By Lemmas 5.9 and 5.15, there is a natural unital algebra embedding

$$\zeta_{\mathrm{cont}(\rho),d\delta}\colon R^{\Lambda_0}_{\mathrm{cont}(\rho)} \to R^{\Lambda_0}_{\mathrm{cont}(\rho),d\delta} = 1_{\mathrm{cont}(\rho),d\delta} R^{\Lambda_0}_{\alpha} 1_{\mathrm{cont}(\rho),d\delta}.$$

We always identify $R^{\Lambda_0}_{\mathrm{cont}(\rho)}$ with a subalgebra of $R^{\Lambda_0}_{\mathrm{cont}(\rho),d\delta}$ via this embedding. We consider the centralizer of $R^{\Lambda_0}_{\mathrm{cont}(\rho)}$ in $R^{\Lambda_0}_{\mathrm{cont}(\rho),d\delta}$:

$$\mathcal{Z}_{\rho,d} := \mathcal{Z}_{R^{\Lambda_0}_{\mathrm{cont}(\rho),d\delta}}(R^{\Lambda_0}_{\mathrm{cont}(\rho)}).$$

Lemma 5.16. *We have an algebra isomorphism $R^{\Lambda_0}_{\mathrm{cont}(\rho)} \otimes \mathcal{Z}_{\rho,d} \xrightarrow{\sim} R^{\Lambda_0}_{\mathrm{cont}(\rho),d\delta}$ given by $a \otimes b \mapsto ab$.*

Proof. This follows from Lemma 5.8 using [**Ev**, Proposition 4.10] (whose proof goes through over $\mathbb{Z}$). □

Recalling (5.12), we denote

$$C_{\rho,d} := \hat{C}_{d\delta} / \ker \Omega. \tag{5.17}$$

We have the induced embedding $\bar{\Omega}\colon C_{\rho,d} \to R^{\Lambda_0}_{\mathrm{cont}(\rho),d\delta}$. By Theorem 4.14, $R^{\Lambda_0}_{\mathrm{cont}(\rho)+d\delta}$ is $\mathbb{Z}$-free, so

Lemma 5.18. *The $\mathbb{Z}$-module $C_{\rho,d}$ is free of finite rank.*

Lemma 5.19. *We have $\mathcal{Z}_{\rho,d} = \bar{\Omega}(C_{\rho,d})$.*

Proof. It is clear from the definitions that $\bar{\Omega}(C_{\rho,d}) = \Omega(\hat{C}_{d\delta}) \subseteq \mathcal{Z}_{\rho,d}$. Conversely, let $x \in \mathcal{Z}_{\rho,d}$. We can write $x = \sum_{i=1}^{m} a_i b_i$ for some $a_1, \dots, a_m \in R^{\Lambda_0}_{\mathrm{cont}(\rho)}$ and $b_1, \dots, b_m \in \Omega(\hat{C}_{d\delta}) = \bar{\Omega}(C_{\rho,d})$, and we may assume that $a_1, \dots, a_m$ are linearly independent with $a_1 = 1$. By Lemma 5.16, $x = b_1$, so $x \in \bar{\Omega}(C_{\rho,d})$. □

In view of Lemma 5.16, we deduce:

Corollary 5.20. *We have:*

(i) *The map $\bar{\Omega}\colon C_{\rho,d} \to \mathcal{Z}_{\rho,d}$ is an algebra isomorphism.*
(ii) *There is an algebra isomorphism $R^{\Lambda_0}_{\mathrm{cont}(\rho)} \otimes C_{\rho,d} \xrightarrow{\sim} R^{\Lambda_0}_{\mathrm{cont}(\rho),d\delta}$ given by $a \otimes b \mapsto a\bar{\Omega}(b)$.*

Remark 5.21. By Lemma 5.8, the algebra $R^{\Lambda_0}_{\mathrm{cont}(\rho)}$ is isomorphic to a graded matrix algebra. Consider the homogeneous matrix unit $E_{1,1}$ in $R^{\Lambda_0}_{\mathrm{cont}(\rho)} \subseteq R^{\Lambda_0}_{\mathrm{cont}(\rho),d\delta}$. By Corollary 5.20, we have $C_{\rho,d} \cong E_{1,1} R^{\Lambda_0}_{\mathrm{cont}(\rho),d\delta} E_{1,1}$. So by Lemma 5.15, we have $C_{\rho,d} \cong E_{1,1} 1_{\mathrm{cont}(\rho),d\delta} R^{\Lambda_0}_{\alpha} E_{1,1} 1_{\mathrm{cont}(\rho),d\delta}$. Note that $\mathbf{e} := E_{1,1} 1_{\mathrm{cont}(\rho),d\delta}$ is an idempotent in $R^{\Lambda_0}_{\alpha}$, so $C_{\rho,d} \cong \mathbf{e} R^{\Lambda_0}_{\alpha} \mathbf{e}$ is an idempotent truncation of $R^{\Lambda_0}_{\alpha}$.

The definition of $C_{\rho,d}$, Lemma 5.8 and Corollary 5.20 make sense and can be proved over an arbitrary unital commutative ring $\Bbbk$, so the algebra $C_{\rho,d}$ defined over $\Bbbk$ is isomorphic to the idempotent truncation

$$(\mathbf{e} \otimes 1) R^{\Lambda_0}_{\alpha,\Bbbk} (\mathbf{e} \otimes 1) \cong (\mathbf{e} R^{\Lambda_0}_{\alpha} \mathbf{e}) \otimes \Bbbk \cong C_{\rho,d,\Bbbk}.$$

Corollary 5.22. *For any field $\Bbbk$, the algebra $C_{\rho,d,\Bbbk}$ is symmetric. More precisely, it admits a symmetrizing form of degree $-2d$.*

Proof. By Remark 5.21, $C_{\rho,d}$ is an idempotent truncation of $R^{\Lambda_0}_\alpha$. By [**SY**, Theorem IV.4.1], an idempotent truncation of a symmetric algebra is symmetric, with a symmetrizing form obtained by restriction. So it suffices to prove that $R^{\Lambda_0}_{\alpha,\Bbbk}$ is symmetric with a symmetrizing form of degree $-2d$. But this follows from Theorem 4.15 and an easy Lie-theoretic computation, see [$\mathbf{K}_1$, Lemma 11.1.4]. □

6. Dimensions

Throughout the section we fix $d \in \mathbb{Z}_{>0}$, a d-Rouquier core ρ of residue κ, and $n \in \mathbb{Z}_{>0}$. We also fix an integer $N \geq |\rho| + de$ such that $N + e\mathbb{Z} = -\kappa$ and assume in this section that *all abaci have N beads*, cf. §5.1.

The main goal of this section is to compute dimensions of certain idempotent truncations of the algebras $C_{\rho,d}$. The idempotents we use here are the so-called Gelfand-Graev idempotents first considered in [$\mathbf{KM}_1$].

6.1. **Gelfand-Graev idempotents.** Recall from §4.5 that for all $j \in J$, we have defined special R_δ-modules $L_{\delta,j}$ with $\mathrm{ch}_q\, L_{\delta,j} = \sum_{\boldsymbol{i} \in I^{\delta,j}} \boldsymbol{i}$. From now on, for every $j \in J$, we fix an arbitrary word

$$\boldsymbol{l}^j = (l_{j,1}, \dots, l_{j,e}) \in I^{\delta,j}. \tag{6.1}$$

Consider the divided power words

$$\boldsymbol{l}^j(d) := (l^{(d)}_{j,1}, \dots, l^{(d)}_{j,e}) \in I^{d\delta}_{\mathrm{div}} \qquad (j \in J). \tag{6.2}$$

Recall the notation (3.7) and let $(\lambda, \boldsymbol{c}) \in \Lambda^{\mathrm{col}}(n,d)$. We set

$$\boldsymbol{l}(\lambda, \boldsymbol{c}) := \boldsymbol{l}^{c_1}(\lambda_1) \dots \boldsymbol{l}^{c_n}(\lambda_n) \in I^{d\delta}_{\mathrm{div}}.$$

Now, we define the *Gelfand-Graev idempotent* $\gamma^{\lambda,\boldsymbol{c}}$ and the integer a_λ as follows:

$$\gamma^{\lambda,\boldsymbol{c}} := 1_{\boldsymbol{l}(\lambda,\boldsymbol{c})} \in R_{d\delta}, \tag{6.3}$$

$$a_\lambda := -\langle \boldsymbol{l}(\lambda, \boldsymbol{c}) \rangle = -e \sum_{t=1}^{n} \lambda_t(\lambda_t - 1)/2, \tag{6.4}$$

cf. §4.4. In the special case $n = 1$, $\lambda = (d)$, $\boldsymbol{c} = (j)$, we also use the notation

$$\gamma^{d,j} := 1_{\boldsymbol{l}^j(d)}. \tag{6.5}$$

We set

$$\omega := (1, \dots, 1) \in \Lambda(d,d), \tag{6.6}$$

$$\gamma^\omega := \sum_{\boldsymbol{b} \in J^d} \gamma^{\omega,\boldsymbol{b}} \in 1_{\omega\delta} R_{d\delta} 1_{\omega\delta}. \tag{6.7}$$

Lemma 6.8. *For any $(\lambda, \boldsymbol{c}), (\lambda', \boldsymbol{c}') \in \Lambda^{\mathrm{col}}(n,d)$, we have*

$$\dim_q(\gamma^{\lambda,\boldsymbol{c}} C_{\rho,d} \gamma^{\lambda',\boldsymbol{c}'}) = q^{a_\lambda - a_{\lambda'}} \sum_{\substack{\mu \in \mathscr{P}_{\rho,d} \\ \mathtt{t} \in \mathsf{Std}(\mu\backslash\rho, \boldsymbol{l}(\lambda,\boldsymbol{c})^{+\kappa}) \\ \mathtt{t}' \in \mathsf{Std}(\mu\backslash\rho, \boldsymbol{l}(\lambda',\boldsymbol{c}')^{+\kappa})}} q^{\deg(\mathtt{t}) + \deg(\mathtt{t}')}. \tag{6.9}$$

Proof. It follows from Lemma 5.4, Theorem 5.6 and Corollary 5.20 that

$$\dim_q(R^{\Lambda_0}_{\mathrm{cont}(\rho)})\dim_q(\gamma^{\lambda,\boldsymbol{c}}C_{\rho,d}\gamma^{\lambda',\boldsymbol{c}'}) = \\ = q^{a_\lambda - a_{\lambda'}} \sum_{\substack{\mu\in\mathscr{P}_{\rho,d},\,\boldsymbol{j},\boldsymbol{j}'\in I^{\mathrm{cont}(\rho)}\\ \mathtt{t}\in\mathsf{Std}(\mu,\boldsymbol{j}(\boldsymbol{l}(\lambda,\boldsymbol{c})^{+\kappa}))\\ \mathtt{t}'\in\mathsf{Std}(\mu,\boldsymbol{j}(\boldsymbol{l}(\lambda',\boldsymbol{c}')^{+\kappa}))}} q^{\deg(\mathtt{t})+\deg(\mathtt{t}')}. \tag{6.10}$$

For each $\mu\in\mathscr{P}_{\rho,d}$ and $\boldsymbol{j}\in I^{\mathrm{cont}(\rho)}$, in view of Lemma 5.4, we have a bijection

$$\mathsf{Std}(\mu,\boldsymbol{j}(\boldsymbol{l}(\lambda,\boldsymbol{c})^{+\kappa}))\xrightarrow{\sim}\mathsf{Std}(\rho,\boldsymbol{j})\times\mathsf{Std}(\mu\setminus\rho,\boldsymbol{l}(\lambda,\boldsymbol{c})^{+\kappa}),\ \mathtt{t}\mapsto(\mathtt{t}_0,\mathtt{t}_1)$$

where $\mathtt{t}_0=\mathtt{t}|_{[\![\rho]\!]}$ and $\mathtt{t}_1(u)=\mathtt{t}(u)-|\rho|$ for all $u\in[\![\mu]\!]\setminus[\![\rho]\!]$. Moreover, by definition, $\deg(\mathtt{t})=\deg(\mathtt{t}_0)+\deg(\mathtt{t}_1)$. Hence, the right-hand side of (6.10) is equal to the right-hand side of (6.9) multiplied by

$$\sum_{\mathtt{t}_0,\mathtt{t}_0'\in\mathsf{Std}(\rho)} q^{\deg(\mathtt{t}_0)+\deg(\mathtt{t}_0')}=\dim_q(R^{\Lambda_0}_{\mathrm{cont}(\rho)}),$$

and the result follows after dividing both sides of (6.10) by $\dim_q(R^{\Lambda_0}_{\mathrm{cont}(\rho)})$. □

The main aim of the rest of this section is to determine the rank of the free $\mathbb{Z}$-module $\gamma^{\lambda,\boldsymbol{c}}C_{\rho,d}\gamma^{\omega}$ for any $(\lambda,\boldsymbol{c})\in\Lambda^{\mathrm{col}}(n,d)$, see Corollaries 6.30 and 6.31.

6.2. **Colored tableaux.** A *horizontal strip* is a convex subset U of N such that whenever $(r,s)\neq(k,l)$ are in U we have $s\neq l$. A *vertical strip* is a convex subset U of N such that whenever $(r,s)\neq(k,l)$ are in U we have $r\neq k$.

Recalling the notation of §2.1, for any $i\in I$, we set $\mathsf{N}^{I,i}=\mathbb{Z}_{>0}\times\mathbb{Z}_{>0}\times\{i\}\subset\mathsf{N}^I$. Identifying $\mathsf{N}^{I,i}$ with N, we have a notion of what it means for a subset of $\mathsf{N}^{I,i}$ to be a *horizontal* or *vertical* strip. Given $j\in J$, we say that a subset U of N^I is a *j-bend* if the following conditions are satisfied:

(i) $U\subset\mathsf{N}^{I,j-1}\cup\mathsf{N}^{I,j}$;
(ii) $U\cap\mathsf{N}^{I,j-1}$ is a horizontal strip in $\mathsf{N}^{I,j-1}$, and $U\cap\mathsf{N}^{I,j}$ is a vertical strip in $\mathsf{N}^{I,j}$.

Now let $\boldsymbol{\mu}\in\mathscr{P}^I(d)$. Given $(\lambda,\boldsymbol{c})\in\Lambda^{\mathrm{col}}(n,d)$, we denote by $\mathsf{CT}(\boldsymbol{\mu};\lambda,\boldsymbol{c})$ the set of all weakly increasing maps $\mathtt{T}\colon[\![\boldsymbol{\mu}]\!]\to\{1,\dots,n\}$ such that for all $r=1,\dots,n$ the set $\mathtt{T}^{-1}(r)$ is a c_r-bend and $|\mathtt{T}^{-1}(r)|=\lambda_r$. We refer to the elements of $\mathsf{CT}(\boldsymbol{\mu};\lambda,\boldsymbol{c})$ as the *colored tableaux of shape $\boldsymbol{\mu}$ and type $(\lambda,\boldsymbol{c})$*.

Colored tableaux will play the role of a combinatorial intermediary connecting the standard tableaux appearing in Lemma 6.8 and the explicit expression for $\dim\gamma^{\lambda,\boldsymbol{c}}C_{\rho,d}\gamma^{\omega}$ appearing on the right hand side of the formula in Corollary 6.30.

6.3. **Counting standard tableaux in terms of colored tableaux.** Given $0\leq i<e$ and $u\in\mathbb{Z}\times\mathbb{Z}$, we call the image of $[\![(i+1,1^{e-i-1})]\!]$ under the translation of $\mathbb{Z}\times\mathbb{Z}$ sending $(1,1)$ to u the *e-hook with vertex u and arm length i*, or simply an *e-hook*.

Recall the abacus notation from §5.1. For any $i\in I$, let $b_i=b_i(\rho)$, $b_{>i}=\sum_{j=i+1}^{e-1}b_j$ and $b_{<i}=\sum_{j=0}^{i-1}b_j$. Since ρ is d-Rouquier, we have $b_{i+1}\geq b_i+d-1$ for $i=0,\dots,e-2$, and hence, for all $i\leq j$ in I,

$$b_{>i}-b_{>j}\geq(b_i+d-1)(j-i), \tag{6.11}$$

$$b_{<j} - b_{<i} \leq (b_j - d + 1)(j - i). \tag{6.12}$$

Given $(r, s, i) \in \mathsf{N}^I$, define the integers

$$x(r, s, i) := r - (e - i - 1)(b_i - r + s) + b_{>i},$$
$$y(r, s, i) := s + i(b_i - r + s) - b_{<i}.$$

Define

$$\mathsf{H}(r, s, i) \subset \mathbb{Z} \times \mathbb{Z}$$

be the e-hook with arm length i and vertex $(x(r, s, i), y(r, s, i))$. The following lemma is a refinement of [**CK**, Lemma 4] and [**Ev**, Lemma 4.3].

Lemma 6.13. *Let $\mu \in \mathscr{P}_{\rho,d}$ and $\boldsymbol{\mu} = \mathrm{quot}(\mu)$. Then*

$$[\![\mu]\!] = [\![\rho]\!] \sqcup \bigsqcup_{u \in [\![\boldsymbol{\mu}]\!]} \mathsf{H}(u).$$

Moreover, every $\mathsf{H}(u)$ with $u \in [\![\boldsymbol{\mu}]\!]$ has vertex of residue κ.

Proof. It is easy to check that $y(r, s, i) - x(r, s, i) \equiv -N \pmod{e}$, so the second statement holds.

For the first statement, there is nothing to prove when $|\boldsymbol{\mu}| = 0$, so we assume that $|\boldsymbol{\mu}| \geq 1$ and choose $(r, s, i) \in [\![\boldsymbol{\mu}]\!]$ such that $[\![\boldsymbol{\mu}]\!] \setminus \{(r, s, i)\} = [\![\boldsymbol{\nu}]\!]$ for some $\boldsymbol{\nu} \in \mathscr{P}^I(d-1)$. Arguing by induction on d, we may assume that the lemma holds for the partition $\nu \in \mathscr{P}_{\rho,d-1}$ determined from $\mathrm{quot}(\nu) = \boldsymbol{\nu}$, so it is enough to show that $[\![\mu]\!] \setminus [\![\nu]\!] = \mathsf{H}(r, s, i)$.

Let $\boldsymbol{\mu} = (\mu^{(0)}, \dots, \mu^{(e-1)})$ and $\boldsymbol{\nu} = (\nu^{(0)}, \dots, \nu^{(e-1)})$. Then $[\![\mu^{(i)}]\!] \setminus [\![\nu^{(i)}]\!] = \{(r, s)\}$ and $[\![\mu^{(j)}]\!] = [\![\nu^{(j)}]\!]$ for all $j \in I \setminus \{i\}$. We have

$$\mathsf{A}^e_N(\mu) = \big(\mathsf{A}^e_N(\nu) \setminus \{(a-1, i)\}\big) \cup \{(a, i)\} \tag{6.14}$$

for some $a \in \mathbb{Z}_{>0}$. In view of Lemma 5.2, $\mathsf{A}^e_N(\mu)$ has $b_i - r$ beads and s non-beads belonging to the runner R_i and preceding (a, i), so $a = b_i - r + s$. By [**CK**, Lemma 4(1)], we have

$$\mathsf{A}^e_N(\mu) \supseteq [0, a-1] \times [i+1, e-1], \tag{6.15}$$
$$\mathsf{A}^e_N(\mu) \cap (\mathbb{Z}_{\geq a} \times [0, i-1]) = \varnothing. \tag{6.16}$$

In particular, each of $(a-1, i+1), \dots, (a-1, e-1)$ is a bead of $\mathsf{A}^e_N(\mu)$, and each of $(a, 0), \dots, (a, i-1)$ is a non-bead of $\mathsf{A}^e_N(\mu)$. By (6.14) and Lemma 5.2, it follows that $[\![\mu]\!] \setminus [\![\nu]\!]$ is an e-hook with arm length i and vertex (x, y) where x is the number of the bead (a, i) and y is the number of the non-bead $(a-1, i)$ of $\mathsf{A}^e_N(\mu)$, cf. the proof of [**CK**, Lemma 4(2)]. Using (6.15), (6.16) and the fact that there are $r - 1$ beads of $\mathsf{A}^e_N(\mu)$ on R_i succeeding (a, i), we obtain

$$x = r + b_{>i} - a(e - i - 1) = x(r, s, i).$$

Similarly, $y = s + ia - b_{<i} = y(r, s, i)$. □

Corollary 6.17. *Let $0 \leq f \leq d$, and $\mu \in \mathscr{P}_{\rho,d}$, $\nu \in \mathscr{P}_{\rho,f}$ be partitions with the e-quotients $\boldsymbol{\mu}$, $\boldsymbol{\nu}$ respectively. Then $[\![\nu]\!] \subseteq [\![\mu]\!]$ if and only if $[\![\boldsymbol{\nu}]\!] \subseteq [\![\boldsymbol{\mu}]\!]$.*

Proof. The if-part follows from Lemma 6.13. For the only-if-part, we apply induction on $d-f$, the case $d=f$ being obvious. Let $d-f>0$. If $[\![\boldsymbol{\nu}]\!] \not\subseteq [\![\boldsymbol{\mu}]\!]$, then there is a node $(r,s,i) \in [\![\boldsymbol{\mu}]\!] \setminus [\![\boldsymbol{\nu}]\!]$ such that $[\![\boldsymbol{\nu}]\!] \cup \{(r,s,i)\} = [\![\mathrm{quot}(\lambda)]\!]$ for some $\lambda \in \mathscr{P}_{\rho,f+1}$. Then $[\![\lambda]\!] = [\![\nu]\!] \sqcup \mathsf{H}(r,s,i) \subseteq [\![\mu]\!]$ by Lemma 6.13. By induction, $[\![\mathrm{quot}(\lambda)]\!] \subseteq [\![\boldsymbol{\mu}]\!]$, which is a contradiction. □

Lemma 6.18. *For any $j \in J$, the set of standard $\boldsymbol{l}^j$-tableaux whose shape is a partition consists of exactly two elements, $\mathtt{t}$ and $\mathtt{s}$, where*

(a) *$\mathtt{t}$ has shape $(j, 1^{e-j})$, with $\mathtt{t}(e-j+1,1) = e$, and $\deg(\mathtt{t}) = 0$.*
(b) *$\mathtt{s}$ has shape $(j+1, 1^{e-j-1})$, with $\mathtt{s}(1,j+1) = e$, and $\deg(\mathtt{s}) = 1$.*

Proof. By Lemma 5.4, the shape of any standard tableau in question must be an element of $\mathscr{P}_{\varnothing,1}$, and the rest is easy to see. □

The graded dimension of $C_{\rho,d}$ for $d=1$ can be easily computed:

Lemma 6.19. *For any $k,j \in J$, we have:*

$$\dim_q (1_{\boldsymbol{l}^k} C_{\rho,1} 1_{\boldsymbol{l}^j}) = \begin{cases} 1+q^2 & \text{if } k=j, \\ q & \text{if } k \text{ and } j \text{ are neighbors,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Lemma 6.8, we have

$$\dim_q (1_{\boldsymbol{l}^k} C_{\rho,1} 1_{\boldsymbol{l}^j}) = \sum_{\substack{\mu \in \mathscr{P}_{\rho,1} \\ \mathtt{t} \in \mathsf{Std}(\mu \setminus \rho, (\boldsymbol{l}^k)^{+\kappa}) \\ \mathtt{t}' \in \mathsf{Std}(\mu \setminus \rho, (\boldsymbol{l}^j)^{+\kappa})}} q^{\deg(\mathtt{t}) + \deg(\mathtt{t}')}.$$

Let $\mu \in \mathscr{P}_{\rho,1}$. By Lemma 6.13, the set $[\![\mu]\!] \setminus [\![\rho]\!]$ is an e-hook with a vertex v of residue κ. Let i be the arm length of this e-hook and $\nu = (i+1, 1^{e-i-1})$. Denoting by τ the translation of $\mathbb{Z} \times \mathbb{Z}$ which maps $(1,1)$ to v, we have a bijection

$$\mathsf{Std}(\mu \setminus \rho, (\boldsymbol{l}^k)^{+\kappa}) \xrightarrow{\sim} \mathsf{Std}(\nu, \boldsymbol{l}^k)$$

given by $\mathtt{t} \mapsto \mathtt{s}$ where $\mathtt{s}(u) = \mathtt{t}(\tau(u))$ for all $u \in [\![\nu]\!]$ (and similarly for $\boldsymbol{l}^j$). Moreover, we have $\deg(\mathtt{s}) = \deg(\mathtt{t})$ by [**Ev**, (4.10)]. Hence,

$$\dim_q (1_{\boldsymbol{l}^k} C_{\rho,1} 1_{\boldsymbol{l}^j}) = \sum q^{\deg(\mathtt{s}) + \deg(\mathtt{s}')},$$

where the sum is over all $\mu \in \mathscr{P}_{\varnothing,1}$ and all pairs $(\mathtt{s}, \mathtt{s}') \in \mathsf{Std}(\mu, \boldsymbol{l}^k) \times \mathsf{Std}(\mu, \boldsymbol{l}^j)$. The result now follows by Lemma 6.18. □

Let H be an e-hook with arm length i and vertex $(x,y) \in \mathbb{Z} \times \mathbb{Z}$, and let G be another e-hook. We refer to the node $(x, y+i)$ as the *hand* of H and to $(x+e-i-1, y)$ as the *foot* of H. We call G a *right extension* of H if the foot of G is the right neighbor of the hand of H. We call G a *bottom extension* of H if the hand of G is the bottom neighbor of the foot of H. The following is deduced from the definition of $\mathsf{H}(r,s,i)$ by an easy calculation:

Lemma 6.20. *Let $(r,s,i) \in \mathsf{N}^I$.*

(i) *The hook $\mathsf{H}(r,s+1,i)$ is a right extension of $\mathsf{H}(r,s,i)$.*
(ii) *The hook $\mathsf{H}(r+1,s,i)$ is a bottom extension of $\mathsf{H}(r,s,i)$.*

Lemma 6.21. *Let $\boldsymbol{\mu} \in \mathscr{P}^I(d)$. If nodes $(r,s,i),(k,l,j) \in [\![\boldsymbol{\mu}]\!]$ are independent, then $\mathsf{H}(r,s,i)$ and $\mathsf{H}(k,l,j)$ are independent.*

Proof. First, suppose that $i \neq j$. Without loss of generality, $i < j$. Since $|\boldsymbol{\mu}| = d$, we have $k + s \leq d$. Also, $b_j - b_i \geq d - 1$ as ρ is d-Rouquier. We have

$$\begin{aligned} y(k,l,j) - y(r,s,i) &= (l-s) + i(b_j - k + l - b_i + r - s) + (j-i)(b_j - k + l) \\ &\quad - (b_{<j} - b_{<i}) \\ &\geq 1 - s + i(b_j - b_i + 2 - k - s) + (j-i)(-k + l + d - 1) \\ &\geq 1 + i + d - k - s \\ &\geq 1 + i, \end{aligned}$$

where we have used (6.12) for the second step. Hence, the vertex of $\mathsf{H}(k,l,j)$ has a greater second coordinate than the hand of $\mathsf{H}(r,s,i)$. A similar calculation using (6.11) shows that

$$x(r,s,i) - x(k,l,j) \geq e - j,$$

hence the vertex of $\mathsf{H}(r,s,i)$ has first coordinate greater than that of the foot of $\mathsf{H}(k,l,j)$. Thus, $\mathsf{H}(r,s,i)$ and $\mathsf{H}(k,l,j)$ are independent.

Now let $i = j$. Without loss of generality, $k < r$ and $l > s$. We have

$$\begin{aligned} y(k,l,i) - y(r,s,i) &= (l-s) + i(l - k + r - s) \geq 1 + 2i \geq 1 + i, \\ x(r,s,i) - x(k,l,i) &= (r-k) + (e - i - 1)(r - s + l - k) \geq e - i, \end{aligned}$$

and it follows again that $\mathsf{H}(r,s,i)$ and $\mathsf{H}(k,l,j)$ are independent. □

Recall from (6.1) that for every $j \in J$, we have fixed a word $\boldsymbol{l}^j = (l_{j,1}, \dots, l_{j,e}) \in I^{\delta,j}$. Define the map $q\colon J \times I \to \{1, \dots, e\}$ by the condition that $l_{j,q(j,i)} = i$ for all $j \in J$ and $i \in I$. Let $\mu \in \mathscr{P}_{\rho,d}$ and $0 < f \leq d$. Suppose that $\nu \in \mathscr{P}_{\rho,d-f}$ is a partition with $[\![\nu]\!] \subseteq [\![\mu]\!]$. Note that $\operatorname{cont}(\mu \setminus \nu) = f\delta$. For any $j \in J$, define the function

$$\mathtt{t}_{\mu\setminus\nu,j}\colon [\![\mu]\!] \setminus [\![\nu]\!] \to \{1, \dots, e\},\ u \mapsto q(j, \operatorname{res}(u) - \kappa).$$

For the notation $\boldsymbol{l}^j(f) = (l^{(f)}_{j,1}, \dots, l^{(f)}_{j,e}) \in I^{f\delta}_{\mathrm{div}}$ in the following lemma see (6.2).

Lemma 6.22. *Let $j \in J$ and $\mu \in \mathscr{P}_{\rho,d}$ with e-quotient $\boldsymbol{\mu}$. Let $0 < f \leq d$ and $\nu \in \mathscr{P}_{\rho,d-f}$ with e-quotient $\boldsymbol{\nu}$ satisfy $[\![\nu]\!] \subseteq [\![\mu]\!]$. Then*

$$\mathsf{Std}(\mu \setminus \nu, \boldsymbol{l}^j(f)^{+\kappa}) = \begin{cases} \varnothing & \text{if } [\![\boldsymbol{\mu}]\!] \setminus [\![\boldsymbol{\nu}]\!] \text{ is not a } j\text{-bend;} \\ \{\mathtt{t}_{\mu\setminus\nu,j}\} & \text{if } [\![\boldsymbol{\mu}]\!] \setminus [\![\boldsymbol{\nu}]\!] \text{ is a } j\text{-bend.} \end{cases}$$

Proof. Since $l_{j,1}, \dots, l_{j,e} \in I$ are all distinct, any element of $\mathsf{Std}(\mu \setminus \nu, \boldsymbol{l}^j(f)^{+\kappa})$ must assign $q(j, i - \kappa)$ to every node of residue i, i.e. such an element must be $\mathtt{t}_{\mu\setminus\nu,j}$. So it suffices to prove the following:

Claim. We have $\mathtt{t}_{\mu\setminus\nu,j} \in \mathsf{Std}(\mu \setminus \nu, \boldsymbol{l}^j(f)^{+\kappa})$ if and only if $[\![\boldsymbol{\mu}]\!] \setminus [\![\boldsymbol{\nu}]\!]$ is a j-bend.

For the claim, by Lemma 6.13 and Corollary 6.17, we have $\boldsymbol{\nu} \subset \boldsymbol{\mu}$ and $[\![\mu]\!] \setminus [\![\nu]\!] = \bigsqcup_{u \in [\![\boldsymbol{\mu}]\!] \setminus [\![\boldsymbol{\nu}]\!]} \mathsf{H}(u)$. Suppose that $\mathtt{t}_{\mu\setminus\nu,j} \in \mathsf{Std}(\mu \setminus \nu, \boldsymbol{l}^j(f)^{+\kappa})$. Then, for every $u \in [\![\boldsymbol{\mu}]\!] \setminus [\![\boldsymbol{\nu}]\!]$, the restriction $\mathtt{t}_{\mu\setminus\nu,j}|_{\mathsf{H}(u)}$ is $(\boldsymbol{l}^j)^{+\kappa}$-standard. By Lemmas 6.18 and 6.13, we deduce that $[\![\boldsymbol{\mu}]\!] \setminus [\![\boldsymbol{\nu}]\!] \subset \mathsf{N}^{I,j-1} \cup \mathsf{N}^{I,j}$. Suppose for contradiction that $([\![\boldsymbol{\mu}]\!] \setminus [\![\boldsymbol{\nu}]\!]) \cap \mathsf{N}^{I,j}$ is not a vertical strip. Then $(r,s,j),(r,s+1,j) \in [\![\boldsymbol{\mu}]\!] \setminus [\![\boldsymbol{\nu}]\!]$

for some r, s. Let u be the hand of $\mathsf{H}(r, s, j)$ and v be the foot of $\mathsf{H}(r, s+1, j)$. By Lemma 6.20(i), the node v is the right neighbor of u. By Lemma 6.18(b), we have $\mathtt{t}_{\mu\setminus\nu,j}(u) = e > \mathtt{t}_{\mu\setminus\nu,j}(v)$, which contradicts the standardness of $\mathtt{t}_{\mu\setminus\nu,j}$. Hence, $([\![\boldsymbol{\mu}]\!] \setminus [\![\boldsymbol{\nu}]\!]) \cap \mathsf{N}^{I,j}$ is a vertical strip. A similar argument, using Lemmas 6.20(ii) and 6.18(a), shows that $([\![\boldsymbol{\mu}]\!] \setminus [\![\boldsymbol{\nu}]\!]) \cap \mathsf{N}^{I,j-1}$ is a horizontal strip.

Conversely, suppose that $[\![\boldsymbol{\mu}]\!] \setminus [\![\boldsymbol{\nu}]\!]$ is a j-bend. By Lemmas 6.21 and 6.20, $[\![\mu]\!] \setminus [\![\nu]\!]$ is a disjoint union of independent sets of two types: (1) consecutive right extensions of hooks with arm length $j-1$; (2) consecutive bottom extensions of hooks with arm length j. In fact, we may assume that either $[\![\mu]\!] \setminus [\![\nu]\!]$ is of type (1) or $[\![\mu]\!] \setminus [\![\nu]\!]$ is of type (2). If $[\![\mu]\!] \setminus [\![\nu]\!]$ is of type (1), i.e. $[\![\mu]\!] \setminus [\![\nu]\!]$ is a union $\mathsf{H}_1 \sqcup \cdots \sqcup \mathsf{H}_m$ of hooks with arm length $j-1$, then by Lemma 6.18(a), $\mathtt{t}_{\mu\setminus\nu,j}(v) = e$ for any v which is a foot of H_a for $a = 1, \dots, m$. So $\mathtt{t}_{\mu\setminus\nu,j}$ is standard if $\mathtt{t}_{\mu\setminus\nu,j}|_{\mathsf{H}_a}$ is standard for all a. Hence we may assume that $m = 1$. But in this case $\mathtt{t}_{\mu\setminus\nu,j}$ is easily seen to be standard using Lemma 6.18(a) one more time. The case where $[\![\mu]\!] \setminus [\![\nu]\!]$ is of type (2) is similar. □

Recall the set $\Lambda^{\mathrm{col}}(n, d)$ of colored compositions defined by (3.7) and the set $\mathsf{CT}(\boldsymbol{\mu}; \lambda, \boldsymbol{c})$ of colored tableaux of shape $\boldsymbol{\mu}$ and type $(\lambda, \boldsymbol{c})$ from §6.2.

Corollary 6.23. *Let $\mu \in \mathscr{P}_{\rho,d}$, $\boldsymbol{\mu} = \mathrm{quot}(\mu)$ and $(\lambda, \boldsymbol{c}) \in \Lambda^{\mathrm{col}}(n, d)$. Then*

$$|\mathsf{Std}(\mu \setminus \rho, \boldsymbol{l}(\lambda, \boldsymbol{c})^{+\kappa})| = |\mathsf{CT}(\boldsymbol{\mu}; \lambda, \boldsymbol{c})|.$$

Proof. Recall that $\boldsymbol{l}(\lambda, \boldsymbol{c}) = \boldsymbol{l}^{c_1}(\lambda_1) \dots \boldsymbol{l}^{c_{n-1}}(\lambda_{n-1}) \boldsymbol{l}^{c_n}(\lambda_n)$. We have

$$\boldsymbol{l}^{c_1}(\lambda_1) \dots \boldsymbol{l}^{c_{n-1}}(\lambda_{n-1}) = \boldsymbol{l}(\lambda', \boldsymbol{c}')$$

for $\lambda' = (\lambda_1, \dots, \lambda_{n-1})$ and $\boldsymbol{c}' = (c_1, \dots, c_{n-1})$. Then

$$|\mathsf{Std}(\mu \setminus \rho, \boldsymbol{l}(\lambda, \boldsymbol{c})^{+\kappa})| = \sum_{[\![\rho]\!] \subseteq [\![\nu]\!] \subseteq [\![\mu]\!]} |\mathsf{Std}(\nu \setminus \rho, \boldsymbol{l}(\lambda', \boldsymbol{c}')^{+\kappa})| |\mathsf{Std}(\mu \setminus \nu, \boldsymbol{l}^{c_n}(\lambda_n))|.$$

If $|\mathsf{Std}(\nu \setminus \rho, \boldsymbol{l}(\lambda', \boldsymbol{c}')^{+\kappa})| \neq 0$, then $\mathrm{cont}(\nu) = \mathrm{cont}(\rho) + (d - \lambda_n)\delta$, whence by Lemma 5.4, we have $\nu \in \mathscr{P}_{\rho, d-\lambda_n}$. Arguing by induction on n, for such ν we have

$$|\mathsf{Std}(\nu \setminus \rho, \boldsymbol{l}(\lambda', \boldsymbol{c}')^{+\kappa})| = |\mathsf{CT}(\boldsymbol{\nu}; \lambda', \boldsymbol{c}')|,$$

where $\boldsymbol{\nu} = \mathrm{quot}(\nu)$. Moreover, by Lemma 6.22, we have

$$|\mathsf{Std}(\mu \setminus \nu, \boldsymbol{l}^{c_n}(\lambda_n))| = \begin{cases} 1 & \text{if } [\![\boldsymbol{\mu}]\!] \setminus [\![\boldsymbol{\nu}]\!] \text{ is a } c_n\text{-bend,} \\ 0 & \text{otherwise.} \end{cases}$$

The result follows. □

6.4. **Counting colored tableaux.** In view of Lemma 6.8 and Corollary 6.23, we can understand the dimensions of $\gamma^{\lambda,\boldsymbol{c}} C_{\rho,d} \gamma^{\lambda',\boldsymbol{c}'}$ for $(\lambda, \boldsymbol{c}), (\lambda', \boldsymbol{c}') \in \Lambda^{\mathrm{col}}(n, d)$ by counting appropriate colored tableaux. The first main goal of this subsection is a formula for $|\mathsf{CT}(\boldsymbol{\mu}; \lambda, \boldsymbol{c})|$.

Recall that $J = \{1, \dots, e-1\} = I \setminus \{0\}$. For $j \in J$, we define

$$\mathsf{Inc}(j) := \{j, j-1\} \subseteq I.$$

Remark 6.24. The notation $\mathsf{Inc}(j)$ is motivated by the following considerations. The irreducible semicuspidal $R_{\delta,\mathbb{F}}$-modules are exactly the irreducible $R_{\delta,\mathbb{F}}$-modules which factor through $R^{\Lambda_0}_{\delta,\mathbb{F}}$, see [$\mathbf{K}_2$, Lemma 5.1]. The algebra $R^{\Lambda_0}_{\delta,\mathbb{F}}$ is a

Brauer tree algebra of type A_e with vertices I and edges in natural bijection with J, so that $\mathsf{Inc}(j)$ is just the set of vertices incident to the edge j.

Let $\mathsf{Char} := \bigoplus_{t\in\mathbb{Z}_{\geq 0}} \mathbb{Z}\mathrm{Irr}(\mathfrak{S}_t)$ be the $\mathbb{Z}$-module of all formal $\mathbb{Z}$-linear combinations of irreducible characters of $\mathfrak{S}_t$ for $t = 0, 1, \dots$. We have the inner product $\langle\cdot,\cdot\rangle$ on Char such that on each summand it is the standard inner product on (generalized) characters and $\mathbb{Z}\mathrm{Irr}(\mathfrak{S}_t)$ is orthogonal to $\mathbb{Z}\mathrm{Irr}(\mathfrak{S}_u)$ for $t \neq u$. Let

$$\mathsf{Char}^I := \bigotimes_{i\in I} \mathsf{Char},$$

with the induced inner product. For every $\boldsymbol{\mu} = (\mu^{(0)}, \dots, \mu^{(e-1)}) \in \mathscr{P}^I$, we define

$$\chi^{\boldsymbol{\mu}} := \chi^{\mu^{(0)}} \otimes \cdots \otimes \chi^{\mu^{(e-1)}} \in \mathsf{Char}^I,$$

where χ^μ denotes the irreducible character of $\mathfrak{S}_t$ corresponding to the partition $\mu \in \mathscr{P}(t)$.

Let S, T be finite sets and $m, l \in \mathbb{Z}_{>0}$. We denote by $\mathcal{M}(S,T)$ the set of all matrices $A = (a_{s,t})_{s\in S, t\in T}$ with non-negative integer entries. Given $A \in \mathcal{M}(S,T)$, we set

$$\alpha_s(A) := \sum_{t\in T} a_{s,t} \qquad (s \in S),$$
$$\beta_t(A) := \sum_{s\in S} a_{s,t} \qquad (t \in T).$$

We write $\mathcal{M}(m,T) := \mathcal{M}([1,m],T)$, $\mathcal{M}(S,m) := \mathcal{M}(S,[1,m])$, etc. Given $\mu \in \Lambda(m)$ and $\lambda \in \Lambda(l)$, we define

$$\begin{aligned}
{}_\mu\mathcal{M}(m,T) &:= \{A \in \mathcal{M}(m,T) \mid \alpha_r(A) = \mu_r \text{ for all } r \in [1,m]\},\\
\mathcal{M}(S,m)_\mu &:= \{A \in \mathcal{M}(S,m) \mid \beta_r(A) = \mu_r \text{ for all } r \in [1,m]\},\\
{}_\lambda\mathcal{M}(m,l)_\mu &:= {}_\lambda\mathcal{M}(m,l) \cap \mathcal{M}(m,l)_\mu.
\end{aligned}$$

Let $(\lambda, \boldsymbol{c}) \in \Lambda^{\mathrm{col}}(n,d)$. We define

$${}_{(\lambda,\boldsymbol{c})}\mathcal{M}(n,I) := \{A = (a_{r,i}) \in {}_\lambda\mathcal{M}(n,I) \mid a_{r,i} = 0 \text{ if } i \notin \mathsf{Inc}(c_r)\} \tag{6.25}$$

Let $A = (a_{r,i}) \in {}_{(\lambda,\boldsymbol{c})}\mathcal{M}(n,I)$. For each $i \in I$, define the parabolic subgroup

$$\mathfrak{S}_{A,i} := \mathfrak{S}_{a_{1,i}} \times \cdots \times \mathfrak{S}_{a_{n,i}} \leq \mathfrak{S}_{\beta_i(A)}$$

and the induced character

$$\chi^{A,i} := \mathrm{ind}_{\mathfrak{S}_{A,i}}^{\mathfrak{S}_{\beta_i(A)}} \left(\mathtt{sgn}_{a_{1,i}}^{\delta_{c_1,i}} \boxtimes \cdots \boxtimes \mathtt{sgn}_{a_{n,i}}^{\delta_{c_n,i}}\right),$$

where, for $a \in \mathbb{Z}_{\geq 0}$ and $j \in J$, we interpret $\mathtt{sgn}_a^{\delta_{j,i}}$ as the trivial character of $\mathfrak{S}_a$ when $j \neq i$ and as the sign character of $\mathfrak{S}_a$ when $j = i$. Then set

$$\chi^A := \chi^{A,0} \otimes \cdots \otimes \chi^{A,e-1} \in \mathsf{Char}^I,$$
$$\chi^{(\lambda,\boldsymbol{c})} := \sum_{A\in {}_{(\lambda,\boldsymbol{c})}\mathcal{M}(n,I)} \chi^A.$$

Lemma 6.26. *Let $\boldsymbol{\mu} \in \mathscr{P}^I(d)$ and $(\lambda, \boldsymbol{c}) \in \Lambda^{\mathrm{col}}(n,d)$. Then*

$$|\mathsf{CT}(\boldsymbol{\mu}; \lambda, \boldsymbol{c})| = \langle \chi^{(\lambda,\boldsymbol{c})}, \chi^{\boldsymbol{\mu}} \rangle.$$

Proof. We apply induction on $n = 1, 2, \dots$, the induction base being immediate from the definitions. Let $n > 1$. Set $\lambda' = (\lambda_1, \dots, \lambda_{n-1}) \in \Lambda(n-1)$ and $\boldsymbol{c}' = (c_1, \dots, c_{n-1})$. We denote ${}_{((\lambda_n),(c_n))}\mathcal{M}(1, I)$ by ${}_{(\lambda_n, c_n)}\mathcal{M}(1, I)$. For a matrix $A' \in {}_{(\lambda', \boldsymbol{c}')}\mathcal{M}(n-1, I)$ and a one-row matrix $B \in {}_{(\lambda_n, c_n)}\mathcal{M}(1, I)$ we denote by $\begin{pmatrix} A' \\ B \end{pmatrix} \in {}_{(\lambda, \boldsymbol{c})}\mathcal{M}(n, I)$ the vertical concatenation of A' and B. Then

$$ {}_{(\lambda', \boldsymbol{c}')}\mathcal{M}(n-1, I) \times {}_{(\lambda_n, c_n)}\mathcal{M}(1, I) \to {}_{(\lambda, \boldsymbol{c})}\mathcal{M}(n, I), \ (A', B) \mapsto \begin{pmatrix} A' \\ B \end{pmatrix} $$

is a bijection. Denoting the entries of $B \in {}_{(\lambda_n, c_n)}\mathcal{M}(1, I)$ by b_i, we have by transitivity of induction

$$ \chi^{(\lambda, \boldsymbol{c})} = \sum_{\substack{A' \in {}_{(\lambda', \boldsymbol{c}')}\mathcal{M}(n-1, I) \\ B \in {}_{(\lambda_n, c_n)}\mathcal{M}(1, I)}} \bigotimes_{i \in I} \operatorname{ind}_{\mathfrak{S}_{\beta_i(A')} \, b_i}^{\mathfrak{S}_{\beta_i(A)}} \left(\chi^{A', i} \boxtimes \mathtt{sgn}_{b_i}^{\delta_{c_n, i}} \right). $$

The proof is concluded by the following computation:

$$ \begin{aligned} \langle \chi^{\boldsymbol{\mu}}, \chi^{(\lambda, \boldsymbol{c})} \rangle &= \sum_{\substack{A' \in {}_{(\lambda', \boldsymbol{c}')}\mathcal{M}(n-1, I) \\ B \in {}_{(\lambda_n, c_n)}\mathcal{M}(1, I)}} \prod_{i \in I} \left\langle \chi^{\mu^{(i)}}, \operatorname{ind}_{\mathfrak{S}_{\beta_i(A')} \, b_i}^{\mathfrak{S}_{\beta_i(A)}} \left(\chi^{A', i} \boxtimes \mathtt{sgn}_{b_i}^{\delta_{c_n, i}} \right) \right\rangle \\ &= \sum_{\substack{A' \in {}_{(\lambda', \boldsymbol{c}')}\mathcal{M}(n-1, I) \\ B \in {}_{(\lambda_n, c_n)}\mathcal{M}(1, I)}} \prod_{i \in I} \delta_{\beta_i(A), |\mu^{(i)}|} \langle \operatorname{res}_{\mathfrak{S}_{\beta_i(A')} \, b_i} \chi^{\mu^{(i)}}, \chi^{A', i} \boxtimes \mathtt{sgn}_{b_i}^{\delta_{c_n, i}} \rangle \\ &= \sum_{\substack{A' \in {}_{(\lambda', \boldsymbol{c}')}\mathcal{M}(n-1, I) \\ B \in {}_{(\lambda_n, c_n)}\mathcal{M}(1, I)}} \sum_{\substack{\boldsymbol{\nu} \subseteq \boldsymbol{\mu} \\ |\mu^{(i)}| - |\nu^{(i)}| = b_i, \ \forall i \in I \\ \boldsymbol{\mu} \backslash \boldsymbol{\nu} \text{ is a } c_n\text{-bend}}} \prod_{i \in I} \langle \chi^{\nu^{(i)}}, \chi^{A', i} \rangle \\ &= \sum_{B \in {}_{(\lambda_n, c_n)}\mathcal{M}(1, I)} \sum_{\substack{\boldsymbol{\nu} \subseteq \boldsymbol{\mu} \\ |\mu^{(i)}| - |\nu^{(i)}| = b_i, \ \forall i \in I \\ \boldsymbol{\mu} \backslash \boldsymbol{\nu} \text{ is a } c_n\text{-bend}}} \langle \chi^{\boldsymbol{\nu}}, \chi^{(\lambda', \boldsymbol{c}')} \rangle \\ &= \sum_{B \in {}_{(\lambda_n, c_n)}\mathcal{M}(1, I)} \sum_{\substack{\boldsymbol{\nu} \subseteq \boldsymbol{\mu} \\ |\mu^{(i)}| - |\nu^{(i)}| = b_i, \ \forall i \in I \\ \boldsymbol{\mu} \backslash \boldsymbol{\nu} \text{ is a } c_n\text{-bend}}} |\mathsf{CT}(\boldsymbol{\nu}; \lambda', \boldsymbol{c}')| \\ &= \sum_{\substack{\boldsymbol{\nu} \subseteq \boldsymbol{\mu} \\ \boldsymbol{\mu} \backslash \boldsymbol{\nu} \text{ is a } c_n\text{-bend}}} |\mathsf{CT}(\boldsymbol{\nu}; \lambda', \boldsymbol{c}')| \\ &= |\mathsf{CT}(\boldsymbol{\mu}; \lambda, \boldsymbol{c})|, \end{aligned} $$

where the second equality holds by Frobenius reciprocity, the third equality comes from the Littlewood-Richardson rule, the fifth equality holds by the inductive assumption and the remaining equalities are clear. □

Let $\boldsymbol{b} \in J^d$ so that $(\omega, \boldsymbol{b}) \in \Lambda^{\mathrm{col}}(d, d)$, cf. (6.6). Recalling (6.25), we set

$$ \mathcal{M}(I, d)_{(\omega, \boldsymbol{b})} := \{ B \in \mathcal{M}(I, d) \mid B^{\mathrm{tr}} \in {}_{(\omega, \boldsymbol{b})}\mathcal{M}(d, I) \}. $$

Define the set

$$ {}_{(\lambda,\boldsymbol{c})}\mathcal{M}(n,I,d)_{(\omega,\boldsymbol{b})} = \{(A,B) \in {}_{(\lambda,\boldsymbol{c})}\mathcal{M}(n,I) \times \mathcal{M}(I,d)_{(\omega,\boldsymbol{b})} \mid \beta_i(A) = \alpha_i(B)\ \forall i \in I\}. $$

Lemma 6.27. *For any $(\lambda,\boldsymbol{c}) \in \Lambda^{\mathrm{col}}(n,d)$, we have*

$$ \langle \chi^{(\lambda,\boldsymbol{c})}, \chi^{(\omega,\boldsymbol{b})} \rangle = \sum_{(A,B) \in {}_{(\lambda,\boldsymbol{c})}\mathcal{M}(n,I,d)_{(\omega,\boldsymbol{b})}} \prod_{i \in I} |\mathfrak{S}_{\beta_i(A)} : \mathfrak{S}_{A,i}|. $$

Proof. Denoting by $\mathtt{reg}_{\mathfrak{S}_t}$ the regular character of $\mathfrak{S}_t$, we have

$$ \begin{aligned} \langle \chi^{(\lambda,\boldsymbol{c})}, \chi^{(\omega,\boldsymbol{b})} \rangle &= \Big\langle \sum_{A \in {}_{(\lambda,\boldsymbol{c})}\mathcal{M}(n,I)} \chi^A, \sum_{B \in {}_{(\omega,\boldsymbol{b})}\mathcal{M}(d,I)} \chi^B \Big\rangle \\ &= \sum_{\substack{A \in {}_{(\lambda,\boldsymbol{c})}\mathcal{M}(n,I) \\ B \in {}_{(\omega,\boldsymbol{b})}\mathcal{M}(d,I)}} \prod_{i \in I} \langle \chi^{A,i}, \mathtt{reg}_{\mathfrak{S}_{\beta_i(B)}} \rangle, \end{aligned} $$

which implies the lemma since

$$ \langle \chi^{A,i}, \mathtt{reg}_{\mathfrak{S}_{\beta_i(B)}} \rangle = \begin{cases} \chi^{A,i}(1) = |\mathfrak{S}_{\beta_i(A)} : \mathfrak{S}_{A,i}| & \text{if } \beta_i(A) = \beta_i(B), \\ 0 & \text{otherwise} \end{cases} $$

for any $i \in I$. □

For $(\lambda,\boldsymbol{c}) \in \Lambda^{\mathrm{col}}(n,d)$ and $(\omega,\boldsymbol{b}) \in \Lambda^{\mathrm{col}}(d,d)$ as above, we define the set ${}_{(\lambda,\boldsymbol{c})}\underline{\mathcal{M}}(n,d)_{(\omega,\boldsymbol{b})}$ of tuples $(T^0,\dots,T^{e-1})$ such that

(1) $T^i = (t^i_{r,s}) \in \mathcal{M}(n,d)$ for all $i \in I$;
(2) $T^0 + \cdots + T^{e-1} \in {}_{\lambda}\mathcal{M}(n,d)_{\omega}$;
(3) $t^i_{r,s} = 0$ unless $i \in \mathsf{Inc}(c_r) \cap \mathsf{Inc}(b_s)$.

Lemma 6.28. *For any $(\lambda,\boldsymbol{c}) \in \Lambda^{\mathrm{col}}(n,d)$ and $(\omega,\boldsymbol{b}) \in \Lambda^{\mathrm{col}}(d,d)$, we have*

$$ |{}_{(\lambda,\boldsymbol{c})}\underline{\mathcal{M}}(n,d)_{(\omega,\boldsymbol{b})}| = \sum_{(A,B) \in {}_{(\lambda,\boldsymbol{c})}\mathcal{M}(n,I,d)_{(\omega,\boldsymbol{b})}} \prod_{i \in I} |\mathfrak{S}_{\beta_i(A)} : \mathfrak{S}_{A,i}|. $$

Proof. Consider the map

$$ \theta\colon {}_{(\lambda,\boldsymbol{c})}\underline{\mathcal{M}}(n,d)_{(\omega,\boldsymbol{b})} \to \mathcal{M}(n,I) \times \mathcal{M}(I,d) $$

defined as follows. Given $\mathbf{T} = (T^0,\dots,T^{e-1}) \in {}_{(\lambda,\boldsymbol{c})}\underline{\mathcal{M}}(n,d)_{(\omega,\boldsymbol{b})}$, we set $\theta(\mathbf{T}) = (A,B)$ where $A = (a_{r,i}) \in \mathcal{M}(n,I)$ and $B = (b_{i,s}) \in \mathcal{M}(I,d)$ are given by $a_{r,i} := \alpha_r(T^i)$ and $b_{i,s} := \beta_s(T^i)$. Clearly, the image of θ is contained in ${}_{(\lambda,\boldsymbol{c})}\mathcal{M}(n,I,d)_{(\omega,\boldsymbol{b})}$.

Let $(A,B) \in {}_{(\lambda,\boldsymbol{c})}\mathcal{M}(n,I,d)_{(\omega,\boldsymbol{b})}$. Then the preimage $\theta^{-1}(A,B)$ consists of all tuples $(T^0,\dots,T^{e-1})$ of matrices in $\mathcal{M}(n,d)$ such that $\alpha_r(T^i) = a_{r,i}$ and $\beta_s(T^i) = b_{i,s}$ for all $i \in I$, $r \in [1,n]$ and $s \in [1,d]$. So, denoting

$$ S_i := \{T \in \mathcal{M}(n,d) \mid \alpha_r(T) = a_{r,i},\ \beta_s(T) = b_{i,s} \text{ for all } r \in [1,n],\ s \in [1,d]\} $$

for any $i \in I$, we have $|\theta^{-1}(A,B)| = \prod_{i \in I} |S_i|$.

To compute $|S_i|$ for a fixed $i \in I$, let $X = \{s \in [1,d] \mid b_{i,s} = 1\}$, so that $|X| = \alpha_i(B) = \beta_i(A)$. Then the set of partitions of X into a disjoint union of subsets X_r, $r \in [1,n]$, with $|X_r| = a_{r,i}$ for each r, is in bijection with the set S_i:

a bijection is given by assigning to each such partition $X = \bigsqcup_{r=1}^{n} X_r$ the matrix $T = (t_{r,s})$ given by

$$t_{r,s} = \begin{cases} 1 & \text{if } s \in X_r, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $|S_i| = |\mathfrak{S}_{\beta_i(A)} : \mathfrak{S}_{A,i}|$, proving the lemma. □

Theorem 6.29. *For any $(\lambda, \boldsymbol{c}) \in \Lambda^{\mathrm{col}}(n,d)$ and $\boldsymbol{b} \in J^d$, we have*

$$\dim(\gamma^{\lambda,\boldsymbol{c}} C_{\rho,d} \gamma^{\omega,\boldsymbol{b}}) = |{}_{(\lambda,\boldsymbol{c})}\underline{\mathcal{M}}(n,d)_{(\omega,\boldsymbol{b})}|.$$

Proof. We have

$$\begin{aligned}
\dim(\gamma^{\lambda,\boldsymbol{c}} C_{\rho,d} \gamma^{\omega,\boldsymbol{b}}) &= \sum_{\mu \in \mathscr{P}_{\rho,d}} |\mathsf{Std}(\mu \setminus \rho, \boldsymbol{l}(\lambda,\boldsymbol{c})^{+\kappa})|\, |\mathsf{Std}(\mu \setminus \rho, \boldsymbol{l}(\omega,\boldsymbol{b})^{+\kappa})| \\
&= \sum_{\boldsymbol{\mu} \in \mathscr{P}^I(d)} |\mathsf{CT}(\boldsymbol{\mu}; \lambda, \boldsymbol{c})|\, |\mathsf{CT}(\boldsymbol{\mu}; \omega, \boldsymbol{b})| \\
&= \sum_{\boldsymbol{\mu} \in \mathscr{P}^I(d)} \langle \chi^{(\lambda,\boldsymbol{c})}, \chi^{\boldsymbol{\mu}} \rangle \langle \chi^{(\omega,\boldsymbol{b})}, \chi^{\boldsymbol{\mu}} \rangle \\
&= \langle \chi^{(\lambda,\boldsymbol{c})}, \chi^{(\omega,\boldsymbol{b})} \rangle \\
&= |{}_{(\lambda,\boldsymbol{c})}\underline{\mathcal{M}}(n,d)_{(\omega,\boldsymbol{b})}|,
\end{aligned}$$

where the first equality comes from Lemma 6.8, the second equality uses Corollary 6.23 and Lemma 5.3, the third equality uses Lemma 6.26, the fourth equality holds since the elements $\chi^{\boldsymbol{\mu}}$ form an orthonormal basis of Char^I, and the final equality comes from Lemmas 6.27 and 6.28. □

Corollary 6.30. *Let $(\lambda, \boldsymbol{c}) \in \Lambda^{\mathrm{col}}(n,d)$. For all $j \in J$, set*

$$d_j = \sum_{\substack{1 \le r \le n \\ c_r = j}} \lambda_r.$$

Then

$$\dim(\gamma^{\lambda,\boldsymbol{c}} C_{\rho,d} \gamma^{\omega}) = \begin{cases} |\mathfrak{S}_d : \mathfrak{S}_\lambda|\, 3^{d_1 + d_{e-1}}\, 4^{\sum_{j=2}^{e-2} d_j} & \text{if } e > 2, \\ |\mathfrak{S}_d : \mathfrak{S}_\lambda|\, 2^{d_1} & \text{if } e = 2. \end{cases}$$

Proof. In this paragraph we fix $\boldsymbol{b} \in J^d$. Let $Y_{\boldsymbol{b}}$ be the set of all maps $\varphi\colon [1,d] \to [1,n] \times I$ such that

(1) $|\varphi^{-1}(\{r\} \times I)| = \lambda_r$ for all $r \in [1,n]$;
(2) for all $s \in [1,d]$, if $\varphi(s) = (r,i)$, then $i \in \mathsf{Inc}(c_r) \cap \mathsf{Inc}(b_s)$.

Observe that there is a bijection $f\colon {}_{(\lambda,\boldsymbol{c})}\underline{\mathcal{M}}(n,d)_{(\omega,\boldsymbol{b})} \xrightarrow{\sim} Y_{\boldsymbol{b}}$ such that $f(\mathbf{T})(s)$ is the unique $(r,i) \in [1,n] \times I$ such that $t^i_{r,s} = 1$, if we write $\mathbf{T} = (T^0, \dots, T^{e-1})$ with $T^i = (t^i_{r,s})$.

Now, let $Y := \{(\varphi, \boldsymbol{b}) \mid \boldsymbol{b} \in J^d,\ \varphi \in Y_{\boldsymbol{b}}\}$. By Theorem 6.29 and (6.7), we have $\dim(\gamma^{\lambda,\boldsymbol{c}} C_{\rho,d} \gamma^{\omega}) = |Y|$. Let W be the set of all set partitions $[1,d] = \bigsqcup_{r \in [1,n]} \Omega_r$ such that $|\Omega_r| = \lambda_r$ for all r. Note that $|W| = |\mathfrak{S}_d : \mathfrak{S}_\lambda|$.

We define the map $\xi\colon Y \to W$ by setting $\xi(\varphi, \boldsymbol{b})$ to be the partition $[1,d] = \bigsqcup_{r\in[1,n]} \varphi^{-1}(\{r\} \times I)$. To complete the proof, we fix a set partition $\Omega\colon [1,d] = \bigsqcup_{r\in[1,n]} \Omega_r$ in W and compute $|\xi^{-1}(\Omega)|$. Given $j \in J$, set

$$\mathsf{Inc}^2(j) := \{(i,l) \in I \times J \mid i \in \mathsf{Inc}(j) \cap \mathsf{Inc}(l)\}.$$

Note that

$$\mathsf{Inc}^2(j) = \begin{cases} \{(j,j),(j,j+1),(j-1,j),(j-1,j-1)\} & \text{if } 1 < j < e-1, \\ \{(1,1),(1,2),(0,1)\} & \text{if } j = 1 \text{ and } e > 2, \\ \{(e-1,e-1),(e-2,e-1),(e-2,e-2)\} & \text{if } j = e-1 \text{ and } e > 2, \\ \{(0,1),(1,1)\} & \text{if } j = 1 \text{ and } e = 2. \end{cases}$$

Note that $\xi^{-1}(\Omega)$ consists of all pairs $(\varphi, \boldsymbol{b})$ where $\varphi\colon [1,d] \to [1,n] \times I$ and $\boldsymbol{b} \in J^d$ are such that for any $r \in J \times [1,n]$ and any $s \in \Omega_r$ we have $\varphi(s) = (r,i)$ with $(i, b_s) \in \mathsf{Inc}^2(c_r)$. So

$$\begin{aligned} |\xi^{-1}(\Omega)| &= \prod_{r\in[1,n]} \prod_{s\in\Omega_r} |\mathsf{Inc}^2(c_r)| = \prod_{j\in J} |\mathsf{Inc}^2(j)|^{d_j} \\ &= \begin{cases} 3^{d_1+d_{e-1}}\, 4^{\sum_{j=2}^{e-2} d_j} & \text{if } e > 2, \\ 2^{d_1} & \text{if } e = 2, \end{cases} \end{aligned}$$

and the corollary follows. □

Recall the algebra W_d and the right W_d-modules $M_{\lambda,\boldsymbol{c}}$ defined in §3.1. Combining Lemma 3.12 and Corollary 6.30, we obtain:

Corollary 6.31. *For all $(\lambda, \boldsymbol{c}) \in \Lambda^{\mathrm{col}}(n,d)$, we have $\dim(\gamma^{\lambda,\boldsymbol{c}} C_{\rho,d} \gamma^{\omega}) = \dim M_{\lambda,\boldsymbol{c}}$.*

Corollary 6.32. *We have $\dim(\gamma^{\omega} C_{\rho,d} \gamma^{\omega}) = d!(4e-6)^d = \dim W_d$.*

Proof. This can be derived from the algebra isomorphism in [**Ev**, Theorem 3.4]. We give a more direct proof for the reader's convenience. By (3.3), we have $\dim \mathsf{Z} = 4e-6$, and the second equality follows. For any $\boldsymbol{c} \in J^d$ and $j \in J$, set $d_j(\boldsymbol{c}) := |\{r \in [1,d] \mid c_r = j\}|$. For $e > 2$, we compute:

$$\begin{aligned} \dim(\gamma^{\omega} C_{\rho,d} \gamma^{\omega}) &= \sum_{\boldsymbol{c}\in J^d} \dim(\gamma^{\omega,\boldsymbol{c}} C_{\rho,d} \gamma^{\omega}) \\ &= d! \sum_{\boldsymbol{c}\in J^d} 3^{d_1(\boldsymbol{c})+d_{e-1}(\boldsymbol{c})} 4^{d_2(\boldsymbol{c})+\cdots+d_{e-2}(\boldsymbol{c})} \\ &= d!(3+4(e-3)+3)^d = d!(4e-6)^d, \end{aligned}$$

where the second equality is due to Corollary 6.30. For $e = 2$, the same computation yields $\dim(\gamma^{\omega} C_{\rho,d} \gamma^{\omega}) = d!2^d = d!(4e-6)^d$. □

7. The semicuspidal algebra

As usual, $d \in \mathbb{Z}_{>0}$ is fixed. Recall the semicuspidal algebra $\hat{C}_{d\delta}$ from §4.5. In this section we prove some results on the structure of $\hat{C}_{d\delta}$. These results are used in Section 8 to study the quotient $C_{\rho,d}$ of $\hat{C}_{d\delta}$ in the context of a RoCK block, cf. (5.17).

7.1. **Preliminary results on the semicuspidal algebra.** We have the parabolic subalgebra

$$\hat{C}_{\omega\delta} \cong \hat{C}_\delta \otimes \cdots \otimes \hat{C}_\delta \subseteq \hat{C}_{d\delta}$$

with the identity element $1_{\omega\delta}$, cf. Lemma 4.33.

Lemma 7.1. *We have:*

(i) *The algebra $1_{\omega\delta}\hat{C}_{d\delta}1_{\omega\delta}$ is non-negatively graded.*
(ii) $1_{\omega\delta}\hat{C}^{>0}_{d\delta}1_{\omega\delta} = 1_{\omega\delta}\hat{C}_{d\delta}\hat{C}^{>0}_{\omega\delta} = 1_{\omega\delta}\hat{C}_{d\delta}1_{\omega\delta}\hat{C}^{>0}_{\omega\delta}$.

Proof. (i) follows from [**Ev**, Lemma 6.9(iii)]. The second equality in (ii) is obvious and the first one follows from [**Ev**, Lemma 6.9(i)(ii)]. □

For $\boldsymbol{j} = (j_1, \dots, j_d) \in J^d$, we define

$$e_{\boldsymbol{j}} := 1_{\boldsymbol{l}^{j_1} \dots \boldsymbol{l}^{j_d}} \in R_{d\delta}. \tag{7.2}$$

In particular, for $j \in J$, we interpret e_j as $1_{\boldsymbol{l}^j}$. In fact, the idempotent $e_{\boldsymbol{j}}$ is also known as $\gamma^{\omega,\boldsymbol{j}}$, cf. (6.6), (6.7). So we have $\gamma^\omega = \sum_{\boldsymbol{j}\in J^d} e_{\boldsymbol{j}}$.

Following [$\mathbf{KM}_3$], we consider the R_δ-modules $\Delta_{\delta,j} := \hat{C}_\delta e_j$ for every $j \in J$. Note that R_δ and hence $\hat{C}_\delta$ is non-negatively graded. Recalling the modules $L_{\delta,j}$ with basis $\{v_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^{\delta,j}\}$ from §4.5, the following is immediate from [$\mathbf{KM}_3$, Proposition 5.13]:

Lemma 7.3. *Let $j \in J$. Then $\Delta_{\delta,j}$ is non-negatively graded and there is an isomorphism of R_δ-modules*

$$\Delta_{\delta,j}/\Delta^{>0}_{\delta,j} \xrightarrow{\sim} L_{\delta,j},\ e_j + \Delta^{>0}_{\delta,j} \mapsto v_{\boldsymbol{l}^j}.$$

Lemma 7.4. *For any $\boldsymbol{j} \in J^d$, we have an isomorphism of $R_{d\delta}$-modules*

$$\hat{C}_{d\delta}e_{\boldsymbol{j}} \xrightarrow{\sim} \Delta_{\delta,j_1} \circ \cdots \circ \Delta_{\delta,j_d},\ e_{\boldsymbol{j}} \mapsto 1_{\omega\delta} \otimes e_{j_1} \otimes \cdots \otimes e_{j_d}.$$

Proof. This follows from Lemma 4.32. □

Let $\boldsymbol{j} \in J^d$. We consider the following submodule of the $\hat{C}_{d\delta}$-module $\hat{C}_{d\delta}e_{\boldsymbol{j}}$:

$$N_{\boldsymbol{j}} := \hat{C}_{d\delta}(\hat{C}^{>0}_{\omega\delta})e_{\boldsymbol{j}}. \tag{7.5}$$

Lemma 7.6. *For any $\boldsymbol{j} \in J^d$, we have an isomorphism of $R_{d\delta}$-modules*

$$\hat{C}_{d\delta}e_{\boldsymbol{j}}/N_{\boldsymbol{j}} \xrightarrow{\sim} L_{\delta,j_1} \circ \cdots \circ L_{\delta,j_d},\ e_{\boldsymbol{j}} + N_{\boldsymbol{j}} \mapsto 1_{\omega\delta} \otimes v_{\boldsymbol{l}^{j_1}} \otimes \cdots \otimes v_{\boldsymbol{l}^{j_d}}.$$

Proof. By Lemmas 4.25, 7.3 and 7.4, there is a surjective $R_{d\delta}$-module homomorphism as in the statement of the lemma. That the homomorphism is injective follows from Lemmas 4.33 and 7.3 again. □

Lemma 7.7. *If $\boldsymbol{j} \in J^d$, then $e_{\boldsymbol{j}}\hat{C}^0_{\omega\delta}e_{\boldsymbol{j}} = \mathbb{Z}e_{\boldsymbol{j}}$.*

Proof. Clearly, it suffices to prove the lemma in the case $d = 1$. For any word in $\boldsymbol{i} \in I^\delta$, the entries $i_1, \dots, i_e$ are distinct. Hence, by Theorem 4.13, we have $e_{\boldsymbol{j}}R_\delta e_{\boldsymbol{j}} = \mathbb{Z}[y_1, \dots, y_e]e_{\boldsymbol{j}}$, and the lemma follows. □

7.2. **Some explicit elements of $\gamma^\omega \hat{C}_{d\delta}\gamma^\omega$.** Let $\Delta_\delta := \bigoplus_{j\in J} \Delta_{\delta,j}$. In view of Lemma 7.4, we have an isomorphism

$$\Delta_\delta^{\circ d} \cong \hat{C}_{d\delta}\gamma^\omega \tag{7.8}$$

of left $\hat{C}_{d\delta}$-modules. More precisely, we can explicitly identity $\Delta_\delta^{\circ d}$ with $\hat{C}_{d\delta}\gamma^\omega$ so that the element $1_{\omega\delta}\otimes e_{j_1}\otimes\cdots\otimes e_{j_d}$ of the natural direct summand $\Delta_{\delta,j_1}\circ\cdots\circ\Delta_{\delta,j_d}$ of $\Delta_\delta^{\circ d}$ corresponds to $e_{\boldsymbol{j}} = e_{\boldsymbol{j}}\gamma^\omega \in \hat{C}_{d\delta}\gamma^\omega$ for all $\boldsymbol{j} = (j_1,\dots,j_d) \in J^d$. So $\gamma^\omega\hat{C}_{d\delta}\gamma^\omega$ is naturally identified with $\mathrm{End}_{\hat{C}_{d\delta}}(\Delta_\delta^{\circ d})^{\mathrm{op}}$. The algebra $\mathrm{End}_{\hat{C}_{d\delta}}(\Delta_\delta^{\circ d})$ is described in [**KM**$_3$] as an *affine zigzag algebra* of rank d, so we can reinterpret this as a description of $\gamma^\omega\hat{C}_{d\delta}\gamma^\omega$. We now define some explicit elements of $\gamma^\omega\hat{C}_{d\delta}\gamma^\omega$ which correspond (up to an antiautomorphism and signs) to the elements of $\mathrm{End}_{\hat{C}_{d\delta}}(\Delta_\delta^{\circ d})$ with the same names introduced in [**KM**$_3$, §6.1].

For neighbors $k, j \in J$, we define $w_{k,j} \in \mathfrak{S}_e$ to be the unique permutation such that $w_{k,j}\boldsymbol{l}^j = \boldsymbol{l}^k$. Set

$$a^{k,j} := \psi_{w_{k,j}} e_j \in \hat{C}_\delta.$$

Further, define

$$e_J := \sum_{j\in J} e_j \in \hat{C}_\delta, \tag{7.9}$$

and set

$$c := (y_1 - y_e)e_J \in \hat{C}_\delta, \quad z := y_1 e_J \in \hat{C}_\delta.$$

Recall that, in view of Lemma 4.33, we have identified the parabolic subalgebra $\hat{C}_{\omega\delta} \subseteq \hat{C}_{d\delta}$ with $\hat{C}_\delta \otimes \cdots \otimes \hat{C}_\delta$. For $t = 1,\dots,d$ and $x \in \hat{C}_\delta$, we define

$$x_t := e_J^{\otimes t-1} \otimes x \otimes e_J^{\otimes d-t} \in \hat{C}_{\omega\delta} \subseteq \hat{C}_{d\delta}.$$

In particular we have the elements $c_t, z_t, a_t^{j,k} \in \gamma^\omega\hat{C}_{\omega\delta}\gamma^\omega$.

Recall the algebra W_d and the signs $\zeta_1,\dots,\zeta_{e-1}$ defined by (3.5) and (3.9). Let $1 \le t < d$. Consider the product of transpositions

$$w_t := \prod_{a=(t-1)e+1}^{te} (a, a+e) \in \mathfrak{S}_{de}, \tag{7.10}$$

and let $w := w_1 \in \mathfrak{S}_{2e}$. We set

$$\hat{r}_t := \sum_{j,k\in J} e_J^{\otimes t-1} \otimes (-\psi_w - \delta_{k,j}\zeta_k)(e_k \otimes e_j) \otimes e_J^{\otimes d-t-1} \in \gamma^\omega\hat{C}_{d\delta}\gamma^\omega. \tag{7.11}$$

Note that the sign here is opposite to the one in [**KM**$_3$], which is technically more convenient for us, but does not affect the result below.

Theorem 7.12. *We have:*

(i) *There is an injective algebra homomorphism $\Theta\colon W_d \to \gamma^\omega\hat{C}_{d\delta}\gamma^\omega$ with*

$$\mathsf{e}_{\boldsymbol{j}} \mapsto e_{\boldsymbol{j}}, \quad s_u \mapsto \hat{r}_u, \quad \mathsf{c}^{(j)}[t] \mapsto \pm(ce_j)_t, \quad \mathsf{a}^{k,j}[t] \mapsto \pm a_t^{k,j}$$

for all $\boldsymbol{j} \in J^d$, $1 \le u < d$, $1 \le t \le d$, and all admissible $k, j \in J$, where the signs depend on k and j.

(ii) *For each $a \in \{0,1\}$, the map Θ restricts to a $\mathbb{Z}$-module isomorphism of graded components $W_d^a \xrightarrow{\sim} \gamma^\omega\hat{C}_{d\delta}^a\gamma^\omega$.*

(iii) *The algebra $\gamma^\omega\hat{C}_{d\delta}\gamma^\omega$ is generated by $\Theta(W_d)$ together with $y_1\gamma^\omega$.*

Proof. Part (i) follows from [**KM**$_3$, Theorem 6.16] together with the fact that $a^{j,k}a^{k,j} = \pm ce_j$ for all neighbors $k, j \in J$, as observed in [**KM**$_3$, Theorem 5.24]. Parts (ii) and (iii) follow from [**KM**$_3$, Theorem 6.16] and the easy facts that the affine zigzag algebra is non-negatively graded and is generated by the finite zigzag algebra isomorphic to W_d and a homogeneous element z_1 of degree 2, see [**KM**$_3$, §4.2]. □

Considering scalar extensions to a field $\Bbbk$, we also have the following result. Here and in the sequel, we denote $\gamma^\omega := \gamma^\omega \otimes 1 \in \hat{C}_{d\delta,\Bbbk}$,

Lemma 7.13. *Let $\Bbbk$ be a field with $\operatorname{char}\Bbbk = 0$ or $\operatorname{char}\Bbbk > d$. The left $\hat{C}_{d\delta,\Bbbk}$-module $\hat{C}_{d\delta,\Bbbk}\gamma^\omega$ is a projective generator for the algebra $\hat{C}_{d\delta,\Bbbk}$.*

Proof. By [**KM**$_2$, Lemma 6.22], the $\hat{C}_{d\delta,\Bbbk}$-module $\Delta_\delta^{\circ d} \otimes_{\mathbb{Z}} \Bbbk \cong (\Delta_\delta \otimes_{\mathbb{Z}} \Bbbk)^{\circ d}$ is a projective generator. By (7.8), we have $\Delta_\delta^{\circ d} \otimes_{\mathbb{Z}} \Bbbk \cong \hat{C}_{d\delta,\Bbbk}\gamma^\omega$, and the lemma follows. □

7.3. **Imaginary tensor spaces.** Let $j \in J$. Following [**KM**$_1$], we refer to

$$T_{d,j} := L_{\delta,j}^{\circ d}$$

as the *imaginary tensor space of color j*. In [**KM**$_1$, (4.2.9)], an action of the symmetric group $\mathfrak{S}_d$ on $T_{d,j}$ with $R_{d\delta}$-endomorphisms is defined as follows:

$$(1_{\omega\delta} \otimes v_{\boldsymbol{l}^j}^{\otimes d}) \cdot s_t = (\zeta_j \psi_{w_t} + 1_{d\delta}) 1_{\omega\delta} \otimes v_{\boldsymbol{l}^j}^{\otimes d} \qquad (1 \le t < d).$$

Comparing with (7.11), we see that

$$(1_{\omega\delta} \otimes v_{\boldsymbol{l}^j}^{\otimes d}) \cdot s_t = -\zeta_j \hat{r}_t \otimes v_{\boldsymbol{l}^j}^{\otimes d}. \tag{7.14}$$

As in [**KM**$_1$, §5.2], we define

$$Z_{d,j} := \{v \in T_{d,j} \mid v \cdot g = (-1)^{\ell(g)} v \text{ for all } g \in \mathfrak{S}_d\}.$$

Recall the Gelfand-Graev idempotent $\gamma^{d,j}$ from (6.5).

Lemma 7.15. [**KM**$_1$, Lemma 6.4.1(ii)] *We have $Z_{d,j} = R_{d\delta}\gamma^{d,j} T_{d,j}$.*

More generally, fix $(\lambda, \boldsymbol{c}) \in \Lambda^{\mathrm{col}}(n,d)$ for some $n \in \mathbb{Z}_{>0}$. Define the semicuspidal $R_{d\delta}$-module

$$T_{\lambda,\boldsymbol{c}} := T_{\lambda_1,c_1} \circ \cdots \circ T_{\lambda_n,c_n}.$$

By the $n = 1$ case considered above, we have the right action of $\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_n} = \mathfrak{S}_\lambda$ on $T_{\lambda_1,c_1} \boxtimes \cdots \boxtimes T_{\lambda_n,c_n}$ with $R_{\lambda\delta}$-endomorphisms. By functoriality of induction, this induces a right action of $\mathfrak{S}_\lambda$ on $T_{\lambda,\boldsymbol{c}}$ with $R_{d\delta}$-endomorphisms. Define

$$Z_{\lambda,\boldsymbol{c}} := \{v \in T_{\lambda,\boldsymbol{c}} \mid v \cdot g = (-1)^{\ell(g)} v \text{ for all } g \in \mathfrak{S}_\lambda\}.$$

Recall the idempotent $\gamma^{\lambda,\boldsymbol{c}}$ from (6.3), and note that $\gamma^{\lambda,\boldsymbol{c}} = \gamma^{\lambda,\boldsymbol{c}} 1_{\lambda\delta}$.

Lemma 7.16. *We have $Z_{\lambda,\boldsymbol{c}} = R_{d\delta}(\gamma^{\lambda,\boldsymbol{c}} \otimes (T_{\lambda_1,c_1} \boxtimes \cdots \boxtimes T_{\lambda_n,c_n}))$.*

Proof. By Lemma 7.15, we have

$$\{v \in T_{\lambda_1,c_1} \boxtimes \cdots \boxtimes T_{\lambda_n,c_n} \mid v \cdot g = (-1)^{\ell(g)} \text{ for all } g \in \mathfrak{S}_\lambda\}$$
$$= R_{\lambda_1,\delta}\gamma^{\lambda_1,c_1} T_{\lambda_1,c_1} \boxtimes \cdots \boxtimes R_{\lambda_n\delta}\gamma^{\lambda_n,c_n} T_{\lambda_n,c_n}.$$

Moreover, for each $w \in \mathscr{D}^{e\lambda}$, we have an isomorphism of $\mathbb{Z}$-modules

$$T_{\lambda_1,c_1} \boxtimes \cdots \boxtimes T_{\lambda_n,c_n} \xrightarrow{\sim} \psi_w \otimes T_{\lambda_1,c_1} \boxtimes \cdots \boxtimes T_{\lambda_n,c_n},\ v \mapsto \psi_w \otimes v,$$

which is equivariant with respect to the right action of $\mathfrak{S}_\lambda$. Therefore,

$$\begin{aligned} Z_{\lambda,\boldsymbol{c}} &= \sum_{w \in \mathscr{D}^{e\lambda}} \psi_w \otimes R_{\lambda_1,\delta}\gamma^{\lambda_1,c_1}T_{\lambda_1,c_1} \boxtimes \cdots \boxtimes R_{\lambda_n\delta}\gamma^{\lambda_n,c_n}T_{\lambda_n,c_n} \\ &= R_{d\delta}\gamma^{\lambda,\boldsymbol{c}} \otimes (T_{\lambda_1,c_1} \boxtimes \cdots \boxtimes T_{\lambda_n,c_n}), \end{aligned}$$

as required. □

Define the idempotent

$$e_{\lambda,\boldsymbol{c}} := e_{c_1^{\lambda_1}\ldots c_n^{\lambda_n}} \in \hat{C}_{d\delta} \tag{7.17}$$

and the $\hat{C}_{d\delta}$-module

$$\hat{T}_{\lambda,\boldsymbol{c}} := \hat{C}_{d\delta}e_{\lambda,\boldsymbol{c}}. \tag{7.18}$$

Recalling the notation (7.5), define the left $\hat{C}_{d\delta}$-module

$$N_{\lambda,\boldsymbol{c}} := N_{c_1^{\lambda_1}\ldots c_n^{\lambda_n}} = \hat{C}_{d\delta}\hat{C}^{>0}_{\omega\delta}e_{\lambda,\boldsymbol{c}} \subseteq \hat{T}_{\lambda,\boldsymbol{c}}. \tag{7.19}$$

By Lemma 7.6, we have an isomorphism of left $R_{d\delta}$-modules

$$T_{\lambda,\boldsymbol{c}} \xrightarrow{\sim} \hat{T}_{\lambda,\boldsymbol{c}}/N_{\lambda,\boldsymbol{c}},\ 1_{\omega\delta} \otimes v_{\boldsymbol{l}^{c_1}}^{\otimes\lambda_1} \otimes \cdots \otimes v_{\boldsymbol{l}^{c_n}}^{\otimes\lambda_n} \mapsto e_{\lambda,\boldsymbol{c}} + N_{\lambda,\boldsymbol{c}}. \tag{7.20}$$

Let $\Theta\colon W_d \to \gamma^\omega \hat{C}_{d\delta}\gamma^\omega$ be the algebra homomorphism of Theorem 7.12. Recalling the element $\mathsf{e}_{\lambda,\boldsymbol{c}} \in W_d$ defined by (3.8), note that by Theorem 7.12(i) we have

$$\Theta(\mathsf{e}_{\lambda,\boldsymbol{c}}) = e_{\lambda,\boldsymbol{c}}. \tag{7.21}$$

Recall the function $\varepsilon_{\lambda,\boldsymbol{c}}$ from (3.10). Define the left $\hat{C}_{d\delta}$-submodule

$$\tilde{Z}_{\lambda,\boldsymbol{c}} := \{v \in \hat{T}_{\lambda,\boldsymbol{c}} \mid v\Theta(g) - \varepsilon_{\lambda,\boldsymbol{c}}(g)v \in N_{\lambda,\boldsymbol{c}} \text{ for all } g \in \mathfrak{S}_\lambda\} \subseteq \hat{T}_{\lambda,\boldsymbol{c}}. \tag{7.22}$$

Lemma 7.23. *For every $g \in \mathfrak{S}_\lambda$, we have $e_{\lambda,\boldsymbol{c}}\Theta(g) = \Theta(g)e_{\lambda,\boldsymbol{c}}$.*

Proof. Since we have $\mathsf{e}_{\lambda,\boldsymbol{c}}g = g\mathsf{e}_{\lambda,\boldsymbol{c}}$ in W_d, the lemma follows from (7.21). □

Lemma 7.24. *We have $\tilde{Z}_{\lambda,\boldsymbol{c}} = \hat{C}_{d\delta}\gamma^{\lambda,\boldsymbol{c}}\hat{C}_{\lambda\delta}e_{\lambda,\boldsymbol{c}} + N_{\lambda,\boldsymbol{c}}$.*

Proof. Throughout the proof, we identify $T_{\lambda,\boldsymbol{c}}$ with $\hat{T}_{\lambda,\boldsymbol{c}}/N_{\lambda,\boldsymbol{c}}$ via the isomorphism (7.20), so

$$1_{\lambda\delta} \otimes T_{\lambda_1,c_1} \boxtimes \cdots \boxtimes T_{\lambda_n,c_n} = (\hat{C}_{\lambda\delta}e_{\lambda,\boldsymbol{c}} + N_{\lambda,\boldsymbol{c}})/N_{\lambda,\boldsymbol{c}}$$

and we have a right action of $\mathfrak{S}_\lambda$ on $\hat{T}_{\lambda,\boldsymbol{c}}/N_{\lambda,\boldsymbol{c}}$. The space $Z_{\lambda,\boldsymbol{c}}$ of signed invariants under this action becomes a $\hat{C}_{d\delta}$-submodule of $\hat{T}_{\lambda,\boldsymbol{c}}/N_{\lambda,\boldsymbol{c}}$, and by Lemma 7.16, we have

$$Z_{\lambda,\boldsymbol{c}} = (\hat{C}_{d\delta}\gamma^{\lambda,\boldsymbol{c}}\hat{C}_{\lambda\delta}e_{\lambda,\boldsymbol{c}} + N_{\lambda,\boldsymbol{c}})/N_{\lambda,\boldsymbol{c}}.$$

Let $1 \le t < d$ satisfy $s_t \in \mathfrak{S}_\lambda$, and moreover, let $q \in [1,n]$ be defined by the condition that s_t lies in the $\mathfrak{S}_{\lambda_q}$-component of $\mathfrak{S}_\lambda$. By (7.14), we have

$$(e_{\lambda,\boldsymbol{c}} + N_{\lambda,\boldsymbol{c}}) \cdot s_t = -\zeta_{c_q}\hat{r}_t e_{\lambda,\boldsymbol{c}} + N_{\lambda,\boldsymbol{c}}.$$

Let $v = ve_{\lambda,\boldsymbol{c}} \in \hat{T}_{\lambda,\boldsymbol{c}}$. Then

$$(v + N_{\lambda,\boldsymbol{c}}) \cdot s_t = -\zeta_{c_q}v\hat{r}_t e_{\lambda,\boldsymbol{c}} + N_{\lambda,\boldsymbol{c}} = -\zeta_{c_q}v\Theta(s_t)e_{\lambda,\boldsymbol{c}} + N_{\lambda,\boldsymbol{c}}$$

$$= -\zeta_{c_q} v e_{\lambda,\boldsymbol{c}} \Theta(s_t) + N_{\lambda,\boldsymbol{c}} = -\zeta_{c_q} v \Theta(s_t) + N_{\lambda,\boldsymbol{c}},$$

using Lemma 7.23 for the third equality. So for any $g \in \mathfrak{S}_\lambda$, we have

$$(-1)^{\ell(g)}(v + N_{\lambda,\boldsymbol{c}}) \cdot g = \varepsilon_{\lambda,\boldsymbol{c}}(g) v \Theta(g) + N_{\lambda,\boldsymbol{c}}.$$

In particular, $N_{\lambda,\boldsymbol{c}}\Theta(g) \subseteq N_{\lambda,\boldsymbol{c}}$ for all $g \in \mathfrak{S}_\lambda$. It follows that $\tilde{Z}_{\lambda,\boldsymbol{c}}$ is the preimage of $Z_{\lambda,\boldsymbol{c}}$ under the canonical projection $\hat{T}_{\lambda,\boldsymbol{c}} \twoheadrightarrow \hat{T}_{\lambda,\boldsymbol{c}}/N_{\lambda,\boldsymbol{c}}$. So

$$\tilde{Z}_{\lambda,\boldsymbol{c}} = \hat{C}_{d\delta} \gamma^{\lambda,\boldsymbol{c}} \hat{C}_{\lambda\delta} e_{\lambda,\boldsymbol{c}} + N_{\lambda,\boldsymbol{c}}$$

by the first paragraph of the proof. □

7.4. **The structure of $\gamma^{\lambda,\boldsymbol{c}}\hat{C}_{d\delta}\gamma^{\omega}$.** In view of (4.23) to $\boldsymbol{i} \in I^\theta_{\mathrm{div}}$ we associate $\hat{\boldsymbol{i}} \in I^\theta$. Throughout this subsection we drop the hats and usually write $\boldsymbol{i}$ for $\hat{\boldsymbol{i}}$. For example, $\widehat{\boldsymbol{l}^j(d)}$ is written simply as $\boldsymbol{l}^j(d)$.

Let $h_d \in \mathfrak{S}_{ed}$ be defined by $h_d((t-1)e+q) = (q-1)d + t$ for all $t = 1, \dots, d$ and $q = 1, \dots, e$. In other words, h_d is the shortest element of $\mathfrak{S}_{de}$ such that $h_d((\boldsymbol{l}^j)^d) = \boldsymbol{l}^j(d)$ for all $j \in J$. Let $w_{0,d} \in \mathfrak{S}_{ed}$ be the longest element of $\mathfrak{S}_{(d^e)}$, i.e. $w_{0,d}((q-1)d+t) = (q-1)d + d + 1 - t$ for all $q = 1, \dots, e$ and $t = 1, \dots, d$. Let $j \in J$ and note that $e_{j^d} = 1_{(\boldsymbol{l}^j)^d}$. We set

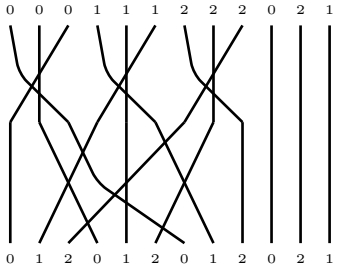
$$u_{d,j} := \psi_{w_{0,d}} \psi_{h_d} e_{j^d} \in \hat{C}_{d\delta}.$$

More generally, fix $n \in \mathbb{Z}_{>0}$ and $(\lambda, \boldsymbol{c}) \in \Lambda^{\mathrm{col}}(n,d)$. Recalling (7.17), we define

$$\begin{aligned} h_\lambda &:= (h_{\lambda_1}, \dots, h_{\lambda_n}) \in \mathfrak{S}_{e\lambda_1} \times \cdots \times \mathfrak{S}_{e\lambda_n} = \mathfrak{S}_{e\lambda} \leq \mathfrak{S}_{ed}, \\ w_{0,\lambda} &:= (w_{0,\lambda_1}, \dots, w_{0,\lambda_n}) \in \mathfrak{S}_{e\lambda_1} \times \cdots \times \mathfrak{S}_{e\lambda_n} = \mathfrak{S}_{e\lambda} \leq \mathfrak{S}_{ed}, \\ u_{\lambda,\boldsymbol{c}} &:= \psi_{w_{0,\lambda}} \psi_{h_\lambda} e_{\lambda,\boldsymbol{c}} \\ &= u_{\lambda_1,c_1} \otimes \cdots \otimes u_{\lambda_n,c_n} \in \hat{C}_{\lambda_1\delta} \otimes \cdots \otimes \hat{C}_{\lambda_n\delta} = \hat{C}_{\lambda\delta} \subseteq \hat{C}_{d\delta}, \end{aligned}$$

where we have used the identification from Lemma 4.33.

Example 7.25. If $e = 3$, then $J = \{1, 2\}$ and the only choice of the words (6.1) is $\boldsymbol{l}^1 = 021 \in I^\delta$ and $\boldsymbol{l}^2 = 012 \in I^\delta$. In this case, if $d = 4$, $n = 5$, $\lambda = (3,0,1,0,0) \in \Lambda(5,4)$ and $\boldsymbol{c} = (2,1,1,1,2) \in J^5$, then in terms of Khovanov-Lauda diagrams [**KL**, §2.1] we have

$u_{\lambda,\boldsymbol{c}} =$ 

In view of (4.20) we get:

Lemma 7.26. *We have $u_{\lambda,\boldsymbol{c}} \in \gamma^{\lambda,\boldsymbol{c}}\hat{C}_{d\delta}e_{\lambda,\boldsymbol{c}}$.*

Recall the integer a_λ defined by (6.4). The following is easily deduced from the definitions:

Lemma 7.27. *The element $u_{\lambda,\boldsymbol{c}} \in \hat{C}_{d\delta}$ is homogeneous of degree a_λ.*

Lemma 7.28. *Let $j \in J$ and $\boldsymbol{i} = \boldsymbol{i}^{(1)} \dots \boldsymbol{i}^{(d)}$ for some $\boldsymbol{i}^{(1)}, \dots, \boldsymbol{i}^{(d)} \in I^\delta$. If $g \in {}^{(d^e)}\mathscr{D}^{(e^d)}$ is such that $g\boldsymbol{i} = \boldsymbol{l}^j(d)$, then $g = h_d$ and $\boldsymbol{i}^{(1)} = \dots = \boldsymbol{i}^{(d)} = \boldsymbol{l}^j$.*

Proof. If $\boldsymbol{i} \in I^\delta$, the letters of $\boldsymbol{i}$ are distinct. The result follows from this observation together with the definition of ${}^{(d^e)}\mathscr{D}^{(e^d)}$. □

Given $\lambda \in \Lambda(n,d)$, define the composition

$$\lambda^{\{e\}} := (\lambda_1^e, \dots, \lambda_n^e) \in \Lambda(ne, de).$$

Define the *block permutation group* $B_d \cong \mathfrak{S}_d$ as the subgroup of $\mathfrak{S}_{de}$ generated by the involutions $w_1, \dots, w_{d-1}$ defined by (7.10).

Lemma 7.29. *Let $\boldsymbol{i}^{(1)}, \dots, \boldsymbol{i}^{(d)} \in I^\delta_{\mathrm{sc}}$ and $\boldsymbol{i} = \boldsymbol{i}^{(1)} \dots \boldsymbol{i}^{(d)}$. If $g \in {}^{\lambda^{\{e\}}}\mathscr{D}^{(e^d)}$ is such that $g\boldsymbol{i} = \boldsymbol{l}(\lambda, \boldsymbol{c})$ for some $\boldsymbol{c} \in J^n$, then $g = h_\lambda b$ for some $b \in B_d$ such that $\ell(g) = \ell(h_\lambda) + \ell(b)$.*

Proof. We apply the induction on n, the case $n = 1$ being Lemma 7.28. Let $n > 1$. By the inductive hypothesis, we may assume that $\lambda_n > 0$. Note that $\boldsymbol{l}(\lambda, \boldsymbol{c}) = \boldsymbol{l}(\lambda', \boldsymbol{c}')\boldsymbol{l}^{c_n}(\lambda_n)$, where $\lambda' = (\lambda_1, \dots, \lambda_{n-1})$ and $\boldsymbol{c}' = (c_1, \dots, c_{n-1})$. Let $\boldsymbol{l}^{c_n} = (l_1, \dots, l_e)$ so that $\boldsymbol{l}^{c_n}(\lambda_n) = (l_1^{\lambda_n}, \dots, l_e^{\lambda_n})$. We know that $l_1 = 0$ and $i_1^{(t)} = 0$ for $t = 1, \dots, d$, see Corollary 4.29. Note that the positions $(d - \lambda_n)e + q$ for $q = 1, \dots, \lambda_n$ in $\boldsymbol{l}(\lambda, \boldsymbol{c})$ correspond to the first λ_n positions in $\boldsymbol{l}^{c_n}(\lambda_n)$, and so they are occupied with 0s. So there exist $1 \le a_1, \dots, a_{\lambda_n} \le d$ such that g sends the first position of the word $\boldsymbol{i}^{(a_q)}$ to the qth position of $\boldsymbol{l}^{c_n}(\lambda_n)$, i.e. $g((a_q - 1)e + 1) = (d - \lambda_n)e + q$ for $q = 1, \dots, \lambda_n$. Since $g \in {}^{\lambda^{\{e\}}}\mathscr{D}$, we have $a_1 < \dots < a_{\lambda_n}$. Since $g \in \mathscr{D}^{(e^d)}$, it sends the remaining positions in the words $\boldsymbol{i}^{(a_1)}, \dots, \boldsymbol{i}^{(a_{\lambda_n})}$ to the remaining positions of $\boldsymbol{l}^{c_n}(\lambda_n)$, i.e. to the last $\lambda_n(e-1)$ positions of $\boldsymbol{l}(\lambda, \boldsymbol{c})$. It follows that $\boldsymbol{i}^{(a_1)} = \dots = \boldsymbol{i}^{(a_{\lambda_n})} = \boldsymbol{l}^{c_n}$.

Let $b' \in B_d$ be the block permutation which moves the blocks $\boldsymbol{i}^{(a_1)}, \dots, \boldsymbol{i}^{(a_{\lambda_n})}$ to the end in the same order and preserves the order of the remaining blocks. Let $g' = g(b')^{-1}$. We claim that $\ell(g') = \ell(g) - \ell(b')$. To prove this, it suffices to show that $g(r) > g(s)$ for all $1 \le r < s \le ed$ such that $b'(r) > b'(s)$, which is clear since for any such r, s, the element r is in one of the blocks corresponding to $\boldsymbol{i}^{(a_1)}, \dots, \boldsymbol{i}^{(a_{\lambda_n})}$, whereas s is not.

We have $g' \in {}^{\lambda^{\{e\}}}\mathscr{D}^{(e^d)}$. Indeed, it is obvious that $g' \in \mathscr{D}^{(e^d)}$, and $g' \in {}^{\lambda^{\{e\}}}\mathscr{D}$ because $g = g'b' \in {}^{\lambda^{\{e\}}}\mathscr{D}$ and $\ell(g) = \ell(g') + \ell(b')$. Moreover, $g' \in \mathfrak{S}_{(d-\lambda_n)e, \lambda_n e}$, so the result follows by the inductive assumption. □

Lemma 7.30. *For any $y \in \mathbb{Z}[y_1, \dots, y_{de}]$ there exists $y' \in \mathbb{Z}[y_1, \dots, y_{de}]$ such that $1_{\boldsymbol{l}(\lambda,\boldsymbol{c})} y \psi_{h_\lambda} = 1_{\boldsymbol{l}(\lambda,\boldsymbol{c})} \psi_{h_\lambda} y'$.*

Proof. This follows from the observation that the Khovanov–Lauda diagram [**KL**, §2.1] of $1_{\boldsymbol{l}(\lambda,\boldsymbol{c})}\psi_{h_\lambda}$ does not have any crossings of two strings with the same label and the relations (4.7), (4.8). □

Lemma 7.31. *For any $(\lambda, \boldsymbol{c}) \in \Lambda^{\mathrm{col}}(n,d)$, we have:*

(i) $\gamma^{\lambda,\boldsymbol{c}} \hat{C}_{d\delta} 1_{\omega\delta} = u_{\lambda,\boldsymbol{c}} \hat{C}_{d\delta} 1_{\omega\delta} = u_{\lambda,\boldsymbol{c}} \gamma^\omega \hat{C}_{d\delta} 1_{\omega\delta}$;

(ii) $\gamma^{\lambda,\boldsymbol{c}}\hat{C}_{\lambda\delta}1_{\omega\delta} = u_{\lambda,\boldsymbol{c}}\hat{C}_{\omega\delta}$.

Proof. By Lemma 7.26,

$$u_{\lambda,\boldsymbol{c}}\gamma^{\omega}\hat{C}_{d\delta}1_{\omega\delta} = u_{\lambda,\boldsymbol{c}}\hat{C}_{d\delta}1_{\omega\delta} \subseteq \gamma^{\lambda,\boldsymbol{c}}\hat{C}_{d\delta}1_{\omega\delta},$$

so for (i) it only remains to prove the inclusion $\gamma^{\lambda,\boldsymbol{c}}\hat{C}_{d\delta}1_{\omega\delta} \subseteq u_{\lambda,\boldsymbol{c}}\gamma^{\omega}\hat{C}_{d\delta}1_{\omega\delta}$. Moreover, for (ii) we may assume that $n = 1$ and prove only that $\gamma^{d,j}\hat{C}_{d\delta}1_{\omega\delta} \subseteq u_{d,j}\hat{C}_{\omega\delta}$.

The word $\boldsymbol{l}(\lambda, \boldsymbol{c}) \in I^{d\delta}$ is the concatenation of ne words of the form $(i^s) \in I^{s\alpha_i}$ for various $i \in I$ and $s \in \mathbb{Z}_{\geq 0}$. We denote the corresponding integer multiples $s\alpha_i \in Q_+$ of simple roots by $\theta_1, \dots, \theta_{ne}$, listed in the order of concatenation, i.e. $\theta_{e(t-1)+q} = \lambda_t\alpha_{l_{c_t,q}}$ for all $t = 1, \dots, n$ and $q = 1, \dots, e$, cf. (6.1). By Lemma 4.18, we have

$$1_{\theta_1,\dots,\theta_{ne}}\hat{C}_{d\delta}1_{\omega\delta} = \sum_{g \in {}^{\lambda\{e\}}\mathscr{D}^{(e^d)}} R_{\theta_1,\dots,\theta_{ne}}\psi_g\hat{C}_{\omega\delta}$$

as $(R_{\theta_1,\dots,\theta_{ne}}, \hat{C}_{\omega\delta})$-bimodules, so

$$\gamma^{\lambda,\boldsymbol{c}}\hat{C}_{d\delta}1_{\omega\delta} = \sum_{g \in {}^{\lambda\{e\}}\mathscr{D}^{(e^d)}} \gamma^{\lambda,\boldsymbol{c}}R_{\theta_1,\dots,\theta_{ne}}\psi_g\hat{C}_{\omega\delta}.$$

Consider an element $g \in {}^{\lambda\{e\}}\mathscr{D}^{(e^d)}$ such that the summand

$$U := \gamma^{\lambda,\boldsymbol{c}}R_{\theta_1,\dots,\theta_{ne}}\psi_g\hat{C}_{\omega\delta}$$

on the right hand side is non-zero. Then there exists $\boldsymbol{i} \in I^{d\delta}$ such that $g\boldsymbol{i} = \boldsymbol{l}(\lambda, \boldsymbol{c})$ and $1_{\boldsymbol{i}}1_{\omega\delta} \neq 0$ in $\hat{C}_{\omega\delta}$, whence $\boldsymbol{i} = \boldsymbol{i}^{(1)} \dots \boldsymbol{i}^{(d)}$ for some $\boldsymbol{i}^{(1)}, \dots, \boldsymbol{i}^{(d)} \in I^{\delta}_{\mathrm{sc}}$. Hence, by Lemma 7.29, we have $g = h_\lambda b$ for some $b \in B_d$. Moreover, in the case when $n = 1$, needed for part (ii), we have $b = 1$ by Lemma 7.28. We may assume that preferred reduced decompositions for the elements of $\mathfrak{S}_{de}$ are chosen in such a way that $\psi_g = \psi_{h_\lambda}\psi_b$, so $U = \gamma^{\lambda,\boldsymbol{c}}R_{\theta_1,\dots,\theta_{ne}}\psi_{h_\lambda}\psi_b\hat{C}_{\omega\delta}$.

Let $P \subseteq R_{d\delta}$ be the subalgebra generated by $y_1, \dots, y_{de}$. Then

$$\begin{aligned} U &= \gamma^{\lambda,\boldsymbol{c}}R_{\theta_1,\dots,\theta_{ne}}\psi_{h_\lambda}\psi_b\hat{C}_{\omega\delta} \\ &= \gamma^{\lambda,\boldsymbol{c}}\psi_{w_{0,\lambda}}P\psi_{h_\lambda}\psi_b\hat{C}_{\omega\delta} \\ &= \gamma^{\lambda,\boldsymbol{c}}\psi_{w_{0,\lambda}}\psi_{h_\lambda}P\psi_b\hat{C}_{\omega\delta} \\ &= u_{\lambda,\boldsymbol{c}}\gamma^{\omega}P\psi_b\hat{C}_{\omega\delta}, \end{aligned}$$

where we have used Lemma 4.21 for the second equality, Lemma 7.30 for the third equality, the definition of $u_{\lambda,\boldsymbol{c}}$ and Lemma 7.26 for the fourth equality. Part (i) now follows since $u_{\lambda,\boldsymbol{c}}\gamma^{\omega}P\psi_b\hat{C}_{\omega\delta} \subseteq u_{\lambda,\boldsymbol{c}}\gamma^{\omega}\hat{C}_{d\delta}1_{\omega\delta}$, whereas part (ii) follows since $\psi_b = 1$ in the case $n = 1$ and $P\hat{C}_{\omega\delta} = \hat{C}_{\omega\delta}$. □

Multiplying the equality in Lemma 7.31(i) by γ^{ω} on the right, we obtain $\gamma^{\lambda,\boldsymbol{c}}\hat{C}_{d\delta}\gamma^{\omega} = u_{\lambda,\boldsymbol{c}}\gamma^{\omega}\hat{C}_{d\delta}\gamma^{\omega}$. In particular:

Corollary 7.32. *As a right $\gamma^{\omega}\hat{C}_{d\delta}\gamma^{\omega}$-module, $\gamma^{\lambda,\boldsymbol{c}}\hat{C}_{d\delta}\gamma^{\omega}$ is generated by $u_{\lambda,\boldsymbol{c}}$.*

Note that by Lemmas 7.1 and 7.27 and Corollary 7.32, we have $\gamma^{\lambda,\boldsymbol{c}}\hat{C}_{d\delta}\gamma^{\omega} \subseteq \hat{C}^{\geq a_\lambda}_{d\delta}$. Recall the left $\hat{C}_{d\delta}$-modules $N_{\boldsymbol{j}}$ defined by (7.5).

Lemma 7.33. *For any $(\lambda, \boldsymbol{c}) \in \Lambda^{\mathrm{col}}(n,d)$ and $\boldsymbol{j} \in J^d$, we have $\gamma^{\lambda,\boldsymbol{c}} N_{\boldsymbol{j}} \subseteq \hat{C}_{d\delta}^{>a_\lambda}$.*

Proof. Recall that $u_{\lambda,\boldsymbol{c}} = u_{\lambda,\boldsymbol{c}}\gamma^\omega$. We have

$$\gamma^{\lambda,\boldsymbol{c}} N_{\boldsymbol{j}} = \gamma^{\lambda,\boldsymbol{c}} \hat{C}_{d\delta}(\hat{C}_{\omega\delta}^{>0}) e_{\boldsymbol{j}} = u_{\lambda,\boldsymbol{c}} \hat{C}_{d\delta}(\hat{C}_{\omega\delta}^{>0}) e_{\boldsymbol{j}} = u_{\lambda,\boldsymbol{c}} e_{\lambda,\boldsymbol{c}} \hat{C}_{d\delta}(\hat{C}_{\omega\delta}^{>0}) e_{\boldsymbol{j}},$$

where the second equality comes from Lemma 7.31(i) since $1_{\omega\delta}$ is the identity element of $\hat{C}_{\omega\delta}$, and the last equality holds by Lemma 7.26. Now $e_{\lambda,\boldsymbol{c}} \hat{C}_{d\delta} 1_{\omega\delta}$ is non-negatively graded by Lemma 7.1(i), and $\deg(u_\lambda) = a_\lambda$ by Lemma 7.27, so the lemma follows. □

Recalling (7.18) and the homomorphism Θ from Theorem 7.12, define the $\hat{C}_{d\delta}$-submodule

$$\hat{Z}_{\lambda,\boldsymbol{c}} := \{v \in \hat{T}_{\lambda,\boldsymbol{c}} \mid v\Theta(g) = \varepsilon_{\lambda,\boldsymbol{c}}(g)v \text{ for all } g \in \mathfrak{S}_\lambda\} \subseteq \hat{T}_{\lambda,\boldsymbol{c}}.$$

Clearly, $\hat{Z}_{\lambda,\boldsymbol{c}} \subseteq \tilde{Z}_{\lambda,\boldsymbol{c}}$, cf. (7.22).

Lemma 7.34. *We have $u_{\lambda,\boldsymbol{c}} \in \hat{Z}_{\lambda,\boldsymbol{c}}$.*

Proof. By Lemma 7.26 and the definition of $u_{\lambda,\boldsymbol{c}}$, we have $u_{\lambda,\boldsymbol{c}} \in \gamma^{\lambda,\boldsymbol{c}} \hat{C}_{\lambda\delta} e_{\lambda,\boldsymbol{c}}$. Hence, by Lemma 7.24, we get $u_{\lambda,\boldsymbol{c}} \in \tilde{Z}_{\lambda,\boldsymbol{c}}$. So for any $g \in \mathfrak{S}_\lambda$, we have $u_{\lambda,\boldsymbol{c}}\Theta(g) - \varepsilon_{\lambda,\boldsymbol{c}}(g)u_{\lambda,\boldsymbol{c}} \in \gamma^{\lambda,\boldsymbol{c}} N_{\lambda,\boldsymbol{c}}$. By Lemma 7.27, $u_{\lambda,\boldsymbol{c}}\Theta(g) - \varepsilon_{\lambda,\boldsymbol{c}}(g)u_{\lambda,\boldsymbol{c}}$ is homogeneous of degree a_λ. But $\gamma^{\lambda,\boldsymbol{c}} N_{\lambda,\boldsymbol{c}}^{a_\lambda} = 0$ by Lemma 7.33, so $u_{\lambda,\boldsymbol{c}}\Theta(g) - \varepsilon_{\lambda,\boldsymbol{c}}(g)u_{\lambda,\boldsymbol{c}} = 0$. □

Lemma 7.35. *Let $(\lambda, \boldsymbol{c}), (\mu, \boldsymbol{b}) \in \Lambda^{\mathrm{col}}(n,d)$. If $v \in \gamma^{\mu,\boldsymbol{b}} \hat{Z}_{\lambda,\boldsymbol{c}}$ is a homogeneous element of degree a_μ, then $v = x u_{\lambda,\boldsymbol{c}}$ for some $x \in \gamma^{\mu,\boldsymbol{b}} \hat{C}_{d\delta} \gamma^{\lambda,\boldsymbol{c}}$.*

Proof. By Lemma 7.24, we have $v \in \gamma^{\mu,\boldsymbol{b}} \hat{C}_{d\delta} \gamma^{\lambda,\boldsymbol{c}} \hat{C}_{\lambda\delta} e_{\lambda,\boldsymbol{c}} + \gamma^{\mu,\boldsymbol{b}} N_{\lambda,\boldsymbol{c}}$. By Lemma 7.33, $\gamma^{\mu,\boldsymbol{b}} N_{\lambda,\boldsymbol{c}}^{a_\mu} = 0$, so $v \in \gamma^{\mu,\boldsymbol{b}} \hat{C}_{d\delta} \gamma^{\lambda,\boldsymbol{c}} \hat{C}_{\lambda\delta} e_{\lambda,\boldsymbol{c}}$. Hence, by Lemma 7.31(ii), we have $v \in \gamma^{\mu,\boldsymbol{b}} \hat{C}_{d\delta} u_{\lambda,\boldsymbol{c}} \hat{C}_{\omega\delta} e_{\lambda,\boldsymbol{c}}$. We know that $\hat{C}_{\omega\delta}$ is non-negatively graded, so we have $v = v_1 + v_2$ for some homogeneous elements $v_1 \in \gamma^{\mu,\boldsymbol{b}} \hat{C}_{d\delta} u_{\lambda,\boldsymbol{c}} \hat{C}_{\omega\delta}^0 e_{\lambda,\boldsymbol{c}}$ and $v_2 \in \gamma^{\mu,\boldsymbol{b}} \hat{C}_{d\delta} u_{\lambda,\boldsymbol{c}} \hat{C}_{\omega\delta}^{>0} e_{\lambda,\boldsymbol{c}}$, with $\deg(v_1) = \deg(v_2) = \deg(v) = a_\mu$. By definition (7.19) of $N_{\lambda,\boldsymbol{c}}$, we have $v_2 \in \gamma^{\mu,\boldsymbol{b}} N_{\lambda,\boldsymbol{c}}$, whence $v_2 = 0$ by Lemma 7.33. On the other hand, by Lemma 7.7, we have $u_{\lambda,\boldsymbol{c}} \hat{C}_{\omega\delta}^0 e_{\lambda,\boldsymbol{c}} = \mathbb{Z} u_{\lambda,\boldsymbol{c}}$, so $v = v_1 \in \gamma^{\mu,\boldsymbol{b}} \hat{C}_{d\delta} u_{\lambda,\boldsymbol{c}}$, and the result follows by Lemma 7.26. □

8. RoCK blocks and generalized Schur algebras

As in §5.4, we fix $d \in \mathbb{Z}_{>0}$ and a d-Rouquier core ρ of residue κ.

8.1. **Identifying W_d with $\gamma^\omega C_{\rho,d} \gamma^\omega$.** Recall from (5.17) that we have the natural surjection

$$\Pi \colon \hat{C}_{\rho,d} \to \hat{C}_{d\delta} / \ker \Omega = C_{\rho,d}.$$

This yields the surjections

$$\Pi_\omega \colon \gamma^\omega \hat{C}_{d\delta} \gamma^\omega \to \gamma^\omega C_{\rho,d} \gamma^\omega, \quad \Pi_{\lambda,\boldsymbol{c}} \colon \gamma^{\lambda,\boldsymbol{c}} \hat{C}_{d\delta} \gamma^\omega \to \gamma^{\lambda,\boldsymbol{c}} C_{\rho,d} \gamma^\omega \tag{8.1}$$

for $(\lambda, \boldsymbol{c}) \in \Lambda^{\mathrm{col}}(n,d)$. For any $x \in \hat{C}_{d\delta}$, we often denote by x its image $\Pi(x)$ in $C_{\rho,d}$.

We define the algebra homomorphism

$$\Xi := \Pi_\omega \circ \Theta \colon W_d \to \gamma^\omega C_{\rho,d} \gamma^\omega \tag{8.2}$$

where $\Theta\colon W_d \to \gamma^\omega \hat{C}_{\rho,d}\gamma^\omega$ is as in Theorem 7.12. Our aim is to prove that Ξ is an isomorphism by generalizing the arguments of [**Ev**, Section 7], where a similar statement is proved over a field containing an element of quantum characteristic e (this means that the field contains an element q with $1+q+\dots+q^{e-1}=0$ and e is minimal such). We begin with the case where $d=1$, when $W_d = \mathsf{Z}$.

Lemma 8.3. *For $d=1$ and each $a \in \{0,1\}$, the map Ξ restricts to a $\mathbb{Z}$-module isomorphism of graded components $\mathsf{Z}^a \xrightarrow{\sim} \gamma^\omega C^a_{\rho,1}\gamma^\omega$.*

Proof. By Theorem 7.12(ii), Θ restricts to an isomorphism $\mathsf{Z}^a \xrightarrow{\sim} \gamma^\omega \hat{C}^a_{\rho,1}\gamma^\omega$, whence Ξ restricts to a surjection $\mathsf{Z}^a \to \gamma^\omega C^a_{\rho,1}\gamma^\omega$. Moreover, by (3.3) and Lemmas 5.18 and 6.19, we have that Z^a and $\gamma^\omega C^a_{\rho,1}\gamma^\omega$ are free $\mathbb{Z}$-modules of the same rank, which completes the proof. □

Lemma 8.4. *Let $d=1$ and $j \in J$. Then $C^2_{\rho,1}e_j = \mathbb{Z}(y_1 - y_e)e_j$.*

Proof. By Lemmas 5.18 and 6.19, the $\mathbb{Z}$-module $C^2_{\rho,1}e_j = e_j C^2_{\rho,1}e_j$ is free of rank 1. It suffices to prove that $y := (y_1 - y_e)e_j \otimes 1$ generates $C^2_{\rho,1,\Bbbk}e_j$ over any field $\Bbbk$, i.e. that $y \neq 0$ for any field $\Bbbk$, cf. Remark 5.21. This is proved in [**Ev**, Proposition 7.2] for any field $\Bbbk$ containing an element of quantum characteristic e, in particular for $e=2$. So we may assume that $e>2$. By Corollary 5.22, the algebra $C_{\rho,1,\Bbbk}$ has a symmetrizing form F of degree -2. Since $e>2$, the element j has a neighbor $k \in J$. In the rest of the proof, we write $x := x \otimes 1 \in C_{\rho,1,\Bbbk}$ for $x \in C_{\rho,1}$. Recalling the elements of $\hat{C}_\delta$ introduced in §7.2, note that $a^{k,j} \neq 0$ in $C^1_{\rho,1,\Bbbk}$ by Lemma 8.3. So there must exist an element $x \in C^1_{\rho,1,\Bbbk}$ such that $F(xa^{k,j}) \neq 0$. Using Theorem 7.12 and Lemma 8.3 again, we may assume that $x = \Xi(\mathsf{a}^{j,k})$, and hence $(y_1 - y_e)e_j = \pm\Xi(\mathsf{ce}_j) = \pm\Xi(\mathsf{a}^{j,k}\mathsf{a}^{k,j}) = \pm a^{j,k}a^{k,j} \neq 0$. □

Now we return to the case when $d \in \mathbb{Z}_{>0}$ is arbitrary. By Lemma 4.33, we have an embedding

$$\iota\colon \hat{C}_\delta \to \hat{C}_{d\delta},\ x \mapsto x \otimes 1_{(d-1)\delta} \in \hat{C}_\delta \otimes \hat{C}_{(d-1)\delta} = \hat{C}_{\delta,(d-1)\delta} \subseteq \hat{C}_{d\delta}. \tag{8.5}$$

In view of Theorem 7.12(i), for any $j \in J$, we have

$$\Theta(\mathsf{e}_j[1]) = \iota(e_j). \tag{8.6}$$

Corollary 8.7. *The element $y_1\gamma^\omega \in C_{\rho,d}$ belongs to the image of Ξ.*

Proof. We have the (non-unital) algebra homomorphisms $\Omega_1\colon \hat{C}_\delta \to R^{\Lambda_0}_{\mathrm{cont}(\rho)+\delta}$ and $\Omega_d\colon \hat{C}_{d\delta} \to R^{\Lambda_0}_{\mathrm{cont}(\rho)+d\delta}$ defined as in (5.12). Recall the algebra homomorphism

$$\zeta := \zeta_{\mathrm{cont}(\rho)+\delta,\,(d-1)\delta}\colon R^{\Lambda_0}_{\mathrm{cont}(\rho)+\delta} \to R^{\Lambda_0}_{\mathrm{cont}(\rho)+\delta,\,(d-1)\delta} \subseteq R^{\Lambda_0}_{\mathrm{cont}(\rho)+d\delta}$$

defined by (4.19). It follows easily from the definitions that $\zeta \circ \Omega_1 = \Omega_d \circ \iota\colon \hat{C}_\delta \to R^{\Lambda_0}_{\mathrm{cont}(\rho)+d\delta}$, whence $\iota(\ker\Omega_1) \subseteq \ker\Omega_d$.

Let $j \in J$. Identifying $\hat{C}_\delta \otimes \hat{C}_{(d-1)\delta}$ with $\hat{C}_{\delta,(d-1)\delta} \subseteq \hat{C}_{d\delta}$ as usual, we have in $\hat{C}_{d\delta}$:

$$\begin{aligned} y_1e_j \otimes 1_{(d-1)\delta} = \iota(y_1e_j) &\in \iota(\mathbb{Z}(y_1 - y_e)e_j + \ker\Omega_1) \\ &= \mathbb{Z}(y_1 - y_e)e_j \otimes 1_{(d-1)\delta} + \iota(\ker\Omega_1) \end{aligned}$$

$$\subseteq \mathbb{Z}(y_1 - y_e)e_j \otimes 1_{(d-1)\delta} + \ker \Omega_d,$$

where we have used Lemma 8.4 for the first inclusion. Multiplying by γ^ω, we get

$$y_1\gamma^\omega = \sum_{j\in J}(y_1 e_j \otimes 1_{(d-1)\delta})\gamma^\omega \in \sum_{j\in J}(\mathbb{Z}(y_1 - y_e)e_j \otimes 1_{(d-1)\delta})\gamma^\omega + \ker \Omega_d$$
$$= \sum_{j\in J} \pm\Theta(\mathsf{c}[1]\mathsf{e}_j[1]) + \ker \Omega_d, \tag{8.8}$$

where the last equality holds by Theorem 7.12(i) and (8.6). Now the lemma follows on applying Π. □

Theorem 8.9. *The map $\Xi\colon W_d \to \gamma^\omega C_{\rho,d}\gamma^\omega$ is an isomorphism of graded algebras.*

Proof. By Theorem 7.12(iii), the algebra $\gamma^\omega C_{\rho,d}\gamma^\omega$ is generated by $\Xi(W_d)$ together with the element $y_1\gamma^\omega$. But $y_1\gamma^\omega \in \Xi(W_d)$ by Corollary 8.7, so Ξ is surjective. By Corollary 6.32, the algebras W_d and $\gamma^\omega C_{\rho,d}\gamma^\omega$ are $\mathbb{Z}$-free of the same rank, and the result follows. □

Lemma 8.10. *Let $\Bbbk$ be a field with $\operatorname{char}\Bbbk = 0$ or $\operatorname{char}\Bbbk > d$. The left module $C_{\rho,d,\Bbbk}\gamma^\omega$ is a projective generator for the algebra $C_{\rho,d,\Bbbk}$.*

Proof. As $C_{\rho,d,\Bbbk}\gamma^\omega$ is projective, it is enough to show that for every simple $C_{\rho,d,\Bbbk}$-module L we have $\gamma^\omega L \cong \operatorname{Hom}_{C_{\rho,d,\Bbbk}}(C_{\rho,d,\Bbbk}\gamma^\omega, L) \neq 0$. But L may also be viewed as a simple $\hat{C}_{d\delta,\Bbbk}$-module via the natural surjection $\hat{C}_{d\delta,\Bbbk} \twoheadrightarrow C_{\rho,d,\Bbbk}$. By Lemma 7.13, the module $\hat{C}_{d\delta,\Bbbk}\gamma^\omega$ is a projective generator for $\hat{C}_{d\delta,\Bbbk}$, whence $\gamma^\omega L \cong \operatorname{Hom}_{\hat{C}_{d\delta,\Bbbk}}(\hat{C}_{d\delta,\Bbbk}\gamma^\omega, L) \neq 0$. □

Corollary 8.11. *The $C_{\rho,d}$-module $C_{\rho,d}\gamma^\omega$ is faithful.*

Proof. By Lemma 5.18, the algebra $C_{\rho,d}$ is $\mathbb{Z}$-free, so it is enough to show that the $C_{\rho,d,\mathbb{Q}}$-module $C_{\rho,d,\mathbb{Q}}\gamma^\omega$ is faithful. By Lemma 8.10, this module is a projective generator for $C_{\rho,d,\mathbb{Q}}$, and the result follows. □

8.2. **Identifying $\gamma^{\lambda,\boldsymbol{c}}C_{d\delta}\gamma^\omega$ with $M_{\lambda,\boldsymbol{c}}$.** Let $n \in \mathbb{Z}_{>0}$ and $(\lambda, \boldsymbol{c}) \in \Lambda^{\mathrm{col}}(n,d)$. By Theorem 7.12, the right $\gamma^\omega\hat{C}_{d\delta}\gamma^\omega$-module $\gamma^{\lambda,\boldsymbol{c}}\hat{C}_{d\delta}\gamma^\omega$ becomes a right W_d-module via the map Θ. Moreover, by Theorem 8.9, the right $\gamma^\omega C_{\rho,d}\gamma^\omega$-module $\gamma^{\lambda,\boldsymbol{c}}C_{\rho,d}\gamma^\omega$ becomes a right W_d-module via the map Ξ. In other words:

$$vz := v\Theta(z) \qquad (v \in \gamma^{\lambda,\boldsymbol{c}}\hat{C}_{d\delta}\gamma^\omega,\ z \in W_d), \tag{8.12}$$
$$vz := v\Xi(z) \qquad (v \in \gamma^{\lambda,\boldsymbol{c}}C_{\rho,d}\gamma^\omega,\ z \in W_d). \tag{8.13}$$

It is clear from the definitions that $\Pi_{\lambda,\boldsymbol{c}}\colon \gamma^{\lambda,\boldsymbol{c}}\hat{C}_{d\delta}\gamma^\omega \to \gamma^{\lambda,\boldsymbol{c}}C_{\rho,d}\gamma^\omega$ is a surjective homomorphism of W_d-modules.

Recall the colored permutation W_d-module $M_{\lambda,\boldsymbol{c}}$ with generator $m_{\lambda,\boldsymbol{c}} = 1_{\lambda,\boldsymbol{c}} \otimes \mathsf{e}_{\lambda,\boldsymbol{c}}$ defined by (3.11) and the element $u_{\lambda,\boldsymbol{c}} \in \gamma^{\lambda,\boldsymbol{c}}C_{d\delta}\gamma^\omega$ introduced in §7.4.

Lemma 8.14. *There is a degree-preserving W_d-module homomorphism*

$$\theta_{\lambda,\boldsymbol{c}}\colon M_{\lambda,\boldsymbol{c}} \to q^{-a_\lambda}\gamma^{\lambda,\boldsymbol{c}}\hat{C}_{d\delta}\gamma^\omega,\ m_{\lambda,\boldsymbol{c}} \mapsto u_{\lambda,\boldsymbol{c}}.$$

Proof. By (7.21) and Lemma 7.26, we have $u_{\lambda,\boldsymbol{c}}\Theta(\mathtt{e}_{\lambda,\boldsymbol{c}}) = u_{\lambda,\boldsymbol{c}}e_{\lambda,\boldsymbol{c}} = u_{\lambda,\boldsymbol{c}}$. By Lemma 7.34, for any $g \in \mathfrak{S}_\lambda$ we have $u_{\lambda,\boldsymbol{c}}\Theta(g) = \varepsilon_{\lambda,\boldsymbol{c}}(g)u_{\lambda,\boldsymbol{c}}$. Using Lemma 7.27, we deduce that there is a degree-preserving $W_{\lambda,\boldsymbol{c}}$-module homomorphism $\mathtt{alt}_{\lambda,\boldsymbol{c}} \to q^{-a_\lambda}\gamma^{\lambda,\boldsymbol{c}}\hat{C}_{d\delta}\gamma^\omega,\ 1_{\lambda,\boldsymbol{c}} \mapsto u_{\lambda,\boldsymbol{c}}$. This map induces a W_d-module homomorphism $\theta_{\lambda,\boldsymbol{c}}$ as in the statement of the lemma. □

From now on, we write $\bar{u}_{\lambda,\boldsymbol{c}} := \Pi_{\lambda,\boldsymbol{c}}(u_{\lambda,\boldsymbol{c}}) \in C_{\rho,d}$.

Theorem 8.15. *For any $(\lambda,\boldsymbol{c}) \in \Lambda^{\mathrm{col}}(n,d)$, there is an isomorphism of graded W_d-modules:*

$$\eta_{\lambda,\boldsymbol{c}}\colon M_{\lambda,\boldsymbol{c}} \xrightarrow{\sim} q^{-a_\lambda}\gamma^{\lambda,\boldsymbol{c}}C_{\rho,d}\gamma^\omega,\ m_{\lambda,\boldsymbol{c}} \mapsto \bar{u}_{\lambda,\boldsymbol{c}}.$$

Proof. Let $\theta_{\lambda,\boldsymbol{c}}$ be as in Lemma 8.14. We have a homomorphism of W_d-modules

$$\eta_{\lambda,\boldsymbol{c}} := \Pi_{\lambda,\boldsymbol{c}} \circ \theta_{\lambda,\boldsymbol{c}}\colon M_{\lambda,\boldsymbol{c}} \to q^{-a_\lambda}\gamma^{\lambda,\boldsymbol{c}}C_{\rho,d}\gamma^\omega,\ m_{\lambda,\boldsymbol{c}} \mapsto \bar{u}_{\lambda,\boldsymbol{c}}.$$

By Corollary 7.32, the right $\gamma^\omega C_{\rho,d}\gamma^\omega$-module $q^{-a_\lambda}\gamma^{\lambda,\boldsymbol{c}}C_{\rho,d}\gamma^\omega$ is generated by $\bar{u}_{\lambda,\boldsymbol{c}}$. Using Theorem 8.9, we conclude that $\eta_{\lambda,\boldsymbol{c}}$ is surjective. By Corollary 6.31, the $\mathbb{Z}$-modules $M_{\lambda,\boldsymbol{c}}$ and $q^{-a_\lambda}\gamma^{\lambda,\boldsymbol{c}}C_{\rho,d}\gamma^\omega$ are free of the same (ungraded) rank, and the theorem follows. □

8.3. **The algebra $E(n,d)$ and the double $D_Q(n,d)$.** Fix $n \in \mathbb{Z}_{>0}$. Recall the tuple $\boldsymbol{c}^0 \in J^{n(e-1)}$, the modules $M^\lambda = M_{\lambda,\boldsymbol{c}^0}$ and the algebra $S^{\mathsf{Z}}(n,d)$ from §3.2. Let $\lambda \in \Lambda(n(e-1),d)$. Define the idempotent

$$f^\lambda := \gamma^{\lambda,\boldsymbol{c}^0}.$$

Recall the integer

$$a_\lambda = -e\sum_{t=1}^{n(e-1)} \lambda_t(\lambda_t - 1)/2 = \deg(u_{\lambda,\boldsymbol{c}^0}),$$

see (6.4) and Lemma 7.27. In the sequel, we abbreviate

$$u_\lambda := u_{\lambda,\boldsymbol{c}^0},\quad \bar{u}_\lambda := \bar{u}_{\lambda,\boldsymbol{c}^0},\quad e_\lambda := e_{\lambda,\boldsymbol{c}^0},\quad \mathtt{e}_\lambda := \mathtt{e}_{\lambda,\boldsymbol{c}^0},\quad \varepsilon_\lambda := \varepsilon_{\lambda,\boldsymbol{c}^0},$$
$$\theta_\lambda := \theta_{\lambda,\boldsymbol{c}^0}\colon M^\lambda \to q^{-a_\lambda}f^\lambda\hat{C}_{d\delta}\gamma^\omega,\quad \eta_\lambda := \eta_{\lambda,\boldsymbol{c}^0}\colon M^\lambda \xrightarrow{\sim} q^{-a_\lambda}f^\lambda C_{\rho,d}\gamma^\omega,$$

where $\theta_{\lambda,\boldsymbol{c}^0}$ is the homomorphism of Lemma 8.14 and $\eta_{\lambda,\boldsymbol{c}^0}$ is the isomorphism of Theorem 8.15.

Define the left $C_{\rho,d}$-module

$$\Gamma(n,d) := \bigoplus_{\lambda\in\Lambda(n(e-1),d)} q^{a_\lambda}C_{\rho,d}f^\lambda$$

and the algebra

$$E(n,d) := \mathrm{End}_{C_{\rho,d}}(\Gamma(n,d))^{\mathrm{op}}. \tag{8.16}$$

Let $\lambda,\mu \in \Lambda(n(e-1),d)$. We identify the (graded) $\mathbb{Z}$-module $q^{a_\lambda - a_\mu}f^\mu C_{\rho,d}f^\lambda$ with the $\mathbb{Z}$-submodule of $E(n,d)$ consisting of the endomorphisms that send the summand $q^{a_\mu}C_{\rho,d}f^\mu$ to $q^{a_\lambda}C_{\rho,d}f^\lambda$ and send the other summands to zero. Specifically, an element $x \in q^{a_\lambda - a_\mu}f^\mu C_{\rho,d}f^\lambda$ corresponds to the homomorphism given by the right multiplication:

$$q^{a_\mu}C_{\rho,d}f^\mu \to q^{a_\lambda}C_{\rho,d}f^\lambda,\ v \mapsto vx.$$

Thus,

$$E(n,d) = \bigoplus_{\lambda,\mu\in\Lambda(n(e-1))} q^{a_\lambda - a_\mu} f^\mu C_{\rho,d} f^\lambda. \tag{8.17}$$

Let $x \in q^{a_\lambda - a_\mu} f^\mu C_{\rho,d} f^\lambda$. Recalling the right W_d-module structure (8.13), we have a W_d-module homomorphism

$$q^{-a_\lambda} f^\lambda C_{\rho,d}\gamma^\omega \to q^{-a_\mu} f^\mu C_{\rho,d}\gamma^\omega,\ v \mapsto xv.$$

Identifying $q^{-a_\lambda} f^\lambda C_{\rho,d}\gamma^\omega$ with M^λ and $q^{-a_\mu} f^\mu C_{\rho,d}\gamma^\omega$ with M^μ via the isomorphisms of Theorem 8.15, we obtain an element $\Phi(x) \in \mathrm{Hom}_{W_d}(M^\lambda, M^\mu)$. In other words,

$$\Phi(x)\colon M^\lambda \to M^\mu,\ v \mapsto \eta_\mu^{-1}(x\eta_\lambda(v)) \qquad (v \in M^\lambda).$$

Recall from §3.2 that $\mathrm{Hom}_{W_d}(M^\lambda, M^\mu)$ is identified with $\xi_\mu S^{\mathsf{Z}}(n,d)\xi_\lambda$. The assignments $x \mapsto \Phi(x)$ for all $\lambda,\mu \in \Lambda(n(e-1))$ and all $x \in q^{a_\lambda - a_\mu} f^\mu C_{\rho,d} f^\lambda$ extend uniquely to a $\mathbb{Z}$-linear map

$$\Phi\colon E(n,d) \to S^{\mathsf{Z}}(n,d).$$

Lemma 8.18. *The map $\Phi\colon E(n,d) \to S^{\mathsf{Z}}(n,d)$ is a homomorphism of graded algebras.*

Proof. That Φ is a homomorphism of ungraded algebras follows easily from the definitions. Let $x \in q^{a_\lambda - a_\mu} f^\mu C_{\rho,d} f^\lambda$ be a homogeneous element for some $\lambda,\mu \in \Lambda(n(e-1),d)$. Then, by definitions, $\Phi(x)\colon m^\lambda \mapsto m^\mu z$ for some homogeneous $z \in W_d$ such that $x\bar{u}_\lambda = \bar{u}_\mu \Xi(z)$ in $q^{-a_\mu} f^\mu C_{\rho,d}\gamma^\omega$. Hence, $\Phi(x)$ is homogeneous of degree

$$\deg(z) = \deg(\Xi(z)) = \deg(x) - (a_\lambda - a_\mu) + \deg(\bar{u}_\lambda) - \deg(\bar{u}_\mu) = \deg(x),$$

where the last equality is due to Lemma 7.27. □

Corollary 8.19. *The algebra homomorphism $\Phi\colon E(n,d) \to S^{\mathsf{Z}}(n,d)$ is injective.*

Proof. If not, then there exist $\lambda,\mu \in \Lambda(n(e-1),d)$ and $0 \neq x \in f^\mu C_{\rho,d} f^\lambda$ such that $x f^\lambda C_{\rho,d}\gamma^\omega = 0$, whence $x C_{\rho,d}\gamma^\omega = 0$. But this is impossible by Corollary 8.11. □

Lemma 8.20. *We have $\Phi(E(n,d)) \supseteq S^{\mathsf{Z}}(n,d)^0$.*

Proof. Suppose that $\lambda,\mu \in \Lambda(n(e-1),d)$ and $h \in \mathrm{Hom}_{W_d}(M^\lambda, M^\mu)^0$. Then there exists $z \in W_d^0$ such that $h(m^\lambda) = m^\mu z$. Hence, $m^\mu z \mathtt{e}_\lambda = m^\mu z$ and $m^\mu z g = \varepsilon_\lambda(g) m^\mu z$ for all $g \in \mathfrak{S}_\lambda$. By Lemma 8.14, (8.12) and (7.21), it follows that the element

$$v := \theta_\lambda(m^\mu z) = u_\mu \Theta(z) \in q^{-a_\mu} f^\mu \hat{C}_{d\delta}\gamma^\omega$$

has degree zero and satisfies $v = v e_\lambda$ and $v\Theta(g) = \varepsilon_\lambda(g)v$ for all $g \in \mathfrak{S}_\lambda$. By Lemma 7.35, there exists $x \in f^\mu \hat{C}_{d\delta} f^\lambda$ such that $v = x u_\lambda$. Applying the surjection Π to this equality and writing $\bar{x} := \Pi(x) \in q^{a_\lambda - a_\mu} f^\mu C_{\rho,d} f^\lambda$, we have $\bar{u}_\mu \Xi(z) = \bar{x}\bar{u}_\lambda$, cf. (8.2). But $\bar{u}_\lambda = \eta_\lambda(m^\lambda)$ and $\bar{u}_\mu \Xi(z) = \eta_\mu(m^\mu z)$, so the map $\Phi(\bar{x})$ sends m^λ to $m^\mu z$. Thus, $\Phi(\bar{x}) = h$, so $h \in \Phi(E(n,d))$. The lemma follows. □

Recall the algebra homomorphisms $\mathtt{i}^\lambda\colon \mathsf{Z} \to S^{\mathsf{Z}}(n,d)$ from (3.15).

Lemma 8.21. *For any $\lambda \in \Lambda((n-1)(e-1), d-1)$, we have $\mathtt{i}^\lambda(\mathsf{Z}) \subseteq \Phi(E(n,d))$.*

Proof. Let $z \in \mathtt{e}_j \mathsf{Z} \mathtt{e}_k$ for some $k, j \in J$. Recall the embedding $\iota\colon \hat{C}_\delta \to \hat{C}_{d\delta}$ from (8.5). It follows from Theorem 7.12(i) and (8.6) that there exists $x \in e_j \hat{C}_\delta e_k$ such that $\Theta(z[1]) = \iota(x)$. Note that

$$u_{\hat{\lambda}^k} = e_k \otimes u_\lambda \in \hat{C}_\delta \otimes \hat{C}_{(d-1)\delta} = \hat{C}_{\delta,(d-1)\delta} \subseteq \hat{C}_{d\delta},$$

and $u_{\hat{\lambda}^j}$ is described similarly. Hence, $\iota(x) u_{\hat{\lambda}^k} = u_{\hat{\lambda}^j} \iota(x) = u_{\hat{\lambda}^j} \Theta(z[1])$. Writing $\bar{x} := \Pi(\iota(x)) f^{\hat{\lambda}^k} = f^{\hat{\lambda}^j} \Pi(\iota(x)) \in f^{\hat{\lambda}^j} C_{\rho,d} f^{\hat{\lambda}^k}$, we have

$$\bar{x} \bar{u}_{\hat{\lambda}^k} = \bar{u}_{\hat{\lambda}^j} \bar{x} = \bar{u}_{\hat{\lambda}^j} \Xi(z[1]),$$

whence

$$\begin{aligned} \Phi(\bar{x})(m^{\hat{\lambda}^k}) &= \eta_{\hat{\lambda}^j}^{-1}(\bar{x} \eta_{\hat{\lambda}^k}(m^{\hat{\lambda}^k})) = \eta_{\hat{\lambda}^j}^{-1}(\bar{x} \bar{u}_{\hat{\lambda}^k}) = \eta_{\hat{\lambda}^j}^{-1}(\bar{u}_{\hat{\lambda}^j} \Xi(z[1])) = m^{\hat{\lambda}^j} z[1] \\ &= \mathtt{i}^\lambda(z)(m^{\hat{\lambda}^k}). \end{aligned}$$

So $\Phi(\bar{x}) = \mathtt{i}^\lambda(z)$, and the lemma follows. □

Lemma 8.22. *For every field $\Bbbk$, the $\Bbbk$-algebra $E(n,d)_\Bbbk$ is symmetric.*

Proof. By Corollary 5.22, the algebra $C_{\rho,d,\Bbbk}$ is symmetric. It follows from (8.16) that $E(n,d)_\Bbbk \cong \mathrm{End}_{C_{\rho,d,\Bbbk}}(\Gamma(n,d)_\Bbbk)^{\mathrm{op}}$. Since $\Gamma(n,d)_\Bbbk$ is a projective $C_{\rho,d,\Bbbk}$-module, the lemma follows by [**SY**, Proposition IV.4.4]. □

Recall the subalgebra $T^{\mathsf{Z}}(n,d) \subseteq S^{\mathsf{Z}}(n,d)$ from §3.2.

Theorem 8.23. *Suppose that $n \geq d$. Then we have an isomorphism of graded algebras $\Phi\colon E(n,d) \xrightarrow{\sim} T^{\mathsf{Z}}(n,d)$.*

Proof. By Lemma 8.18 and Corollary 8.19, the map $\Phi\colon E(n,d) \to S^{\mathsf{Z}}(n,d)$ is an injective homomorphism of graded algebras, so $E(n,d) \cong \Phi(E(n,d))$. By Lemmas 8.20 and 8.21, we have $T^{\mathsf{Z}}(n,d) \subseteq \Phi(E(n,d))$. By Lemma 8.22, for every prime p, the algebra $\Phi(E(n,d)) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is symmetric. An application of Theorem 3.17 completes the proof. □

Corollary 8.24. *Let $n \geq d$. Then $E(n,d) \cong D_Q(n,d)$.*

Proof. This follows from Theorems 3.16 and 8.23. □

Example 8.25. Recall the idempotents $\xi_\lambda \in S^{\mathsf{Z}}(n,d)$ defined in §3.2 for any $\lambda \in \Lambda(n(e-1), d)$. It follows from the definitions that for all $\lambda, \mu \in \Lambda(n(e-1), d)$, the homomorphism Φ maps the component $q^{a_\lambda - a_\mu} f^\mu C_{\rho,d} f^\lambda$ of the decomposition (8.17) of $E(n,d)$ into the component $\xi_\mu S^{\mathsf{Z}}(n(e-1),d)\xi_\lambda = \mathrm{Hom}_{W_d}(M^\lambda, M^\mu)$ of $S^{\mathsf{Z}}(n,d)$. In this example, we consider the case when $e=2$, $d=2$, $n=2$ and $\lambda = (2,0)$, and we identify $\Phi(f^\lambda C_{\rho,d} f^\lambda)$ as an explicit subalgebra of $\mathrm{End}_{W_2}(M^\lambda)$.

Let $x_1, x_2 \in \mathrm{End}_{W_2}(M^\lambda)$ be the endomorphisms defined by the properties that $x_1(m^\lambda) = m^\lambda(\mathtt{c}[1] + \mathtt{c}[2])$ and $x_2(m^\lambda) = m^\lambda \mathtt{c}[1]\mathtt{c}[2]$. Then $\{1 := \xi_\lambda, x_1, x_2\}$ is a $\mathbb{Z}$-basis of the commutative algebra $\mathrm{End}_{W_2}(M^\lambda)$, and $x_1^2 = 2x_2$, $x_1 x_2 = 0$. Moreover, it is easy to see as in [**EK**, Example 4.28] that $\xi_\lambda T^{\mathsf{Z}}(n,d)\xi_\lambda$ is the $\mathbb{Z}$-span of $\{1, x_1, 2x_2\}$, so $\xi_\lambda T^{\mathsf{Z}}(n,d)\xi_\lambda$ is isomorphic to the truncated polynomial algebra $\mathbb{Z}[z]/(z^3)$, with x_1 corresponding to z. Thus, Theorem 8.23 asserts, in particular, that $\Phi(f^\lambda C_{\rho,d} f^\lambda) = \mathbb{Z}1 \oplus \mathbb{Z}x_1 \oplus 2\mathbb{Z}x_2$. This assertion can also be

verified by direct calculations using (1) the defining relations of the affine zigzag algebra $1_{0101}\hat{C}_{2\delta}1_{0101}$, see [$\mathbf{KM}_3$, Definition 4.4]; and (2) the fact that $y_1\gamma^\omega = y_1 1_{0101} = a(y_1 - y_2)1_{0101}$ in $C_{\rho,2}$ for some $a \in \mathbb{Z}$, see (8.8).

8.4. **Morita equivalences.** Let A and B be graded $\mathbb{Z}$-algebras. A *graded* functor A-mod $\to$ B-mod is a functor $\mathcal{F}$ equipped with an isomorphism between $q \circ \mathcal{F}$ and $\mathcal{F} \circ q$. A graded functor $\mathcal{F}$ is a *graded equivalence* if it is an equivalence of categories (in the usual sense). The graded algebras A and B are *graded Morita equivalent* if there is a graded equivalence between A-mod and B-mod. As noted for example in [**LVV**, §II.5.3] the graded analogue of Morita theory holds. In particular, A is graded Morita equivalent to B if and only if there exists a graded projective left A-module P which is a projective generator and such that $B \cong \mathrm{End}_A(P)^{\mathrm{op}}$.

For a graded algebra A, recall the notation $\ell(A)$ from §2.3.

Lemma 8.26. *Let A be a graded $\mathbb{Z}$-algebra which is finitely generated as a $\mathbb{Z}$-module, and let $\varepsilon \in A^0$ be an idempotent. Suppose that for every prime p we have $\ell(A_{\bar{\mathbb{F}}_p}) = \ell((\varepsilon \otimes 1)A_{\bar{\mathbb{F}}_p}(\varepsilon \otimes 1))$. Then the algebras A and $\varepsilon A \varepsilon$ are graded Morita equivalent.*

Proof. We write $\varepsilon := \varepsilon \otimes 1 \in A_{\bar{\mathbb{F}}_p}$ for each prime p. It suffices to show that the left A-module $A\varepsilon$ is a projective generator for A or, equivalently, that $A\varepsilon A = A$. Assume that $A\varepsilon A \neq A$. Then there exists a prime p such that $A_{\bar{\mathbb{F}}_p}\varepsilon A_{\bar{\mathbb{F}}_p} \neq A_{\bar{\mathbb{F}}_p}$. If L is a composition factor of $A_{\bar{\mathbb{F}}_p}/A_{\bar{\mathbb{F}}_p}\varepsilon A_{\bar{\mathbb{F}}_p}$, then $\varepsilon L = 0$, which contradicts the assumption that $\ell(A_{\bar{\mathbb{F}}_p}) = \ell(\varepsilon A_{\bar{\mathbb{F}}_p}\varepsilon)$, for example by [**G**, Theorem 6.2(g)]. □

Let $\lambda \in \Lambda(n(e-1), d)$. It follows from the definitions in §6.1 that $\boldsymbol{l}(\lambda, \boldsymbol{c}^0)$ is obtained from $\hat{\boldsymbol{l}}(\lambda, \boldsymbol{c}^0)$ by replacing each subword of the form i^m that is not preceded by or followed by i with $i^{(m)}$. Therefore, for any $\lambda, \mu \in \Lambda(n(e-1), d)$, we either have $f^\lambda = f^\mu$ or $f^\lambda f^\mu = f^\mu f^\lambda = 0$. We have an equivalence relation on $\Lambda(n(e-1), d)$, with λ being equivalent to μ if and only if $f^\lambda = f^\mu$. Let $\mathcal{X} \subseteq \Lambda(n(e-1), d)$ be a set of representatives of equivalence classes. Define

$$f := \sum_{\lambda \in \mathcal{X}} f^\lambda \in C_{\rho,d}.$$

Then $f^2 = f$ is a homogeneous idempotent.

Lemma 8.27. *The algebra $E(n,d)$ is graded Morita equivalent to $fC_{\rho,d}f$.*

Proof. Consider the left $fC_{\rho,d}f$-module

$$f\Gamma(n,d) = \bigoplus_{\lambda \in \Lambda(n(e-1),d)} q^{a_\lambda} fC_{\rho,d}f^\lambda.$$

There is a surjective $fC_{\rho,d}f$-module homomorphism $f\Gamma(n,d) \to fC_{\rho,d}f$ which is the identity on the summands $fC_{\rho,d}f^\lambda$ for $\lambda \in \mathcal{X}$ and zero on the other summands. Hence, $f\Gamma(n,d)$ is a projective generator for $fC_{\rho,d}f$. It is easy to see that $E(n,d) \cong \mathrm{End}_{fC_{\rho,d}f}(f\Gamma(n,d))^{\mathrm{op}}$, since for all $\lambda, \mu \in \Lambda(n(e-1), d)$ we have

$$\mathrm{Hom}_{fC_{\rho,d}f}(q^{a_\mu} fC_{\rho,d}f^\mu, q^{a_\lambda} fC_{\rho,d}f^\lambda) \cong q^{a_\lambda - a_\mu} f^\mu fC_{\rho,d}f^\lambda = q^{a_\lambda - a_\mu} f^\mu C_{\rho,d}f^\lambda.$$

The lemma follows by graded Morita theory. □

Write $\alpha = \mathrm{cont}(\rho) + d\delta$, so that $R^{\Lambda_0}_\alpha$ is the RoCK block of §5.4. For any $m, h \in \mathbb{Z}_{\geq 0}$, we denote by $S(m,h)$ the usual Schur algebra over $\mathbb{Z}$, as in [**G**].

Theorem 8.28. *Suppose that $n \geq d$. Then the $\mathbb{Z}$-algebras $R^{\Lambda_0}_\alpha$ and $D_Q(n,d)$ are graded Morita equivalent.*

Proof. By Remark 5.21, there is a homogeneous idempotent $\mathbf{e} \in R^{\Lambda_0}_\alpha$ such that $C_{\rho,d} \cong \mathbf{e} R^{\Lambda_0}_\alpha \mathbf{e}$. Hence, by Lemma 8.27, there exists a homogeneous idempotent $\varepsilon \in R^{\Lambda_0}_\alpha$ such that $E(n,d)$ is graded Morita equivalent to $\varepsilon R^{\Lambda_0}_\alpha \varepsilon$. By Corollary 8.24, we have $E(n,d) \cong D_Q(n,d)$, so $\varepsilon R^{\Lambda_0}_\alpha \varepsilon$ is graded Morita equivalent to $D_Q(n,d)$. So it suffices to show that $\varepsilon R^{\Lambda_0}_\alpha \varepsilon$ is graded Morita equivalent to $R^{\Lambda_0}_\alpha$.

Let p be a prime, and write $\varepsilon := \varepsilon \otimes 1 \in R^{\Lambda_0}_{\alpha,\bar{\mathbb{F}}_p}$. By the first paragraph, the algebras $\varepsilon R^{\Lambda_0}_{\alpha,\bar{\mathbb{F}}_p} \varepsilon$ and $D_Q(n,d)_{\bar{\mathbb{F}}_p}$ are graded Morita equivalent. In particular, $\ell(\varepsilon R^{\Lambda_0}_{\alpha,\bar{\mathbb{F}}_p} \varepsilon) = \ell(D_Q(n,d)_{\bar{\mathbb{F}}_p})$. By Lemma 8.26, it remains to show only that $\ell(R^{\Lambda_0}_{\alpha,\bar{\mathbb{F}}_p}) = \ell(D_Q(n,d)_{\bar{\mathbb{F}}_p})$.

Since the algebra $D_Q(n,d)_{\bar{\mathbb{F}}_p}$ is non-negatively graded, we have $\ell(D_Q(n,d)_{\bar{\mathbb{F}}_p}) = \ell(D_Q(n,d)^0_{\bar{\mathbb{F}}_p})$. By [**EK**, (7.2) and Lemma 7.3] together with Theorem 3.16,

$$D_Q(n,d)^0 \cong \bigoplus_{(d_1,\dots,d_{e-1}) \in \Lambda(e-1,d)} S(n,d_1) \otimes \dots \otimes S(n,d_{e-1}).$$

By [**G**, Theorem 3.5(a)], for all $h \leq n$ we have $\ell(S(n,h)_{\bar{\mathbb{F}}_p}) = |\mathscr{P}(h)|$. It follows that $\ell(D_Q(n,d)_{\bar{\mathbb{F}}_p}) = |\mathscr{P}^J(d)|$. On the other hand, by Theorem 5.7, we have $\ell(R^{\Lambda_0}_{\alpha,\bar{\mathbb{F}}_p}) = |\mathscr{P}^J(d)|$, and the proof is complete. □

Thus, we have proved Theorem A. In conclusion, we consider the case where we work over a field of sufficiently large characteristic, cf. the discussion in Section 1.

Proposition 8.29. *Suppose that $n \geq d$ and $\Bbbk$ is a field with $\mathrm{char}\,\Bbbk = 0$ or $\mathrm{char}\,\Bbbk > d$. Then the RoCK block $R^{\Lambda_0}_{\alpha,\Bbbk}$, the Turner double $D_Q(n,d)_\Bbbk$ and the wreath products $W_{d,\Bbbk}$ and $(R^{\Lambda_0}_{\delta,\Bbbk})^{\otimes d} \rtimes \Bbbk\mathfrak{S}_d$ are all graded Morita equivalent to each other.*

Proof. We write $x := x \otimes 1 \in A_\Bbbk$ for any algebra A and any $x \in A$. By (the proof of) Theorem 8.28, the algebras $R^{\Lambda_0}_{\alpha,\Bbbk}$, $C_{\rho,d,\Bbbk}$ and $D_Q(n,d)_\Bbbk$ are graded Morita equivalent. By Lemma 8.10, the module $C_{\rho,d,\Bbbk}\gamma^\omega$ is a projective generator for $C_{\rho,d,\Bbbk}$, so $C_{\rho,d,\Bbbk}$ is graded Morita equivalent to

$$\mathrm{End}_{C_{\rho,d,\Bbbk}}(C_{\rho,d,\Bbbk}\gamma^\omega) \cong \gamma^\omega C_{\rho,d,\Bbbk}\gamma^\omega \cong W_{d,\Bbbk}.$$

where the second isomorphism comes from Theorem 8.9. Recall the idempotent e_J from (7.9). By the $d = 1$ case of Lemma 8.10, the module $R^{\Lambda_0}_{\delta,\Bbbk} e_J$ is a projective generator for $R^{\Lambda_0}_{\delta,\Bbbk}$. Hence, setting $\xi := e_J^{\otimes d}$, we have that $((R^{\Lambda_0}_{\delta,\Bbbk})^{\otimes d} \rtimes \Bbbk\mathfrak{S}_d)\xi$ is a projective generator for $(R^{\Lambda_0}_{\delta,\Bbbk})^{\otimes d} \rtimes \Bbbk\mathfrak{S}_d$. So $(R^{\Lambda_0}_{\delta,\Bbbk})^{\otimes d} \rtimes \Bbbk\mathfrak{S}_d$ is graded Morita equivalent to

$$\xi((R^{\Lambda_0}_{\delta,\Bbbk})^{\otimes d} \rtimes \Bbbk\mathfrak{S}_d)\xi \cong (e_J R^{\Lambda_0}_{\delta,\Bbbk} e_J)^{\otimes d} \rtimes \Bbbk\mathfrak{S}_d \cong (\mathsf{Z}_\Bbbk)^{\otimes d} \rtimes k\mathfrak{S}_d \cong W_{d,\Bbbk},$$

where for the second isomorphism we use the fact that $e_J R^{\Lambda_0}_{\delta,\Bbbk} e_J \cong \mathsf{Z}_\Bbbk$, see [$\mathbf{KM}_3$, Theorem 5.24]. □

Index of notation

$1_{\boldsymbol{i}}$	standard idempotent if $\boldsymbol{i} \in I^\theta$	§4.2
$1_{\boldsymbol{i}}$	divided power idempotent if $\boldsymbol{i} \in I^\theta_{\mathrm{div}}$	§4.4
$\hat{C}_{d\delta}$	(imaginary) semicuspidal algebra	(4.26)
$\hat{C}_{\omega\delta}$	parabolic subalgebra of $C_{d\delta}$	§7.1
$C_{\rho,d}$	algebra Morita equivalent to RoCK block $R^{\Lambda_0}_{\mathrm{cont}(\rho),d}$	(5.17)
$\mathsf{CT}(\boldsymbol{\mu};\lambda,\boldsymbol{c})$	colored tableaux of shape $\boldsymbol{\mu}$ and type $(\lambda,\boldsymbol{c})$.	§6.2
δ	null-root	(4.1)
e	fixed element of $\mathbb{Z}_{\geq 2}$; quantum characteristic	
$\mathsf{e}_{\lambda,\boldsymbol{c}}$	indempotent in W_d; identity of the parabolic $W_{\lambda,\boldsymbol{c}} \subseteq W_d$	(3.8)
$e_{\lambda,\boldsymbol{c}}$	idempotent in $C_{d\delta}$	(7.17)
$e_{\boldsymbol{j}}$	idempotent in $R_{d\delta}$	(7.2)
$\varepsilon_{\lambda,\boldsymbol{c}}$	sign function on $\mathfrak{S}_\lambda$	(3.10)
$\mathbb{F}$	(arbitrary) ground field	
Φ	root system of type $A^{(1)}_{e-1}$	§4.1
$\gamma^{\lambda,\boldsymbol{c}}$	Gelfand-Graev idempotent	(6.3)
I	$\mathbb{Z}/e\mathbb{Z} = \{0,\dots,e-1\}$; vertices of Dynkin diagram $A^{(1)}_{e-1}$	§4.1
$I^\theta; I^\theta_{\mathrm{div}}$	words of weight θ; divided power words of weight θ	§4.4
$I^{d\delta}_{\mathrm{sc}}$	semicuspidal words	§4.5
$\mathtt{i}^\lambda$	non-unital algebra homomorphism $\mathsf{Z} \to S^{\mathsf{Z}}(n,d)$	(3.15)
J	$\{1,\dots,e-1\}$ identified with a subset of I	§3.1
κ	residue of Rouquier core ρ	§5.4
$L_{\delta,j}$	special cuspidal R_δ-modules	§4.5
$[\![\boldsymbol{\lambda}]\!]$	Young diagram of a (multi)partition $\boldsymbol{\lambda}$	§2.1
$\Lambda(n,d)$	compositions of d with n parts	§2.1
$\Lambda^{\mathrm{col}}(n,d)$	colored compositions	(3.7)
$\Lambda^S(n,d)$	S-multicompositions of d with n parts	§2.1
$\boldsymbol{l}^j$	fixed word appearing in the character of $L_{\delta,j}$	(6.1)
$\boldsymbol{l}(\lambda,\boldsymbol{c})$	special words used to define Gelfand-Graev idempotents	§6.1
$M_{\lambda,\boldsymbol{c}}$	colored permutation module over W_d	(3.11)
$\mathscr{P}; \mathscr{P}(n)$	partitions; partitions of n	§2.1
$\mathscr{P}_\rho; \mathscr{P}_{\rho,d}$	partitions with core ρ; partitions with core ρ and weight d	§5.1
$\mathscr{P}^S$	S-multipartitions	§2.1
$\mathscr{P}^S(d)$	S-multipartitions of d	§2.1
Q_+	non-negative part of the root lattice	§4.1
$R_\theta; R^{\Lambda_0}_\theta$	KLR algebra; cyclotomic KLR algebra	§4.2
$S^{\mathsf{Z}}(n,d)$	generalized Schur algebra	§3.2
$\mathfrak{S}_n$	symmetric group on n letters	
$\mathsf{Std}(\lambda \setminus \mu, \boldsymbol{i})$	$\boldsymbol{i}$-standard tableaux of shape $\lambda \setminus \mu$	§5.2
$T_{\lambda,\boldsymbol{c}}$	imaginary tensor space $T_{\lambda_1,c_1} \circ \cdots \circ T_{\lambda_n,c_n}$	§7.3
$T^{\mathsf{Z}}(n,d)$	algebra isomorphic to Turner double $D_Q(n,d)$	§3.2
$u_{\lambda,\boldsymbol{c}}$	special element of $\hat{C}_{d\delta}$	§7.4
W_d	wreath product algebra $\mathsf{Z}^{\otimes d} \rtimes \mathbb{Z}\mathfrak{S}_d$	(3.5)
ζ_j	signs corresponding to $j \in J$	(3.9)
Z	zigzag algebra with standard elements $\mathsf{a}^{j,k}, \mathsf{c}^{(j)}, \mathsf{e}_j$	§3.1
$Z_{\lambda,\boldsymbol{c}}$	sign invariants in $T_{\lambda,\boldsymbol{c}}$	§7.3

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School of Mathematics, University of Birmingham, Birmingham B15 2TT, UK

Department of Mathematics, University of Oregon, Eugene, OR 97403, USA
E-mail address: `klesh@uoregon.edu`