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## Positive operators on extended second order cones

Sándor Zoltán Németh · Jiaxin Xie ·  
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**Abstract** A positive operator on a cone is a linear operator that maps the cone to a subcone of itself. The extended second order cones were introduced by Németh and Zhang (J Glob Optim 62(3):443–457, 2015) as a working tool to solve mixed complementarity problems. Sznajder (J Global Optim 66(3): 585–593, 2016) determined the automorphism group and the Lyapunov (or bilinearity) ranks of these cones. Ferreira and Németh (J Global Optim 70(4): 707–718, 2018) reduced the problem of projecting onto the second order cone to a piecewise linear equation. Németh and Xiao (J Optim Theory Appl 176(2): 269–288, 2018) solved linear complementarity problems on the extended second order cone (motivated by portfolio optimization models) by reducing them to mixed complementarity problems with respect to the nonnegative orthant. As an extension of Sznajder’s results this paper aims to be a first work about finding necessary conditions and sufficient conditions for a linear operator to be a positive operator (which extends the notion of an automorphism) on an extended second order cone. Although, in the particular case of second order

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cones a necessary and sufficient condition is known, for extended second order cone such a condition is very difficult to find without restricting the structure of the linear operator. If the matrix of the linear operator is block-diagonal, we give such a necessary and sufficient condition.

**Keywords** Extended second order cone · Positive operator

**Mathematics Subject Classification (2000)** 65K05 · 90C25 · 90C46

## 1 Introduction

The second order cone (also called ice-cream or Lorentz cone) plays an important role in optimization problems. It has a variety of applications in financial management [15, 26], signal processing [12, 29] and machine learning [11, 20, 27]. We refer to the survey [1] and the references therein. In recent papers, the authors show the characteristic conditions the spherical convexity of quadratic functions defined on Lorentz spherically convex sets [10]. In [17] and [18], we generalized the  $n$ -dimensional second order cone to mutually dual  $(p + q)$ -dimensional cones. In these papers we used the extended second order cones as a tool to solve mixed complementarity problems defined on general cones and variational inequalities defined on cylinders. Sznajder [23] determined the automorphism group and the Lyapunov (or bilinearity) ranks of these cones. He also proved that the extended second order cones are irreducible. In [9], the authors gave explicit formulas for projecting onto the extended second order cones. In the most recent research about extended second order cones, Németh and Xiao [16] discussed the properties and solutions of linear complementarity problems on extended second order cones. They also formulated a class of mean-variance portfolio optimization problems via extended second order cones. It is expected that the theory of extended second order cones will be further developed and applied in various areas.

The study of positive operators (mappings) on a cone is an attractive topic in the theory of fixed points and Hilbert spaces. Birkhoff [6] defined the contraction ratio of a positive operator and proved its connection with the projective diameter. Later Bushell [7] found a simpler proof. One may refer to [28] for a simple review. Loewy and Schneider [14] discussed the necessary and sufficient conditions of positive operators on an  $n$ -dimensional second order cone. They showed that a matrix is a positive operator if and only if it satisfies a positive semidefinite condition (we will show the details of this result in the next section). Tam [24] showed properties of positive operators on polyhedral and simplicial cones and studied the structure and properties of the cone of positive operators [25]. A recent work [2] showed how the (semi)positivity influences the solutions of semidefinite linear complementarity problems.

Considering the positive operators on extended second order cones is a very natural and interesting idea, which is motivated by the theoretical and practical aspects described above. What properties should these positive operators satisfy? What is the difference between the positive operators on second

order cones and the positive operators on extended second order cones? Can the elegant necessary and sufficient conditions of second order cones in [14] be directly extended to extended second order cones? These questions are the key motivators of this article. As it is shown in previous papers, in general the extended second order cones do not have good properties of self-duality or polyhedrality, though they are a useful tool for solving complementarity problems and variational inequalities [17, 18].

This paper is the first work which aims to answer the above questions. Our investigations are also related to the famous  $\mathcal{S}$ -Lemma which is first illustrated in control theory by Yakubovič [31, 32]. Yuan [33] showed an equivalent form of this lemma and an elementary proof. The study of the  $\mathcal{S}$ -Lemma is still an essential topic in optimization theory [30]. Its development and applications are reviewed in [4, 19].

The paper is organized as follows. In Section 2, we list the definitions and terminology used throughout this paper. In this section we also present the necessary and sufficient conditions for a linear operator to be a positive operator on an  $n$ -dimensional second order cone (Theorem 2.3 [14]). In Section 3, we first shows some algebraic properties of the positive operators on extended second order cones. This section contains various necessary and sufficient conditions for these operators. Section 4 presents a necessary and sufficient condition for the block diagonal case. In Section 5 we show some numerical examples. In Section 6 we give some conclusions and outline the proposed future work.

## 2 Preliminaries

Let  $\ell, m, p, q$  be positive integers such that  $m = p+q$ . Define the direct product space  $\mathbb{R}^p \times \mathbb{R}^q$  as the set of all pairs of vectors  $(x, u)$ , where  $x \in \mathbb{R}^p$  and  $u \in \mathbb{R}^q$ . We identify the vectors of  $\mathbb{R}^m$  by  $m \times 1$  real matrices and define the canonical inner product on  $\mathbb{R}^m$  by the mapping

$$\mathbb{R}^m \times \mathbb{R}^m \ni (x, y) \mapsto \langle x, y \rangle := x^\top y \in \mathbb{R}$$

with the induced norm

$$\mathbb{R}^m \ni x \mapsto \|x\| = \sqrt{\langle x, x \rangle} \in \mathbb{R}.$$

The inner product in  $\mathbb{R}^p \times \mathbb{R}^q \equiv \mathbb{R}^{p+q}$  is given by

$$\langle (x, u), (y, v) \rangle := \langle x, y \rangle + \langle u, v \rangle.$$

A set  $\mathcal{K} \subseteq \mathbb{R}^\ell$  is called *cone* if  $\lambda\mathcal{K} \subseteq \mathcal{K}$  for any  $\lambda \geq 0$ . For convenience we identify a linear operator  $A$  of  $\mathbb{R}^\ell$  with its matrix. A cone  $\mathcal{K}$  is called a *closed convex cone* if it is a closed and convex set. A closed convex cone is called *proper cone* if it has nonempty interior and  $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$ . For any cone  $\mathcal{K} \subseteq \mathbb{R}^\ell$ , its *dual cone* is defined by

$$\mathcal{K}^* := \{y \in \mathbb{R}^\ell : \langle x, y \rangle \geq 0, \quad \forall x \in \mathcal{K}\}.$$

A cone  $\mathcal{K} \subseteq \mathbb{R}^\ell$  is called *subdual* if  $\mathcal{K} \subseteq \mathcal{K}^*$ , *superdual* if  $\mathcal{K}^* \subseteq \mathcal{K}$  and *self-dual* if  $\mathcal{K}^* = \mathcal{K}$ . If  $\mathcal{K}, \mathcal{D} \subseteq \mathbb{R}^\ell$  are closed convex cones such that  $\mathcal{D} = \mathcal{K}^*$ , then  $\mathcal{D}^* = \mathcal{K}$  and the cones  $\mathcal{K}, \mathcal{D}$  are called *mutually dual*. The set  $\Gamma(\mathcal{K})$  of *positive operators* on a cone  $\mathcal{K} \subseteq \mathbb{R}^\ell$  (see [14]) is defined by

$$\Gamma(\mathcal{K}) = \{B \in \mathbb{R}^{\ell \times \ell} : Bx \in \mathcal{K} \text{ for any } x \in \mathcal{K}\}. \quad (1)$$

The set of positive operators is a cone in  $\mathbb{R}^{\ell \times \ell}$  [22]. It can be easily checked that  $B$  is a positive operator on  $\mathcal{K}$  if and only if  $B^\top$  is a positive operator on  $\mathcal{K}^*$ .

The *complementarity set* of a closed convex cone  $\mathcal{K} \subseteq \mathbb{R}^\ell$  is defined by

$$\mathcal{C}(\mathcal{K}) = \{(x, y) : x \in \mathcal{K}, y \in \mathcal{K}^*, \langle x, y \rangle = 0\}.$$

A matrix  $B \in \mathbb{R}^{\ell \times \ell}$  is said to be *Lyapunov-like* on  $\mathcal{K}$  if  $\langle Bx, y \rangle = 0$  for all  $(x, y) \in \mathcal{C}(\mathcal{K})$  (see [21]). We call a matrix  $B$  an *automorphism* on the cone  $\mathcal{K}$  if it is invertible and  $B\mathcal{K} = \mathcal{K}$ , where  $B\mathcal{K} = \{Bx : x \in \mathcal{K}\}$ . The group of such matrices on  $\mathcal{K}$  is denoted by  $\text{Aut}(\mathcal{K})$ . A Lyapunov-like matrix  $B$  can also be characterized by the condition (see [8], [22])

$$e^{tB} \in \text{Aut}(\mathcal{K}) \text{ for all } t \in \mathbb{R}.$$

Note that any automorphism is a positive operator.

For  $z \in \mathbb{R}^\ell$ , we denote  $z = (z_1, \dots, z_\ell)^\top$ . Let  $\geq$  denote the *component-wise order* in  $\mathbb{R}^\ell$ , that is, the order defined by  $\mathbb{R}^\ell \ni x \geq y \in \mathbb{R}^\ell$  if and only if  $x_i \geq y_i$  for any  $i = 1, \dots, \ell$ . Depending on the context, denote by  $0$  either the zero vector in  $\mathbb{R}^\ell$  or the scalar zero; by  $e$  the vector of ones in  $\mathbb{R}^\ell$ , i.e.,  $e = (1, \dots, 1)^\top$ ; by  $e_i$  the  $i$ th vector of the standard canonical basis of  $\mathbb{R}^\ell$ ; and by  $\mathbb{R}_+^\ell = \{x \in \mathbb{R}^\ell : x \geq 0\}$  the nonnegative orthant. We call a matrix  $B \in \mathbb{R}^{\ell \times \ell}$  *positive semidefinite* if  $x^\top Bx \geq 0$  holds for any  $x \in \mathbb{R}^\ell$ . For any two matrices  $B$  and  $D$ , we write  $B \succeq D$  to represent  $B - D$  is positive semidefinite.

We recall from [17] the following definition of a pair of mutually dual *extended second order cones*  $\mathcal{L}(p, q)$  and  $\mathcal{M}(p, q)$ .

**Definition 1** The extended second order cones  $\mathcal{L}(p, q)$  and  $\mathcal{M}(p, q)$  are defined by

$$\mathcal{L}(p, q) := \{(x, u) \in \mathbb{R}^p \times \mathbb{R}^q : x \geq \|u\|e\} \quad (2)$$

and

$$\mathcal{M}(p, q) := \{(x, u) \in \mathbb{R}^p \times \mathbb{R}^q : \langle x, e \rangle \geq \|u\|, x \geq 0\}, \quad (3)$$

respectively. Here  $\geq$  denotes the component-wise order.

It can be easily checked that both  $\mathcal{L}(p, q)$  and  $\mathcal{M}(p, q)$  are proper cones. Actually, they are natural extensions of the second order cones, as both  $\mathcal{L}(p, q)$  and  $\mathcal{M}(p, q)$  will become second order cones exactly in the special case  $p = 1$ .  $\mathcal{M}(p, q)$  is superdual and  $\mathcal{L}(p, q) = \mathcal{M}(p, q)$  becomes selfdual exactly when  $p = 1$ . It is easy to show that  $\mathcal{L}(p, q)$  is polyhedral if and only if  $q = 1$ .

We now present the famous  $\mathcal{S}$ -Lemma, which will be used in the next section.

**Lemma 1 (S-Lemma, [4])** *Let  $B, D$  be two symmetric real  $n \times n$  matrices, and assume that the quadratic inequality*

$$x^\top Bx \geq 0 \quad (4)$$

*is strictly feasible, i.e., there exists  $u \in \mathbb{R}^n$  such that  $u^\top Bu > 0$ . Then the quadratic inequality*

$$x^\top Dx \geq 0$$

*is a consequence of (4) if and only if there exists a nonnegative  $\lambda$  such that*

$$D \succeq \lambda B.$$

### 3 Conditions for positive operators on extended second order cones

In this section, we will show some algebraic properties of the positive operators on  $\mathcal{L}(p, q)$  and  $\mathcal{M}(p, q)$ . In addition, we also present a kind of extension of [22, Theorem 2.3], which shows that one can use a positive semidefinite condition to characterize a positive operator on a second order cone.

The following theorem states necessary conditions for a linear operator to be a positive operator on the extended second order cone  $\mathcal{M}(p, q) = (\mathcal{L}(p, q))^*$ , where  $\mathcal{L}(p, q)$  is defined by (2) and  $\mathcal{M}(p, q)$  is defined by (3).

**Theorem 1 (Necessary conditions for positive operators on  $\mathcal{M}(p, q)$ )**  
*Let  $p \geq 1$  and  $q \geq 1$  be two integers. If  $A \in \mathbb{R}^{(p+q) \times (p+q)}$  is a positive operator on  $\mathcal{M}(p, q)$ , then the following hold:*

- (i) *The transpose of the first  $p$  rows of  $A$  are in  $\mathcal{L}(p, q)$ .*
- (ii) *The first  $p$  columns of  $A$  are in  $\mathcal{M}(p, q)$ .*
- (iii) *By adding any  $i$ -th column  $i = 1, \dots, p$  to the linear combination of the columns  $p+1, \dots, p+q$  with coefficients  $u_1, \dots, u_q$  such that the Euclidean norm of  $u = (u_1, \dots, u_q)^\top$  is one, we obtain an element in  $\mathcal{M}(p, q)$ .*
- (iv) *The sum of any  $i$ -th column  $i = 1, \dots, p$  with any  $(p+j)$ -th column  $j = 1, \dots, q$  is in  $\mathcal{M}(p, q)$ .*

*Proof* (i) Since  $A$  is a positive operator on  $\mathcal{M}(p, q)$ , the first  $p$  entries of  $Az$  are nonnegative for any  $z \in \mathcal{M}(p, q)$ . Hence, the inner product of  $z$  and the transpose of any row vector of the first  $p$  rows of  $A$  is nonnegative. Therefore, the transpose of these row vectors must be in the dual cone of  $\mathcal{M}(p, q)$ , i.e.,  $\mathcal{L}(p, q)$ .

(ii) Since  $A^\top$  is a positive operator on  $\mathcal{L}(p, q) = (\mathcal{M}(p, q))^*$ , (ii) follows similarly to (i).

(iii) To state conveniently, let the columns of  $A$  be the vectors  $a_1, a_2, \dots, a_{p+q} \in \mathbb{R}^{p+q}$ . For  $i = 1, \dots, p$ , we set

$$b_i := a_i + \sum_{j=1}^q u_j a_{p+j}.$$

It suffices to prove that for any  $z \in \mathcal{L}(p, q)$  and  $i \in \{1, \dots, p\}$ ,

$$\langle z, b_i \rangle = \langle z, a_i \rangle + \sum_{j=1}^q u_j \langle z, a_{p+j} \rangle \geq 0. \quad (5)$$

Note that

$$\sum_{j=1}^q u_j \langle z, a_{p+j} \rangle \geq -\sqrt{\sum_{j=1}^q \langle z, a_{p+j} \rangle^2} \sqrt{\sum_{j=1}^q u_j^2} = -\sqrt{\sum_{j=1}^q \langle z, a_{p+j} \rangle^2}$$

where the first inequality follows by the Cauchy-Schwarz inequality and the equality follows from  $\|u\|_2 = 1$ . So for any  $z \in \mathcal{L}(p, q)$ , we have

$$\langle z, b_i \rangle \geq \langle z, a_i \rangle - \sqrt{\sum_{j=1}^q \langle z, a_{p+j} \rangle^2}.$$

Recall that  $A$  is a positive operator on  $\mathcal{M}(p, q)$ , which implies that  $A^\top$  is a positive operator on  $\mathcal{L}(p, q)$ . Thus  $A^\top z \in \mathcal{L}(p, q)$ . Observe that  $\langle z, a_k \rangle$  is the  $k$ -th entry of  $A^\top z$ , and by the definition of  $\mathcal{L}(p, q)$  we know that

$$\langle z, a_i \rangle \geq \sqrt{\sum_{j=1}^q \langle z, a_{p+j} \rangle^2}, \quad i = 1, \dots, p.$$

Thus we arrive at (5), which implies that  $b_i \in \mathcal{L}^*(p, q) = \mathcal{M}(p, q)$ .

(iv) Obviously, it is a special case of the above assertion.  $\square$

Next, we will present the sufficient conditions for positive operators on  $\mathcal{M}(p, q)$ . Let us begin with the following lemma, which tells us that  $\mathcal{M}(p, q)$  can be formulated by using a quadratic form with respect to the matrix  $J_{p,q}$  defined below.

**Lemma 2** *Let*

$$J_{p,q} = \begin{bmatrix} ee^\top & 0 \\ 0 & -I_q \end{bmatrix}, \quad (6)$$

where  $e \in \mathbb{R}^p$  is the vector with all entries 1 and  $I_q$  denotes the identity mapping in  $\mathbb{R}^q$ . Then

$$\mathcal{M}(p, q) = \{z = (x, u) \in \mathbb{R}^p \times \mathbb{R}^q : z^\top J_{p,q} z \geq 0, x \in \mathbb{R}_+^p\}. \quad (7)$$

*Proof* Set

$$\Omega := \{z = (x, u) \in \mathbb{R}^p \times \mathbb{R}^q : z^\top J_{p,q} z \geq 0, x \in \mathbb{R}_+^p\}.$$

We have  $z = (x, u) \in \Omega$  if and only if  $z^\top J_{p,q} z \geq 0$  and  $x \geq 0$ , or equivalently,

$$z^\top J_{p,q} z = x^\top ee^\top x - u^\top u = \langle x, e \rangle^2 - \|u\|^2 \geq 0$$

and  $x \geq 0$ . Recall that  $z = (x, u) \in \mathcal{M}(p, q)$  if and only if  $\langle x, e \rangle \geq \|u\|$  and  $x \geq 0$ . Hence  $\Omega = \mathcal{M}(p, q)$ .  $\square$

We have the following result show the sufficient conditions for positive operators on  $\mathcal{M}(p, q)$ .

**Theorem 2 (Sufficient conditions for positive operators on  $\mathcal{M}(p, q)$ )**  
 Suppose that  $A \in \mathbb{R}^{(p+q) \times (p+q)}$ , where  $p \geq 1$  and  $q \geq 1$  are two integers. Then the following hold:

- (i) If  $A$  is  $\mathcal{M}(p, q)$ -Lyapunov like, then for any  $t \in \mathbb{R}$ ,  $e^{tA} \in \text{Aut}(\mathcal{M}(p, q))$ .
- (ii) Let  $J_{p,q}$  be defined by (6). If there exists a  $\lambda \geq 0$  such that  $A^\top J_{p,q} A - \lambda J_{p,q}$  is positive semidefinite and the first  $p$  rows of  $A$  (more precisely, their transposes) are in  $\mathcal{L}(p, q)$ , then  $A$  is a positive operator on  $\mathcal{M}(p, q)$ .

*Proof* (i) See [8], [22].

(ii) To this end, we need prove that for any  $z \in \mathcal{M}(p, q)$ ,  $Az \in \mathcal{M}(p, q)$ . Note that  $A^\top J_{p,q} A - \lambda J_{p,q}$  is positive semidefinite, we obtain that

$$z^\top (A^\top J_{p,q} A - \lambda J_{p,q}) z = (Az)^\top J_{p,q} (Az) - \lambda z^\top J_{p,q} z \geq 0.$$

Let  $z \in \mathcal{M}(p, q)$ . From Lemma 2, we know that  $z^\top J_{p,q} z \geq 0$ . As  $\lambda \geq 0$ , we have  $(Az)^\top J_{p,q} (Az) \geq 0$ . Recall that the first  $p$  rows of  $A$  are in  $\mathcal{L}(p, q)$ , then the first  $p$  entries of  $Az$  are nonnegative for any  $z \in \mathcal{M}(p, q)$ . Hence by Lemma 2, we get  $Az \in \mathcal{M}(p, q)$  for any  $z \in \mathcal{M}(p, q)$ . We conclude that  $A$  is a positive operator on  $\mathcal{M}(p, q)$ .  $\square$

The rest of this section aims to show that one can use a *positive semidefinite condition* to characterize a positive operator on the extended second order cone. This work is inspired by the results in [22], where they show that a positive operator on a second order cone can be characterized by a positive semidefinite condition. Recall that  $\Gamma(\cdot)$  given in (1) is the set of all the positive operators on a certain cone. Specifically, they prove the following result.

**Lemma 3 ( [22], Theorem 2.3)** Let  $B \in \mathbb{R}^{(1+q) \times (1+q)}$ . If  $B \in \Gamma(\mathcal{L}(1, q)) \cup \Gamma(-\mathcal{L}(1, q))$ , then there exists a  $\mu \geq 0$  such that

$$B^\top J_{1,q} B \succeq \mu J_{1,q}. \quad (8)$$

Conversely, if  $\text{rank}(B) \neq 1$  and (8) holds for some  $\mu \geq 0$ , then

$$B \in \Gamma(\mathcal{L}(1, q)) \cup \Gamma(-\mathcal{L}(1, q)).$$

Before inducing the main result, we first prove the following.

**Lemma 4** For  $i = 1, \dots, p$ , let

$$H_i = \begin{bmatrix} \text{diag}(e_i) & 0 \\ 0 & -I_q \end{bmatrix}, \quad (9)$$

where  $e_i \in \mathbb{R}^p$  is the  $i$ th vector of the standard basis of  $\mathbb{R}^p$ , and  $I_q$  denotes the identity mapping in  $\mathbb{R}^q$ . Then

$$\mathcal{L}(p, q) = \{z \in \mathbb{R}^p \times \mathbb{R}^q : z^\top H_i z \geq 0, z_i \geq 0, i = 1, \dots, p\}. \quad (10)$$

*Proof* Note that the definition of  $\mathcal{L}(p, q)$  means the first  $p$  entries of the vectors in  $\mathcal{L}(p, q)$  are nonnegative. So we can conclude the above directly.  $\square$

For  $i = 1, \dots, p$ , we also introduce the following cones

$$\mathcal{N}_i(p, q) = \{(x, u) \in \mathbb{R}^p \times \mathbb{R}^q : |x_i| \geq \|u\|\}, \quad (11)$$

and

$$\mathcal{N}(p, q) = \{(x, u) \in \mathbb{R}^p \times \mathbb{R}^q : |x_i| \geq \|u\|, i = 1, \dots, p\}. \quad (12)$$

It is obvious that  $\cap_{i=1}^p \Gamma(\mathcal{N}_i) \subseteq \Gamma(\cap_{i=1}^p \mathcal{N}_i) = \Gamma(\mathcal{N})$ . Our theorem is based on this simple observation.

**Theorem 3** *Suppose that  $A \in \mathbb{R}^{(p+q) \times (p+q)}$ . Then  $A \in \cap_{i=1}^p \Gamma(\mathcal{N}_i)$  if and only if there exist  $p$  nonnegative numbers  $\lambda_1, \dots, \lambda_p$  such that for any  $(\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p$ ,*

$$A^\top \left( \sum_{i=1}^p \alpha_i H_i \right) A \succeq \sum_{i=1}^p \alpha_i \lambda_i H_i. \quad (13)$$

*Moreover, if the first  $p$  rows of  $A$  are in  $\mathcal{M}(p, q)$ , then  $A$  is a positive operator on  $\mathcal{L}(p, q)$ .*

*Proof* If  $z \in \mathcal{N}_i(p, q)$ , then  $z^\top H_i z \geq 0$ . If  $A \in \Gamma(\mathcal{N}_i(p, q))$ , then we have

$$(Az)^\top H_i (Az) = z^\top A^\top H_i A z \geq 0, \quad i = 1, \dots, p.$$

We can easily find  $v \in \mathcal{N}_i$  such that  $v^\top H_i v > 0$  for  $i = 1, \dots, p$ . Hence, by the  $\mathcal{S}$ -Lemma, there exists a  $\lambda_i \geq 0$  such that

$$A^\top H_i A \succeq \lambda_i H_i. \quad (14)$$

Then, just multiply both sides of (14) by  $\alpha_i$  and sum up with respect to  $i$ , we get (13).

Conversely, given the inequality (13), let  $\alpha_i = 1$  when  $j = i$  and  $\alpha_i = 0$  when  $j \neq i$ , we will get (14). Then, by the converse of  $\mathcal{S}$ -lemma, we get  $A \in \Gamma(\mathcal{N}_i)$  for any  $i$ . Moreover, if the first  $p$  rows of  $A$  are in the dual cone of  $\mathcal{L}(p, q)$ , the first  $p$  entries of  $Az$  are positive for any  $z \in \mathcal{L}(p, q)$ . Together with the definition of  $\mathcal{N}(p, q)$ , we have  $A$  is a positive operator on  $\mathcal{L}(p, q)$ .  $\square$

*Remark 1* If  $p = 1$ , then  $\mathcal{L}(1, q)$  is a second order cone and  $\cap_{i=1}^p \Gamma(\mathcal{N}_i) = \Gamma(\cap_{i=1}^p \mathcal{N}_i) = \Gamma(\mathcal{N})$ . In this case Theorem 3 reduces to Lemma 3.

#### 4 A necessary and sufficient condition for block diagonal cases

In this section we consider the special case when  $A$  is a block-diagonal matrix. We can apply our previous result in [18, Proposition 6.1].

**Theorem 4** *Let  $A \in \mathbb{R}^{(p+q) \times (p+q)}$  be given by*

$$A = \begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix},$$

where  $S \in \mathbb{R}^{p \times p}$ ,  $T \in \mathbb{R}^{q \times q}$  and  $\|T\| = \alpha$  for some  $\alpha \geq 0$ . Then,

- (i)  $A$  is a positive operator on  $\mathcal{L}(p, q)$  if and only if each entry of  $S$  is nonnegative and the sum of each row of  $S$  is at least  $\alpha$ .
- (ii) There exist  $\lambda \geq 0$  such that  $AJ_{p,q}A^\top - \lambda J_{p,q}$  is positive semidefinite if and only if the sum of each row of  $S$  are the same and at least  $\alpha$ .
- (iii) If there exist  $\lambda \geq 0$  such that  $AJ_{p,q}A^\top - \lambda J_{p,q}$  is positive semidefinite and each entry of  $S$  is nonnegative, then  $A$  is a positive operator on  $\mathcal{L}(p, q)$ .

*Proof* (i) “ $\Leftarrow$ ”: Suppose  $(x, u) \in \mathcal{L}(p, q)$ . Thus,  $x_j \geq \|u\|$  where  $x_j$  is the  $j$ th entry of  $x$ . Let  $(y, v) = A(x, u) = (Sx, Tu) \in \mathbb{R}^p \times \mathbb{R}^q$ . It follows that

$$y_i = \sum_{j=1}^p s_{ij}x_j \geq \|u\| \sum_{j=1}^p s_{ij} \geq \|T\|\|u\| \geq \|Tu\| = \|v\|,$$

for any  $i$ th entry of  $y$ . Then  $A(x, u) \in \mathcal{L}(p, q)$  and therefore  $A$  is a positive operator on  $\mathcal{L}(p, q)$ .

“ $\Rightarrow$ ”: Suppose  $A$  is a positive operator on  $\mathcal{L}(p, q)$ . By Theorem 1, it is obvious that each entry of  $S$  is nonnegative. Suppose that the sum of the  $k$ th row of  $S$  is strictly less than  $\alpha$ . Let  $u_0$  be the vector with  $\|Tu_0\| = \alpha\|u_0\|$  (such an  $u_0$  always exists). Then  $(\|u_0\|e, u_0) \in \mathcal{L}(p, q)$ . If the sum of the  $j$ th row of  $S$  is  $s_j < \alpha$ , then  $A(\|u_0\|e, u_0) = (\|u_0\|s_j e, Tu_0)$  and the  $j$ th entry of this vector will be  $s_j\|u_0\| < \alpha\|u_0\| = \|Tu_0\|$ . That means  $A(\|u_0\|e, u_0) \notin \mathcal{L}(p, q)$ , which is a contradiction.

(ii) “ $\Leftarrow$ ”: By the assumptions, if the sum of each row of  $S$  is  $s$ , we get that  $AJ_{p,q}A^\top - \lambda J_{p,q}$  is

$$\begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} ee^\top & 0 \\ 0 & -I_q \end{bmatrix} \begin{bmatrix} S^\top & 0 \\ 0 & T^\top \end{bmatrix} - \lambda \begin{bmatrix} ee^\top & 0 \\ 0 & -I_q \end{bmatrix} = \begin{bmatrix} (s^2 - \lambda)E & 0 \\ 0 & \lambda I_q - TT^\top \end{bmatrix}.$$

Since  $E$  is positive semidefinite, there exists a  $\lambda \in (\alpha^2, s^2)$  such that  $AJ_{p,q}A^\top - \lambda J_{p,q}$  is positive semidefinite.

“ $\Rightarrow$ ”: Suppose  $s_i \neq s_j$  where  $s_i$  and  $s_j$  are the sums of the  $i$ th and  $j$ th row, respectively. The lower right block of  $AJ_{p,q}A^\top - \lambda J_{p,q}$  will be  $\lambda I_q - TT^\top$ . So if



By using Theorem 4 (i), it is easy to verify that  $A_2$  is a positive operator on  $\mathcal{L}(2, 2)$ . However,

$$A_2^\top J_{2,2} A_2 - \lambda J_{2,2} = \begin{bmatrix} 0.3025 - \lambda & 0.4675 - \lambda & 0 & 0 \\ 0.4675 - \lambda & 0.7225 - \lambda & 0 & 0 \\ 0 & 0 & \lambda - 0.13 & -0.12 \\ 0 & 0 & -0.12 & \lambda - 0.13 \end{bmatrix}.$$

If  $A_2^\top J_{2,2} A_2 - \lambda J_{2,2}$  is positive semidefinite, then  $0.25 \leq \lambda \leq 0.3025$  and  $\det(S_2^\top e e^\top S_2 - \lambda e e^\top) \geq 0$ . So,

$$(0.3025 - \lambda)(0.7225 - \lambda) - (0.4675 - \lambda)(0.4675 - \lambda) \geq 0.$$

Then  $\lambda \leq 0$ , which is a contradiction. That means there is no  $\lambda$  such that  $A_2^\top J_{2,2} A_2 - \lambda J_{2,2}$  is positive semidefinite. Hence the conditions of Theorem 2 are not satisfied which shows that the theorem describes only a sufficient condition for positivity.

The following example is to show the necessity and sufficiency of Theorem 4.

**Example 2** Let  $A_1 \in \mathbb{R}^{4 \times 4}$  provided by

$$A_1 = \begin{bmatrix} S_1 & 0 \\ 0 & T_1 \end{bmatrix} = \begin{bmatrix} 0.25 & 0.45 & 0 & 0 \\ 0.45 & 0.25 & 0 & 0 \\ 0 & 0 & 0.2 & 0.3 \\ 0 & 0 & 0.3 & 0.2 \end{bmatrix}.$$

Here  $S_1$  is a  $2 \times 2$  matrix with the sums of each row at least 0.55 and the sum of each column equal to 0.7. Because the eigenvalues of  $T_1$  are 0.5 and  $-0.1$ , so  $\|T_1\| \leq 0.5 < 0.55$ . Therefore, by the result of Theorem 4 (i), we have  $A_1 \mathcal{L}(2, 2) \subseteq \mathcal{L}(2, 2)$ . Since  $A_1 = A_1^\top$  and  $A_1^\top \in \Gamma(\mathcal{M}(2, 2))$ ,  $A_1$  is a positive operator on  $\mathcal{M}(2, 2)$ .

On the other hand, note that

$$A_1^\top J_{2,2} A_1 - \lambda J_{2,2} = \begin{bmatrix} 0.49 - \lambda & 0.49 - \lambda & 0 & 0 \\ 0.49 - \lambda & 0.49 - \lambda & 0 & 0 \\ 0 & 0 & \lambda - 0.13 & -0.12 \\ 0 & 0 & -0.12 & \lambda - 0.13 \end{bmatrix},$$

we have  $A_1^\top J_{2,2} A_1 - \lambda J_{2,2}$  is positive semidefinite for every  $\lambda \in [0.25, 0.49]$ . Since every entry of  $S$  is positive and the first 2 lines of  $A_1$  are in  $\mathcal{L}(2, 2)$ , it follows from Theorem 2 that  $A_1$  is a positive operator on  $\mathcal{M}(2, 2)$ .

Finally, let us propose two examples in support of Theorem 3.

**Example 3** Let

$$A_3 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then, we can choose  $(\alpha_1, \lambda_1) = (5, 6)$  and  $(\alpha_2, \lambda_2) = (5, 3)$  such that

$$A_3^\top \left( \sum_{i=1}^2 \alpha_i H_i \right) A_3 - \sum_{i=1}^2 \alpha_i \lambda_i H_i$$

is positive definite. It is easy to see that  $A_3$  is a positive operator on  $\mathcal{L}(2, 2)$ .

**Example 4** This example shows a matrix  $B$  that is a positive operator on  $\mathcal{L}(2, 2)$  while  $B \notin \cap_{i=1}^2 \Gamma(\mathcal{N}_i)$ . Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

It is trivial to show that  $B \in \Gamma(\mathcal{L}(2, 2))$ . Let  $z = (2, -2, 1, -1)^\top \in \mathcal{N}_2$ ,  $Bz = (0, 0, 2, 2)^\top \notin \mathcal{N}_2$ .

## 6 Conclusions and future works

In this note, we showed some conditions for the positivity of operators on extended second order cones. Theorem 2.3 in [14] showed a necessary and sufficient condition for a linear operator to be a positive operator on a second order cone. But when  $p > 1$ , the study of positive operators on an extended second order cone  $\mathcal{L}(p, q)$  (or  $\mathcal{M}(p, q)$ ) becomes much more complicated. In Theorem 4, we showed a necessary and sufficient condition for a block diagonal linear operator  $A$  to be a positive operator on  $\mathcal{L}(p, q)$ . In general, if

$$A = \begin{bmatrix} S & R \\ W & T \end{bmatrix},$$

where  $R \in \mathbb{R}^{p \times q}$  and  $W \in \mathbb{R}^{q \times p}$ , then  $A(x^\top, u^\top)^\top = (Sx + Ru, Wx + Tu)^\top$ . Hence, each entry is determined by both  $x$  and  $u$ . It is difficult to ensure the first  $p$  entries of  $A(x^\top, u^\top)^\top$ , that is,  $Sx + Ru$ , to be all positive or all negative simultaneously. Similar difficulties occur in Theorem 3. Though the  $\mathcal{S}$ -lemma provides a necessary and sufficient condition, the matrix  $B$  in Example 4 shows that  $\cap_{i=1}^p \Gamma(\mathcal{N}_i) \neq \Gamma(\mathcal{N})$ . This is because the extended second order cones are irreducible [23].

Since the extended second order cones are not self-dual when  $p > 1$  and are not polyhedral when  $q > 1$  (see [17]), it seems hard to find a unified feature of such a positive operator  $A$ . To improve our results, we may consider some other direction to investigate the necessary conditions and sufficient conditions for a linear operator to be a positive operator. The  $\mathcal{S}$ -Lemma (Yuan's Lemma) was extended to be defined in a first order cone (direct sum of a subspace and a ray) rather than the whole space in the following way (see [3, Corollary 3.2], [13, Lemma 2.1]):

**Lemma 5** *Let  $A, B \in \mathbb{R}^{n \times n}$  be symmetric matrices and  $\mathcal{K} \subseteq \mathbb{R}^n$  be a first-order cone. Then, the following conditions are equivalent:*

1.  $\max\{x^\top Ax, x^\top Bx\} \geq 0, x \in \mathcal{K}$ .
2. *There exists  $t_1 \geq 0, t_2 \geq 0$  with  $t_1 + t_2 = 1$  such that  $t_1A + t_2B$  is positive semidefinite on  $\mathcal{K}$ .*

It is a natural but challenging question to find a version of the  $\mathcal{S}$ -lemma defined on second order cones or extended second order cones which could be used to characterise the positive operators on the latter cones. This direction seems very interesting and challenging and if successful it would lead to many applications.

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