

Equicontinuity, transitivity and sensitivity

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EQUICONTINUITY, TRANSITIVITY AND SENSITIVITY: THE AUSLANDER-YORKE DICHOTOMY REVISITED

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ABSTRACT. We study sensitivity, topological equicontinuity and even continuity in dynamical systems. In doing so we provide a classification of topologically transitive dynamical systems in terms of equicontinuity pairs, give a generalisation of the Auslander-Yorke dichotomy for minimal systems and show there exists a transitive system with an even continuity pair but no equicontinuity point. We define what it means for a system to be eventually sensitive; we give a dichotomy for transitive dynamical systems in relation to eventual sensitivity. Along the way we define a property called splitting and discuss its relation to some existing notions of chaos. The approach we take is topological rather than metric.

1. Introduction. Recently there has been a move towards studying dynamical systems without assuming the underlying phase space is necessarily metric or compact (see, for example, [3, 7, 14, 15, 16, 25, 27, 32, 33]). This paper is part of that project, extending the ideas developed by the first author and Macías [14] and addressing a question raised by the referee of that paper for this journal asking about the Auslander-Yorke dichotomy from a topological view point.

Let (X, f) be a discrete dynamical system, so that $f: X \rightarrow X$ is a (continuous) map on the metric space X . The dynamical system is *equicontinuous at a point* $x \in X$ if for any $\epsilon > 0$ there is a $\delta > 0$ such that the δ -ball around x does not expand to more than diameter ϵ under iteration of f . The system itself is said to be *equicontinuous* if it is equicontinuous at every point. Compactness of the space X ensures that equicontinuity is equivalent to uniform equicontinuity: for

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any $\epsilon > 0$ there is a $\delta > 0$ such that no δ -ball expands to more than diameter ϵ under iteration of f . Equicontinuity is extremely important in mathematical analysis where it provides the primary condition in the Arzelà-Ascoli theorem (see [12, Theorem 8.2.10]). A related concept to equicontinuity is that of sensitivity. The system (X, f) is *sensitive* if every nonempty open set expands to at least diameter δ under iteration of f . It is obvious that the properties of sensitivity and equicontinuity are mutually exclusive. Examining the quantifiers one sees that sensitivity is *almost* a negation of equicontinuity. Indeed, negating the property of equicontinuity at a given point gives a localised version of sensitivity. Auslander and Yorke [4] specify a type of system for which sensitivity is precisely the negation of equicontinuity: a dynamical system (X, f) is said to be *minimal* if the forward orbit of every point is dense in the space. The Auslander-Yorke dichotomy states that a compact metric minimal system is either equicontinuous or sensitive. Various analogues of this theorem have since been offered [17].

Topological transitivity, or simply transitivity, is a weakening of minimality. The system (X, f) is said *transitive* if for any nonempty open sets U and V there is an $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. Under certain conditions (compact metric being sufficient) this is equivalent to the existence of a *transitive point* (i.e. a point with a dense orbit) [2]. Transitivity and sensitivity are often cited as two key ingredients for a system to be chaotic (see, for example [4, 11]). The former prevents the system from being decomposed into multiple invariant open sets (and thereby studied as a collection of subsystems). The latter brings an element of unpredictability to the system; a small error in initial conditions may be exacerbated over time. This is clearly of particular importance in an applied setting where there is almost always going to be an error in one's measurements and computations. In his definition of chaos, along with these two properties, Robert Devaney [11, p. 50] included the condition that the set of periodic points be dense in whole space, thus providing "an element of regularity" in the midst of seemingly random behaviour. Perhaps surprisingly this regularity condition together with transitivity proved sufficient in a compact space to entail sensitivity [5, 13]. Since then, the gap between transitivity and sensitivity has been researched extensively (see, for example, [1, 14, 20, 26]); Akin *et al* [1] gave the following dichotomy: a compact metric transitive system is either sensitive or contains a point of equicontinuity; in 2007, Moothathu [26] generalised results in [5] and [1] by defining stronger notions of sensitivity. These variations on sensitivity have since attracted an array of interest [29, 17, 23, 30].

For a survey on recent developments in the theory of chaotic systems, including results on sensitivity, equicontinuity and transitivity, see [21].

In [14] the authors introduce what they term *Hausdorff sensitivity* of a system $f: X \rightarrow X$, where f is a continuous map on a topological space, showing that this coincides with the usual notion of sensitivity if the phase space is compact metric. Topological equicontinuity was introduced by Royden in [28], which, in general, is weaker than equicontinuity. The concept of even continuity, introduced by Kelley [19, p. 234], dates back further than topological equicontinuity and is even weaker still, although all three concepts (i.e. equicontinuity, topological equicontinuity and even continuity) coincide in the presence of compactness (see [19, Theorem 7.23]). In contrast to equicontinuity, which is an inherently uniform concept, neither topological equicontinuity nor even continuity require the phase space to be anything more than a topological space. Whilst the concepts of topological equicontinuity

and even continuity have gained some attention with regard to topological semi-groups and families of mappings in a general setting (e.g. [8, 9]), little appears to have been done with regard to dynamical systems.

In this paper we take a careful look at the Auslander-Yorke dichotomy via a topological approach which leads to some interesting results: After the preliminaries in Section 2, we build up some theory related to topological equicontinuity in dynamical systems in Section 3. Two fruits of this theory are Corollary 7 - a generalisation of the Auslander-Yorke dichotomy - along with an exposition, with regard to topological equicontinuity, of when a system is transitive (Theorem 3.6). Section 4 starts by building up theory regarding even continuity in dynamical systems. This section culminates in a construction of a compact topologically transitive system with an even continuity pair but no point of even continuity; this provides an element of regularity in a system which is Auslander-Yorke chaotic, densely and strongly Li-Yorke chaotic, but not Devaney chaotic. In Section 5 we discuss a property we call *splitting* and its relationship to topological equicontinuity, even continuity and existing notions of chaos. Finally, in Section 7 we give a dichotomy for compact Hausdorff transitive systems (Theorem 7.2); they are either equicontinuous or *eventually sensitive*.

Throughout this paper X is a topological space. Usually it is assumed to be Hausdorff, while some results rely on the additional assumption of compactness. We will always state the relevant assumptions.

We denote by \mathbb{Z} the set of all integers; the set of positive integers $1, 2, 3, 4, \dots$ is denoted by \mathbb{N} whilst $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

2. Preliminaries.

2.1. Uniform spaces. We start by providing some background on uniformities for those who are unfamiliar. The definitions in this section can be found in [31]. Let X be a set. The diagonal of the Cartesian product $X \times X$ is the set $\Delta = \{(x, x) \mid x \in X\}$. Given two subsets A and B of $X \times X$, we define the composition of these sets as $A \circ B = \{(x, z) \mid \exists y \in X : (x, y) \in B, (y, z) \in A\}$. We write nA to denote $\underbrace{A \circ A \circ \dots \circ A}_{n \text{ times}}$. We define the inverse $A^{-1} = \{(x, y) \mid (y, x) \in A\}$. If $A \subseteq X \times X$ contains the diagonal Δ we say it is an *entourage of the diagonal*.

Definition 2.1. A *uniformity* \mathcal{D} on a set X is a collection of entourages of the diagonal such that the following conditions are satisfied.

- a. $D_1, D_2 \in \mathcal{D} \implies D_1 \cap D_2 \in \mathcal{D}$.
- b. $D \in \mathcal{D}, D \subseteq E \implies E \in \mathcal{D}$.
- c. $D \in \mathcal{D} \implies E \circ E \subseteq D$ for some $E \in \mathcal{D}$.
- d. $D \in \mathcal{D} \implies E^{-1} \subseteq D$ for some $E \in \mathcal{D}$.

We call the pair (X, \mathcal{D}) a *uniform space*. We say \mathcal{D} is *separating* if $\bigcap_{D \in \mathcal{D}} D = \Delta$; in this case we say X is *separated*. A subcollection \mathcal{E} of \mathcal{D} is said to be a *base* for \mathcal{D} if for any $D \in \mathcal{D}$ there exists $E \in \mathcal{E}$ such that $E \subseteq D$. Clearly any base \mathcal{E} for a uniformity will have the following properties:

1. $D_1, D_2 \in \mathcal{D} \implies$ there exists $E \in \mathcal{E}$ such that $E \subseteq D_1 \cap D_2$.
2. $D \in \mathcal{D} \implies E \circ E \subseteq D$ for some $E \in \mathcal{E}$.
3. $D \in \mathcal{D} \implies E^{-1} \subseteq D$ for some $E \in \mathcal{E}$.

If \mathcal{D} is separating then \mathcal{E} will satisfy $\bigcap_{E \in \mathcal{E}} E = \Delta$. A *subbase* for \mathcal{D} is a subcollection such that the collection of all finite intersections from said subcollection form a base. We say an entourage of the diagonal D is *symmetric* if $D = D^{-1}$.

For an entourage $D \in \mathcal{D}$ and a point $x \in X$ we define the set $D[x] = \{y \in X \mid (x, y) \in D\}$. This naturally extends to a subset $A \subseteq X$; $D[A] = \bigcup_{x \in A} D[x]$. We emphasise that (see [31, Section 35.6]):

- For all $x \in X$, the collection $\mathcal{U}_x := \{D[x] \mid D \in \mathcal{D}\}$ is a neighbourhood base at x , making X a topological space. The same topology is produced if any base \mathcal{E} of \mathcal{D} is used in place of \mathcal{D} .
- The topology is Hausdorff if and only if \mathcal{D} is separating.

A topological space is said to be *Tychonoff*, or $T_{3\frac{1}{2}}$, if it is both Hausdorff and *completely regular* (i.e. points and closed sets can be separated by a bounded continuous real-valued function). A topological space is Tychonoff precisely when it admits a separating uniformity. Finally we remark that for a compact Hausdorff space X there is a unique uniformity \mathcal{D} which induces the topology (see [12, Section 8.3.13]).

2.2. Dynamical systems. For those wanting a thorough introduction to topological dynamics, [10] is an excellent resource. Most of the definitions in this section are standard and can be found there.

A *dynamical system* is a pair (X, f) consisting of a topological space X and a continuous function $f: X \rightarrow X$. For any $x \in X$ we denote the set of neighbourhoods of x by \mathcal{N}_x ; the elements of this set are not assumed to be open. We say the orbit of x under f is the set of points $\{x, f(x), f^2(x), \dots\}$; we denote this set by $\text{Orb}_f(x)$. We say x is *periodic* if there exists $n \in \mathbb{N}$ such that $f^n(x) = x$; the least such n is called the *period* of x ; if $n = 1$ we say x is a *fixed point*. A point $x \in X$ is *eventually periodic* if there exists $y \in \text{Orb}_f(x)$ such that y is periodic. It immediately follows that $\text{Orb}_f(x)$ is finite if and only if x is eventually periodic. For $x \in X$, we define the *ω -limit set* of x under f , denoted $\omega_f(x)$, or simply $\omega(x)$ where there is no ambiguity, to be the set of limit points of the sequence $(f^n(x))_{n \in \mathbb{N}}$. Formally

$$\omega_f(x) = \bigcap_{N \in \mathbb{N}} \overline{\{f^n(x) \mid n > N\}}.$$

This means that $y \in \omega_f(x)$ if and only if for every neighbourhood U of y and every $N \in \mathbb{N}$ there exists $n > N$ such that $f^n(x) \in U$. If X is compact $\omega_f(x) \neq \emptyset$ for any $x \in X$ by Cantor's intersection theorem. Notice that $\overline{\text{Orb}_f(x)} = \text{Orb}_f(x) \cup \omega_f(x)$. A point x is said to be *recurrent* if $x \in \omega(x)$. It is said to be *non-wandering* if, for any neighbourhood $U \in \mathcal{N}_x$ and any $N \in \mathbb{N}$ there is $n > N$ such that $f^n(U) \cap U \neq \emptyset$. Clearly a recurrent point is non-wandering. We define the *non-wandering set* of x , denoted $\Omega_f(x)$, by saying that $y \in \Omega_f(x)$ if and only if for any $V \in \mathcal{N}_y$, any $U \in \mathcal{N}_x$ and any $N \in \mathbb{N}$ there exists $n > N$ such that $f^n(U) \cap V \neq \emptyset$. It follows that, for any $x \in X$, $\omega(x) \subseteq \Omega(x)$.

When X is a compact Hausdorff space we will denote the unique uniformity associated with X by \mathcal{D}_X or usually simply \mathcal{D} if there is no ambiguity. Given $A, B \subseteq X$, we denote by $N(A, B)$ the (forward) hitting times of A on B under f ; specifically

$$N(A, B) = \{n \in \mathbb{N} \mid f^n(A) \cap B \neq \emptyset\}. \quad (1)$$

If $x \in X$ and $B \subseteq X$, we will abuse notation by writing $N(x, B)$ instead of $N(\{x\}, B)$. A dynamical system (X, f) is *topologically transitive*, or simply *transitive*, when, for any pair of nonempty open sets U and V , $N(U, V) \neq \emptyset$. It is *weakly mixing* if the product system $(X \times X, f \times f)$ is transitive. A point $x \in X$ is said to be a *transitive point* if $\omega(x) = X$. (NB. Some literature defines a transitive point as a point whose orbit is dense in the space. This is, in general, a weaker condition. However these two versions coincide for a point x if either $x \in f(X)$ or x is not isolated in X (see, for example, [10, Corollary 1.3.3].) A system (X, f) is said to be minimal if $\omega(x) = X$ for all $x \in X$; equivalently, if there are no proper, nonempty, closed, positively-invariant subsets of X . (A subset $A \subseteq X$ is said to be *positively invariant* (under f) if $f(A) \subseteq A$.)

In [2] the authors introduce the concept of a *density basis*; a density basis for a topological space X is a collection \mathcal{V} of nonempty open sets in X such that if $A \subseteq X$ is such that $A \cap V \neq \emptyset$ for any $V \in \mathcal{V}$, then $\bar{A} = X$. They go on to show that if X is of Baire second category (i.e. nonmeagre) and has a countable density basis then topological transitivity is equivalent to the existence of a transitive point. Topologists may be more familiar with the concept of a π -base than a density basis.

Definition 2.2. A π -base for a topological space X is a collection \mathcal{U} of nonempty open sets in X such that if R is any nonempty open set in X then there exists $V \in \mathcal{U}$ such that $V \subseteq R$.

Proposition 1. Let X be a topological space. A collection is a π -base if and only if it is a density basis.

Proof. Note first that both are defined as collections of nonempty open sets.

Suppose \mathcal{U} is a π -base. Suppose $A \subseteq X$ is such that $A \cap U \neq \emptyset$ for all $U \in \mathcal{U}$. Let W be open and nonempty. Then there exists $U \in \mathcal{U}$ such that $U \subseteq W$. Then $A \cap U \neq \emptyset$; therefore $A \cap W \neq \emptyset$ and so $\bar{A} = X$.

Now suppose \mathcal{U} is a density basis. Assume \mathcal{U} is not a π -base. Then there exists a nonempty open set W such that $U \not\subseteq W$ for any $U \in \mathcal{U}$. This means that $U \setminus W \neq \emptyset$ for any $U \in \mathcal{U}$. Take

$$A = \bigcup_{U \in \mathcal{U}} U \setminus W.$$

It follows that $A \cap W = \emptyset$ and, for each $U \in \mathcal{U}$, $A \cap U \neq \emptyset$. Since \mathcal{U} is a density basis the latter entails $\bar{A} = X$, contradicting the fact that $A \cap W = \emptyset$. Hence \mathcal{U} is a π -base. \square

The following lemma is folklore (e.g. [2]) and will be useful throughout.

Lemma 2.3. Let (X, f) be a dynamical system, where X is a Hausdorff space. Then (X, f) is topologically transitive if and only if $N(U, V)$ is infinite for any pair of nonempty open sets U and V .

Remark 1. It follows from Lemma 2.3 that, for a transitive system (X, f) where X is a Hausdorff space, we have $\Omega(x) = X$ for any $x \in X$.

For the rest of this section (X, \mathcal{D}) is a separated uniform space.

We say the system (X, f) is *uniformly rigid* if for any $E \in \mathcal{D}$ there exists $n \in \mathbb{N}$ such that, for any $x \in X$, $(x, f^n(x)) \in E$. Clearly every point in a uniformly rigid system is recurrent.

Let $U \subseteq X$ and let $D \in \mathcal{D}$ be symmetric. Define

$$N_D(U) = \{n \in \mathbb{N} \mid \exists x, y \in U \text{ such that } (f^n(x), f^n(y)) \notin D\}. \quad (2)$$

We say the system (X, f) is *sensitive* if there exists a symmetric entourage $D \in \mathcal{D}$ such that $N_D(U) \neq \emptyset$ for any nonempty open $U \subseteq X$. In this case we say D is a sensitivity entourage (X, f) . If X is a metric space, for $U \subseteq X$ and $\delta > 0$ we define

$$N_\delta(U) = \{n \in \mathbb{N} \mid \exists x, y \in U \text{ such that } d(f^n(x), f^n(y)) \geq \delta\}. \quad (3)$$

In this case we say the system is sensitive if there exists $\delta > 0$ such that $N_\delta(U) \neq \emptyset$ for any nonempty open set U . The definitions for a metric space coincide when it is equipped with the metric uniformity (see [14]). We invite readers unfamiliar with uniformities to notice the similarities in these definitions; it may be helpful for such readers to view the statement, “there exists $D \in \mathcal{D}$ such that $(x, y) \in D$,” as “there exists $\delta > 0$ such that $d(x, y) < \delta$ ”. Similarly “ $(x, y) \notin D$ ” may be read as “ $d(x, y) \geq \delta$ ”. In this way, $D[x]$ may be thought of as $B_\delta(x)$. The uniform structure of a space can be used to mimic existing metric proofs (see, for example, [14]). In the proof of the following lemma, which is folklore, we invite the reader to observe how entourages have simply replaced the real numbers which would have designated distances for a metric version.

Lemma 2.4. *If (X, \mathcal{D}) is a separated uniform space and (X, f) is a sensitive dynamical system, with sensitivity $D \in \mathcal{D}$, then for any nonempty open $U \subseteq X$ the set $N_D(U)$ is infinite.*

Proof. Let $U \subseteq X$ be nonempty open and suppose $N_D(U)$ is finite; let $k \in \mathbb{N}$ be an upper bound for this set. Let $E \in \mathcal{D}$ be such that $2E \subseteq D$. Let $x \in U$. By continuity we may choose a symmetric entourage $D_0 \in \mathcal{D}$ such that, for any $y \in X$, if $(x, y) \in D_0$ then $(f^i(x), f^i(y)) \in E$ for all $i \in \{1, \dots, k\}$. Consider the set $W := U \cap D_0[x]$; W is a neighbourhood of x . Thus $N_D(W) \neq \emptyset$ by sensitivity, but $f^i(W) \subseteq E[f^i(x)]$ for $i \in \{1, \dots, k\}$; in particular if $y, z \in W$ then $(f^i(y), f^i(z)) \in D$ for $i \in \{1, \dots, k\}$. Therefore there exists $n > k$ and $y, z \in W$ such that $(f^n(y), f^n(z)) \notin D$. As $W \subseteq U$ we have a contradiction and the result follows. \square

A point $x \in X$ is said to be an *equicontinuity point* of the system (X, f) if

$$\forall E \in \mathcal{D} \exists D \in \mathcal{D} : \forall n \in \mathbb{N}, y \in D[x] \implies f^n(y) \in E[f^n(x)]. \quad (4)$$

In this case we say (X, f) is *equicontinuous at x* . If (X, f) is equicontinuous at every $x \in X$ then we say the system itself is *equicontinuous*. When X is compact this is equivalent to the system being *uniformly equicontinuous*, that is

$$\forall E \in \mathcal{D} \exists D \in \mathcal{D} : \forall n \in \mathbb{N}, (x, y) \in D \implies (f^n(x), f^n(y)) \in E. \quad (5)$$

We denote the set of all equicontinuity points by $\text{Eq}(X, f)$, so a system is equicontinuous if $\text{Eq}(X, f) = X$.

The following results will be useful; versions for compact metric systems may be found in [1], their proofs may be mimicked to give the following more general versions.

Lemma 2.5. [1] *Let (X, f) be a dynamical system, where X is a compact Hausdorff space. If $x \in \text{Eq}(X, f)$ then $\omega_f(x) = \Omega_f(x)$.*

Theorem 2.6. [1] *Let (X, f) be a transitive dynamical system where X is a compact Hausdorff space. If $\text{Eq}(X, f) \neq \emptyset$ then the set of equicontinuity points coincide with the set of transitive points.*

Corollary 1. *Let (X, f) be a transitive dynamical system where X is a compact Hausdorff space. If X is not separable then $\text{Eq}(X, f) = \emptyset$.*

Proof. If $\text{Eq}(X, f) \neq \emptyset$ then every equicontinuity point is a transitive point. If the system has a transitive point then it has a countable dense subset and is thereby separable. \square

We end this section of the preliminaries with three common notions of chaos. A dynamical system is said to be Auslander-Yorke chaotic (see [4]) if it is both transitive and sensitive. If, in addition, it has a dense set of periodic points it is said to be Devaney chaotic (see [11]).

If X is a metric space and (X, f) a dynamical system, then we say a pair $(x, y) \in X \times X$ is *proximal* if

$$\liminf_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0,$$

and *asymptotic* if

$$\limsup_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0.$$

The pair (x, y) is said to be a *Li-Yorke pair* if they are proximal but not asymptotic. It is said to be a *strong Li-Yorke pair* if it is both a Li-Yorke pair and recurrent in the product system $(X^2, f \times f)$. A set $S \subseteq X$ is said to be *scrambled* if every pair of distinct points in S form a Li-Yorke pair; it is said to be *strongly scrambled* if every pair of distinct points in S form a strong Li-Yorke pair. A system (X, f) is said to be *Li-Yorke chaotic* (see [24]) if there exists an uncountable scrambled set S . If S is strongly scrambled we say (X, f) is *strongly Li-Yorke chaotic*. Finally if S is dense in X then we say the system is *densely Li-Yorke chaotic* [10, Section 7.3].

2.3. Shift spaces. Given a finite set Σ considered with the discrete topology, the *full one sided shift with alphabet* Σ consists of the set of infinite sequences in Σ , that is $\Sigma^{\mathbb{N}_0}$, which we consider with the product topology. This forms a dynamical system with the *shift map* σ , given by

$$\sigma(\mathbf{x})_i = \mathbf{x}_{i+1}.$$

A *shift space* is some compact positively-invariant (under σ) subset of some full shift. Let X be a shift space, with alphabet Σ . Given a finite word, $\mathbf{a}_0\mathbf{a}_1 \dots \mathbf{a}_m$, made up of elements of Σ , we denote by $[\mathbf{a}_0\mathbf{a}_1 \dots \mathbf{a}_m]$ the *cylinder set* induced by the word $\mathbf{a}_0\mathbf{a}_1 \dots \mathbf{a}_m$; this is all points in X which begin with ' $\mathbf{a}_0\mathbf{a}_1 \dots \mathbf{a}_m$ '. The collection of all cylinder sets intersected with X form a base for the induced subspace topology from the Tychonoff product $\Sigma^{\mathbb{N}_0}$. For a symbol $\mathbf{a} \in \Sigma$, we use the notation \mathbf{a}^n , for some $n \in \mathbb{N}$, to mean

$$\underbrace{\mathbf{a}\mathbf{a}\mathbf{a} \dots \mathbf{a}}_{n \text{ times.}}$$

For a word W , we use $|W|$ to denote the length of W . So if $W = \mathbf{w}_0\mathbf{w}_1\mathbf{w}_2 \dots \mathbf{w}_n$, then $|W| = n + 1$. For the word W , we refer to the set $\{\mathbf{w}_k\mathbf{w}_{k+1} \dots \mathbf{w}_{k+j} \mid 0 \leq k \leq n, 0 \leq j \leq n - k\}$ as the *set of all subwords of* W ; the elements of this set are called subwords of W . We refer to any subword of the form $\mathbf{w}_0\mathbf{w}_1 \dots \mathbf{w}_k$, for some $k \leq n$, as an *initial segment* of W . In similar fashion, if $\mathbf{x} = \langle \mathbf{x}_i \rangle_{i \geq 0} \in \Sigma^{\mathbb{N}_0}$ and $n \in \mathbb{N}_0$, we refer to $\mathbf{x}_0\mathbf{x}_1 \dots \mathbf{x}_n$ as an *initial segment* of \mathbf{x} .

For those wanting more information about shift systems, [10, Chapter 5] provides a thorough introduction to the topic.

2.4. A note on taking a topological approach to dynamical systems. Before we proceed a final remark is in order about taking a topological approach to dynamical systems. When seeking to define appropriate topological versions of metric definitions one cannot always ensure they coincide in an arbitrary metric setting. To take an elementary example, consider the dynamical system $f: (0, \infty) \rightarrow (0, \infty): x \mapsto 2x$. Equipped with the Euclidean metric, this system is sensitive. However, a topologically equivalent metric is the following $d_0(x, y) := |1/x - 1/y|$. Equipped with this metric the system is equicontinuous. Since sensitivity and equicontinuity are mutually exclusive in a metric setting, we cannot expect topological variants of these notions to coincide with the metric versions in this instance. As we shall see, however, the presence of compactness is often enough for topological definitions to be equivalent to their metric/uniform cousins when the space is metrizable/uniform.

3. Topological equicontinuity and the Auslander-Yorke dichotomy. Previously we defined equicontinuity for dynamical systems where the phase space is Tychonoff. More generally [31], if X is any topological space and Y a uniform space, we say that a family \mathcal{F} of continuous functions from X to Y is *equicontinuous at $x \in X$* if for each $E \in \mathcal{D}_Y$ there exists $U \in \mathcal{N}_x$ such that, for each $f \in \mathcal{F}$, $f(U) \subseteq E[f(x)]$. We say \mathcal{F} is *equicontinuous* provided it is equicontinuous at each point of X . To generalise this to arbitrary spaces, Royden [28] presents the following concept of *topological equicontinuity*. If X and Y are topological spaces we say a collection of maps \mathcal{F} from X to Y is *topologically equicontinuous* at an ordered pair $(x, y) \in X \times Y$ if for any $O \in \mathcal{N}_y$ there exist neighbourhoods $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that, for any $f \in \mathcal{F}$, if $f(U) \cap V \neq \emptyset$ then $f(U) \subseteq O$; when this is the case we refer to (x, y) as an *equicontinuity pair*. We say \mathcal{F} is *topologically equicontinuous* at a point $x \in X$ if it is topologically equicontinuous at (x, y) for all $y \in Y$. We say the collection is *topologically equicontinuous* if it is topologically equicontinuous at every $x \in X$. If (x, y) is an equicontinuity pair then we will say y is an *equicontinuity partner* of x .

Topological equicontinuity and the usual notion of equicontinuity coincide when Y is a compact Hausdorff space.

Theorem 3.1. [28, p. 364] *Let X and Y be topological spaces, with \mathcal{F} a collection of continuous functions from X to Y . Let $x \in X$. If Y is a Tychonoff space and \mathcal{F} is equicontinuous at x with respect to some compatible uniformity then \mathcal{F} is topologically equicontinuous at x . If Y is a compact Hausdorff space then the collection \mathcal{F} is equicontinuous at $x \in X$ if and only if it is topologically equicontinuous at x .*

If (X, f) is a dynamical system, we will denote the set of equicontinuity pairs by $\text{EqP}(X, f)$. Note that in this case, if we consider the above definitions, we have $Y = X$ and $\mathcal{F} = \{f^n \mid n \in \mathbb{N}\}$. By definition it follows that (X, f) is topologically equicontinuous precisely when $\text{EqP}(X, f) = X \times X$. For $(x, y) \in \text{EqP}(X, f)$, we refer to the condition

$$\forall O \in \mathcal{N}_y \exists U \in \mathcal{N}_x \exists V \in \mathcal{N}_y : \forall n \in \mathbb{N}, f^n(U) \cap V \neq \emptyset \implies f^n(U) \subseteq O, \quad (6)$$

as the *topological equicontinuity condition* for x and y . We say that U and V , as in Equation 6, satisfy the topological equicontinuity condition for x, y and O .

The following simple observation relies solely on continuity and will be useful throughout what follows.

Lemma 3.2. *Let (X, f) be a dynamical system, where X is a Hausdorff space. Let $x, y \in X$ and $n \in \mathbb{N}$. Pick $O \in \mathcal{N}_y$ and let*

$$S = \{k \in \{1, \dots, n\} \mid f^k(x) = y\}.$$

There exist neighbourhoods $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that $N(U, V) \cap \{1, \dots, n\} = S$ and $f^k(U) \subseteq V \subseteq O$ for all $k \in S$.

Proof. Let $S = \{k \in \{1, \dots, n\} \mid f^k(x) = y\}$ (this set may be empty). For all $i \in \{1, \dots, n\} \setminus S$, let $U_i \in \mathcal{N}_{f^i(x)}$ and $V_i \in \mathcal{N}_y$ be such that $U_i \cap V_i = \emptyset$. Define

$$V := \left(\bigcap_{i \in \{1, \dots, n\} \setminus S} V_i \right) \cap O.$$

Then $V \in \mathcal{N}_y$. Now take

$$U := \left(\bigcap_{i \in \{1, \dots, n\} \setminus S} f^{-i}(U_i) \right) \cap \left(\bigcap_{i \in S} f^{-i}(V) \right).$$

Notice $U \in \mathcal{N}_x$.

By construction, $N(U, V) \cap \{1, \dots, n\} = S$ and $f^k(U) \subseteq V \subseteq O$ for all $k \in S$. \square

In particular Lemma 3.2 shows that any pair $(x, y) \in X \times X$, satisfy the following weakened version of the topological equicontinuity condition (Equation 6).

Corollary 2. *Let (X, f) be a dynamical system, where X is a Hausdorff space. Let $x, y \in X$ and $n \in \mathbb{N}$. Then for any $O \in \mathcal{N}_y$ there exist $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that, for any $k \in \{1, \dots, n\}$,*

$$f^k(U) \cap V \neq \emptyset \implies f^k(U) \subseteq O.$$

Proof. Immediate from Lemma 3.2. \square

If (X, f) is a Hausdorff dynamical system and the points $x, y \in X$ are such that there exist neighbourhoods $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that, for all $n \in \mathbb{N}$, $f^n(U) \cap V = \emptyset$ then $(x, y) \in \text{EqP}(X, f)$; this is vacuously true. The following result adds to this.

Proposition 2. *Let (X, f) be a dynamical system where X is Hausdorff space. Let $x, y \in X$ and suppose that $y \notin \Omega(x)$. Then $(x, y) \in \text{EqP}(X, f)$.*

Proof. Let $O \in \mathcal{N}_y$. Take $U \in \mathcal{N}_x$, $V \in \mathcal{N}_y$ and $N \in \mathbb{N}$ such that $f^n(U) \cap V = \emptyset$ for all $n > N$. By Corollary 2, there exist U' and V' such that, for any $k \in \{1, \dots, N\}$, if $f^k(U') \cap V' \neq \emptyset$ then $f^k(U') \subseteq O$; without loss of generality $U' \subseteq U$ and $V' \subseteq V \cap O$. Then, since $f^n(U') \cap V' = \emptyset$ for all $n > N$, U' and V' satisfy the topological equicontinuity condition for x, y and O . As $O \in \mathcal{N}_y$ was picked arbitrarily the result follows. \square

With this in mind we make the following definition.

Definition 3.3. If (X, f) is a dynamical system, where X is a Hausdorff space. We say $(x, y) \in X \times X$ is a *trivial equicontinuity pair* if $y \notin \Omega(x)$.

Remark 2. Proposition 2 tells us that a trivial equicontinuity pair is indeed an equicontinuity pair.

Generally, in a non-compact Tychonoff space, topologically equicontinuity, whilst clearly necessary for equicontinuity (Theorem 3.1), is not sufficient; it is a strictly weaker property than equicontinuity. Example 1 shows this. First, recall that a metric system (X, f) is said to be *expansive* if there exists $\delta > 0$ such that for any x and y , with $x \neq y$, there exists $k \in \mathbb{N}_0$ such that $d(f^k(x), f^k(y)) \geq \delta$. It is easy to see that if X is perfect (i.e. without isolated points) then expansivity implies sensitivity.

Example 1. Consider the dynamical system $(X = \mathbb{R} \setminus \{0\}, f)$, where $f(x) = 2x$. Using Proposition 2 it can be verified that $\text{EqP}(X, f) = X \times X$, hence the system is topologically equicontinuous. However this system is not only sensitive but it is also expansive. Each of these properties (the latter, since X is perfect) are mutually exclusive with the existence of an equicontinuity point, thus $x \notin \text{Eq}(X, f)$ for any $x \in X$.

Lemma 3.4. *Let (X, f) be a dynamical system, where X is a Hausdorff space. If $(x, y) \in \text{EqP}(X, f)$ then either (x, y) is a trivial equicontinuity pair or $y \in \omega(x)$.*

Proof. Suppose (x, y) is a non-trivial equicontinuity pair (otherwise we are done). Now suppose $y \notin \omega(x)$; then there exists $O \in \mathcal{N}_y$ and $N \in \mathbb{N}$ such that for all $n > N$ we have $f^n(x) \notin O$. Since (x, y) is a nontrivial pair, for any neighbourhoods U and V of x and y respectively, the set $N(U, V)$ is infinite. Pick $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ and let $n > N$ be such that $f^n(U) \cap V \neq \emptyset$. Then, as $f^n(x) \notin O$, $f^n(U) \not\subseteq O$. As U and V were arbitrary neighbourhoods this contradicts the fact that $(x, y) \in \text{EqP}(X, f)$. \square

This means that a pair (x, y) is a non-trivial equicontinuity pair if and only if it is an equicontinuity pair and $y \in \omega(x)$.

The statement $(x, y) \notin \text{EqP}(X, f)$, for $x, y \in X$, means precisely

$$\exists O \in \mathcal{N}_y : \forall U \in \mathcal{N}_x \forall V \in \mathcal{N}_y \exists n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset \text{ and } f^n(U) \not\subseteq O. \quad (7)$$

In particular, for any pair of neighbourhoods $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$, we have that U meets V after some number of iterations of f . If $N(U, V)$ were finite, for some such pair, then $(x, y) \in \text{EqP}(X, f)$ by Proposition 2 (it would be a trivial equicontinuity pair), thus $N(U, V)$ is infinite. By definition this means that $y \in \Omega(x)$. (NB. We shall refer to a neighbourhood such as O in Equation (7) as a *splitting neighbourhood* of y with regard to x .) This leads us to the following generalisation of Lemma 2.5.

Lemma 3.5. *Let (X, f) be a dynamical system where X is a Hausdorff space. If (X, f) is topologically equicontinuous at $x \in X$ then $\omega(x) = \Omega(x)$.*

Proof. Pick $y \in X$ arbitrarily; note that $(x, y) \in \text{EqP}(X, f)$ by hypothesis. Since $\omega(x) \subseteq \Omega(x)$ it suffices to consider the case when $y \in \Omega(x)$. In this case we have (x, y) is a non-trivial equicontinuity pair. Hence $y \in \omega(x)$ by Lemma 3.4. \square

We are now in a position to characterise transitive dynamical systems on Hausdorff spaces purely with reference to equicontinuity pairs.

Theorem 3.6. *Let X be a Hausdorff space, and let $f: X \rightarrow X$ be a continuous function. Then (X, f) is a transitive dynamical system if and only if there are no trivial equicontinuity pairs.*

Proof. Suppose first that (X, f) is transitive. Let $(x, y) \in X \times X$ be given and let $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$. By transitivity, $N(U, V)$ is infinite (see Lemma 2.3).

Since U and V were arbitrary neighbourhoods it follows that (x, y) is not a trivial equicontinuity pair.

Now suppose (X, f) has no trivial equicontinuity pairs and let U and V be nonempty open sets. Pick $x \in U$ and $y \in V$; (x, y) is not a trivial equicontinuity pair. If $(x, y) \in \text{EqP}(X, f)$ then, by Lemma 3.4, $y \in \omega(x)$ from which it follows that $N(U, V) \neq \emptyset$. If $(x, y) \notin \text{EqP}(X, f)$ then by Equation (7) there exists $n \in N(U, V)$. In every case, $N(U, V) \neq \emptyset$ and we have transitivity. \square

The following corollary is a direct consequence of putting Lemma 3.4 and Theorem 3.6 together.

Corollary 3. *Let X be a Hausdorff space and (X, f) be a transitive dynamical system. If $(x, y) \in \text{EqP}(X, f)$ then $y \in \omega(x)$.*

We now construct a class of examples which have no isolated points and non-trivial equicontinuity pairs but no points of topological equicontinuity. The information provided on shift spaces in Section 2.3 will be of relevance here.

Example 2. Take $\Sigma = \{0, 1, 2, \dots, m\}$, where $m \geq 2$. For each $k \in \mathbb{N}$, let W_k represent a word of length k containing only the symbols $\{1, 2, \dots, m\}$. Let \mathcal{W} be the collection of all sequences of the form:

$$W_1 0 W_2 0^2 W_3 0^3 \dots 0^{n-1} W_n 0^n \dots,$$

Now take

$$Y = \mathcal{W} \cup \{0^n \mathbf{x} \mid \mathbf{x} \in \mathcal{W}, n \in \mathbb{N}\} \cup \{0^\infty\},$$

and let

$$X := \overline{\{\sigma^k(\mathbf{y}) \mid \mathbf{y} \in Y, k \in \mathbb{N}_0\}},$$

where the closure is taken with regard to the full shift $\Sigma^{\mathbb{N}_0}$. It is worth observing that the ω -limit sets of points in \mathcal{W} are points of the following forms:

$$0^\infty \text{ and } W_k 0^\infty.$$

Notice that, for any $\mathbf{x} \in \mathcal{W}$, $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, $\sigma^k(\mathbf{x}) \in [0^n]$ if and only if, for all $\mathbf{y} \in \mathcal{W}$, $\sigma^k(\mathbf{y}) \in [0^n]$. With this observation in mind, we claim that if $\mathbf{x} \in \mathcal{W}$, then $(\mathbf{x}, 0^\infty) \in \text{EqP}(X, \sigma)$. Indeed, pick such an \mathbf{x} ; write $\mathbf{x} = W_1 0 W_2 0^2 W_3 0^3 \dots$. Now let $O \ni 0^\infty$ be open. Let $V = [0^n] \subseteq O$ and take $U = [W_1 0 W_2]$. If $\mathbf{y} \in U$ then $\mathbf{y} \in \mathcal{W}$ by construction. But by our observation, if $\sigma^k(\mathbf{y}) \in V$ then $\sigma^k(\mathcal{W}) \subseteq V \subseteq O$. Hence $(\mathbf{x}, 0^\infty) \in \text{EqP}(X, \sigma)$.

It remains to observe that $\text{Eq}(X, \sigma) = \emptyset$, because shift systems, with no isolated points, are sensitive. By Theorem 3.1, this means there are no points of topological equicontinuity.

Example 2 demonstrates that, even in a compact metric setting, a point may have non-trivial equicontinuity partners but not be a point of equicontinuity.

We will now build up some results relating to equicontinuity pairs in dynamical systems, this will culminate in a generalisation of the Auslander-Yorke dichotomy.

Lemma 3.7. *Let (X, f) be a dynamical system, where X is a Hausdorff space. Let $x, y \in X$. If $(x, y) \in \text{EqP}(X, f)$, f is open at y and there is a neighbourhood base for y , $\mathcal{B}_y \subseteq \mathcal{N}_y$, such that $f^{-1}(f(O)) = O$ for all $O \in \mathcal{B}_y$, then $(x, f(y)) \in \text{EqP}(X, f)$.*

Proof. Let $O \in \mathcal{N}_{f(y)}$. Then $f^{-1}(O) \in \mathcal{N}_y$. Let $O' \in \mathcal{B}_y$ be such that $f(O') \subseteq O$. Notice that, since f is open at y , $f(O') \in \mathcal{N}_{f(y)}$. Since $(x, y) \in \text{EqP}(X, f)$ there exist $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ satisfying the topological equicontinuity for x, y

and O' ; without loss of generality $V \subseteq O'$ and $V \in \mathcal{B}_y$. If $x \neq y$ then, without loss of generality, $U \cap V = \emptyset$. If $x = y$ then, without loss of generality $U = V$. Because f is open at y , $f(V) \in \mathcal{N}_{f(y)}$. For any $n \in \mathbb{N}$, if $f^n(U) \cap f(V) \neq \emptyset$ then $f^{n-1}(U) \cap f^{-1}(f(V)) \neq \emptyset$. Because $V \in \mathcal{B}_y$ we have $f^{-1}(f(V)) = V$, hence $f^{n-1}(U) \cap V \neq \emptyset$. If $n = 1$ then it follows that $U = V$ and so $U \subseteq O'$. This itself implies $f(U) \subseteq f(O') \subseteq O$. If $n > 1$ then $f^{n-1}(U) \subseteq O'$ by topological equicontinuity at x and y . This implies $f^n(U) \subseteq f(O') \subseteq O$. \square

Corollary 4. *Let (X, f) be a dynamical system, where X is a Hausdorff space. If f is a homeomorphism and $(x, y) \in \text{EqP}(X, f)$ then $(x, f(y)) \in \text{EqP}(X, f)$.*

Proof. Immediate from Lemma 3.7. \square

Lemma 3.8. *Let (X, f) be a dynamical system, where X is Hausdorff space. Suppose $(x, y) \notin \text{EqP}(X, f)$ and let O be a splitting neighbourhood of y with regard to x . Then, for any pair of neighbourhoods U and V of x and y respectively, the set of natural numbers n for which $f^n(U) \cap V \neq \emptyset$ and $f^n(U) \not\subseteq O$ is infinite.*

Proof. The proof is similar to that of Proposition 2.

Let $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$. Take

$$A = \{n \in \mathbb{N} \mid f^n(U) \cap V \neq \emptyset \text{ and } f^n(U) \not\subseteq O\}.$$

Suppose that A is finite; note that $A \neq \emptyset$ as $(x, y) \notin \text{EqP}(X, f)$. Let N be the largest element in A . By Corollary 2, there exist $U' \in \mathcal{N}_x$ and $V' \in \mathcal{N}_y$ such that, for any $k \in \{1, \dots, N\}$, if $f^k(U') \cap V' \neq \emptyset$ then $f^k(U') \subseteq O$; without loss of generality $U' \subseteq U$ and $V' \subseteq V$. But as $(x, y) \notin \text{EqP}(X, f)$ we have $A' \neq \emptyset$, where

$$A' = \{n \in \mathbb{N} \mid f^n(U') \cap V' \neq \emptyset \text{ and } f^n(U') \not\subseteq O\}.$$

Thus there exists $m > N$ with $m \in A' \subseteq A$. \square

Lemma 3.9. *Let (X, f) be a dynamical system, where X is Hausdorff space. Let $x, y, z \in X$ and let $z \in \overline{\text{Orb}(x)}$. If $(x, y) \notin \text{EqP}(X, f)$ and O is a splitting neighbourhood of y with regard to x then $(z, y) \notin \text{EqP}(X, f)$ and O is a splitting neighbourhood of y with regard to z .*

Proof. Let $U \in \mathcal{N}_z$ and $V \in \mathcal{N}_y$. Let $n \in \mathbb{N}$ be such that $W = f^{-n}(U) \ni x$. Take $m > n$ such that $f^m(W) \cap V \neq \emptyset$ and $f^m(W) \not\subseteq O$; such an m exists by Lemma 3.8. \square

Remark 3. The contrapositive of Lemma 3.9 is: If $(y, z) \in \text{EqP}(X, f)$ and $y \in \overline{\text{Orb}(x)}$ then $(x, z) \in \text{EqP}(X, f)$.

Corollary 5. *Let (X, f) be a Hausdorff dynamical system and suppose $x, y, z \in X$. If (y, z) is a trivial (resp. non-trivial) equicontinuity pair and $y \in \overline{\text{Orb}(x)}$ then (x, z) is a trivial (resp. non-trivial) equicontinuity pair. In particular, if (x, y) and (y, z) are non-trivial equicontinuity pairs then so is (x, z) .*

Proof. If (y, z) is a trivial equicontinuity pair then $z \notin \Omega(y)$. Let $U \in \mathcal{N}_y$, $V \in \mathcal{N}_z$ and $N \in \mathbb{N}$ be such that, for any $n > N$, $f^n(U) \cap V = \emptyset$. Now let $m \in \mathbb{N}_0$ be such that $W = f^{-m}(U) \ni x$. Then, for all $n > N + m$, $f^n(W) \cap V = \emptyset$. Thus (x, z) is a trivial equicontinuity pair.

Now suppose that (y, z) is a non-trivial equicontinuity pair. If $y \in \text{Orb}(x)$ then $\omega(x) = \omega(y)$ and so $z \in \omega(x)$. If $y \in \omega(x)$ then, since ω -limit sets are positively

invariant, $z \in \omega(x)$. Therefore we have $z \in \omega(x)$. It now suffices to check $(x, z) \in \text{EqP}(X, f)$; but this is just Remark 3.

Finally, if (x, y) and (y, z) are non-trivial equicontinuity pairs then $y \in \omega(x)$ and the result follows by the above. \square

Remark 4. Corollary 5 shows that the relation given by ‘non-trivial equicontinuity pair’ is transitive.

Remark 5. It follows from Corollary 5 that if a system has a transitive point, say x , then, if (a, b) is an equicontinuity pair then (x, b) is also an equicontinuity pair; every equicontinuity partner is an equicontinuity partner of the transitive point.

Corollary 6. *Let X be a Hausdorff space. If (X, f) is minimal then, for any $x, y \in X$,*

$$(x, y) \in \text{EqP}(X, f) \implies \forall z \in X, (z, y) \in \text{EqP}(X, f),$$

and

$$(x, y) \notin \text{EqP}(X, f) \implies \forall z \in X, (z, y) \notin \text{EqP}(X, f).$$

Proof. The former statement follows from Corollary 5, the latter from Lemma 3.9. \square

The following theorem is a generalisation of [1, Theorem 2.4] (see Theorem 2.6).

Theorem 3.10. *Let (X, f) be a transitive dynamical system, where X is a Hausdorff space. Suppose there exists a topological equicontinuity point. Then the set of topological equicontinuity points coincides with the set of transitive points.*

In particular, if (X, f) is a minimal system and there is a topological equicontinuity point then the system is topologically equicontinuous.

Proof. Let $x \in X$ be a point of topological equicontinuity. By Lemma 3.5, $\omega(x) = \Omega(x)$; but since (X, f) is a transitive system $\Omega(x) = X$ by Remark 1. Hence x is a transitive point.

Now suppose x is a transitive point. Let y be a point of topological equicontinuity. Then $y \in \omega(x)$ as x is a transitive point. Now, $(y, z) \in \text{EqP}(X, f)$ for all $z \in X$, and these are all non-trivial equicontinuity pairs by Theorem 3.6, therefore, by Corollary 5, it follows that $(x, z) \in \text{EqP}(X, f)$ for all $z \in X$; i.e. x is a point of topological equicontinuity. \square

We are now in a position to present a generalised version of the Auslander-Yorke dichotomy for minimal systems; in [4] the authors show that a compact metric minimal system is either equicontinuous or is sensitive. The following definition was given by Good and Macías in [14] where they show it is equivalent to sensitivity if X is a compact Hausdorff space.

Definition 3.11. A dynamical system (X, f) , where X is a Hausdorff space, is said to be *Hausdorff sensitive* if there exists a finite open cover \mathcal{U} such that for any nonempty open set V there exist $x, y \in V$, $x \neq y$, and $k \in \mathbb{N}$ such that $\{f^k(x), f^k(y)\} \not\subseteq U$ for all $U \in \mathcal{U}$.

In similar fashion to metric and uniform settings, for a subset $U \subseteq X$ and an open cover \mathcal{U} of X we define the set $N_{\mathcal{U}}(U)$ as the set of natural numbers k for which there exist $x, y \in U$, $x \neq y$ such that $\{f^k(x), f^k(y)\} \not\subseteq U$ for all $U \in \mathcal{U}$. Thus, a system is Hausdorff sensitive precisely when there is a finite open cover \mathcal{U} for which $N_{\mathcal{U}}(U) \neq \emptyset$ for any nonempty open set U .

Theorem 3.12. *Let (X, f) be a system with a transitive point x , where X is a regular Hausdorff space (i.e. T_3). If there exists $y \in X$ with $(x, y) \notin \text{EqP}(X, f)$ then (X, f) is Hausdorff sensitive.*

Proof. Let x and y be as in the statement. Therefore

$$\exists O \in \mathcal{N}_y : \forall U \in \mathcal{N}_x \forall V \in \mathcal{N}_y \exists n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset \text{ and } f^n(U) \not\subseteq O. \quad (8)$$

Let V_1 and V_2 be open neighbourhoods of y such that $\overline{V_1} \subseteq O$ and $\overline{V_2} \subseteq V_1$; these exist as X is regular. Then $\mathcal{U} := \{V_1, X \setminus \overline{V_2}\}$ is a finite open cover. Now let U be an arbitrary nonempty open set. Let $n \in \mathbb{N}$ be such that $W = f^{-n}(U) \ni x$. Take $m > n$ such that $f^m(W) \cap V_2 \neq \emptyset$ and $f^m(W) \not\subseteq O$; such an m exists by Lemma 3.8. Then $f^{m-n}(U) \cap V_2 \neq \emptyset$ and $f^{m-n}(U) \not\subseteq O$. In particular there exists $a, b \in U$ such that $f^{m-n}(a) \notin O$ and $f^{m-n}(b) \in V_2$. Then $\{f^{m-n}(a), f^{m-n}(b)\} \cap V_1 = \{f^{m-n}(b)\}$ and $\{f^{m-n}(a), f^{m-n}(b)\} \cap X \setminus \overline{V_2} = \{f^{m-n}(a)\}$. \square

Corollary 7. *(Generalised Auslander-Yorke dichotomy I) Let X be a T_3 space. A minimal system (X, f) is either topologically equicontinuous or Hausdorff sensitive.*

Proof. Suppose it is not equicontinuous. Then there exists $x, y \in X$ with $(x, y) \notin \text{EqP}(X, f)$. Since x is a transitive point the result follows from Theorem 3.12. \square

The following theorem is a generalisation of the result by Banks *et al* [5] that the first two ingredients of Devaney chaos (transitivity and dense set of periodic points) entail the third (sensitivity).

Theorem 3.13. *Let (X, f) be a transitive system where X is an infinite T_3 space. If the set of eventually periodic points is dense in X then the system is Hausdorff sensitive.*

Proof. Suppose first that the set of periodic points is not dense in X . Let U be a nonempty open set not containing any periodic points. Let V_1 and V_2 be nonempty open sets such that $\overline{V_1} \subseteq U$ and $\overline{V_2} \subseteq V_1$. We claim the finite open cover $\mathcal{U} := \{V_1, X \setminus \overline{V_2}\}$ bears witness to Hausdorff sensitivity. Indeed, let W be a nonempty open set. Since the set of eventually periodic points is dense there is such a point in W . Because there are no periodic points in U it follows that $N(W, X \setminus U)$ is cofinite in \mathbb{N} . However, by transitivity, $N(W, V_2)$ is infinite. Therefore there exists $k \in N(W, X \setminus U) \cap N(W, V_2)$: i.e. there exist $x, y \in W$ such that $f^k(x) \in X \setminus U$ and $f^k(y) \in V_2$. Notice that $f^k(x) \notin V_1$ and $f^k(y) \notin X \setminus \overline{V_2}$, hence $\{f^k(x), f^k(y)\} \not\subseteq V$ for any V in \mathcal{U} . Since W was picked arbitrarily we are done.

Now suppose that the set of periodic points is dense in X . Let v and w be two periodic points, with periods n and m respectively, belonging to distinct orbits and take disjoint open sets V and W such that $V \supseteq \text{Orb}(v)$ and $W \supseteq \text{Orb}(w)$. By regularity, there are open sets V_1, V_2, W_1 and W_2 such that

$$\text{Orb}(v) \subseteq V_2 \subseteq \overline{V_2} \subseteq V_1 \subseteq \overline{V_1} \subseteq V; \text{ and,}$$

$$\text{Orb}(w) \subseteq W_2 \subseteq \overline{W_2} \subseteq W_1 \subseteq \overline{W_1} \subseteq W.$$

Consider the open cover $\mathcal{U} := \{V_1, W_1, X \setminus (\overline{V_2} \cup \overline{W_2})\}$: this bears witness to Hausdorff sensitivity. To see this, let U be a nonempty open set. Let $p \in U$ be periodic with period l . Notice that either $\text{Orb}(p) \not\subseteq V$ or $\text{Orb}(p) \not\subseteq W$: without loss of generality assume the former and take $k = \max\{l, n\}$. Let

$$V' = \bigcap_{i \in \{0, \dots, k\}} f^{-i}(V_2).$$

Notice $v \in V'$ so $V' \neq \emptyset$. By transitivity there exists $r \in \mathbb{N}$ such that $f^r(U) \cap V' \neq \emptyset$. By the definition of V' it follows that $f^{r+i}(U) \cap V_2 \neq \emptyset$ for all $i \in \{0, \dots, k\}$. However

$$\bigcup_{i \in \{0, \dots, k\}} f^{r+i}(U) \supseteq \text{Orb}(p) \not\subseteq V.$$

Fix $i \in \{0, \dots, k\}$ such that $f^{r+i}(p) \notin V$. Then $f^{r+i}(p) \notin V_1$. Furthermore there is $x \in U$ with $f^{r+i}(x) \in V_2$. Thus $\{f^{r+i}(x), f^{r+i}(p)\} \not\subseteq A$ for any $A \in \mathcal{U}$. \square

We end this section with the following question.

Question 1. Does there exist a transitive system (X, f) , where X is a Hausdorff space, with a non-trivial equicontinuity pair (x, y) but where x is not a topological equicontinuity point?

The following result may help make some headway with Question 1.

Proposition 3. *Suppose (X, f) is a transitive dynamical system where X is an infinite Hausdorff space. If $x \in X$ is an eventually periodic point then $(x, y) \notin \text{EqP}(X, f)$ for any $y \in X$.*

Proof. Write $\text{Orb}(x) = \{x, f(x), \dots, f^l(x)\}$. Suppose $(x, y) \in \text{EqP}(X, f)$. Then by Corollary 3 it follows that $y \in \omega(x)$; as x is eventually periodic this means $y \in \text{Orb}(x)$ and y is periodic. Write $y = f^m(x)$ and let n be the period of y (so $n \leq l$). Let $z \in X \setminus \text{Orb}(x)$; for each $i \in \{0, \dots, n-1\}$ let $W_i \in \mathcal{N}_z$ and $O_i \in \mathcal{N}_{f^i(y)}$ be such that $W_i \cap O_i = \emptyset$. Now let

$$O := \bigcap_{i=0}^{n-1} f^{-i}(O_i),$$

and

$$W := \bigcap_{i=0}^{n-1} W_i.$$

Thus $W \in \mathcal{N}_z$ and $O \in \mathcal{N}_y$. Now let $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ satisfy the equicontinuity condition for x, y and O . Notice that $f^i(O) \cap W = \emptyset$ for all $i \in \{0, \dots, n-1\}$. Since $f^m(U) \cap V \neq \emptyset$ we have $f^m(U) \subseteq O$. Furthermore, $f^{m+an}(U) \cap V \neq \emptyset$ for all $a \in \mathbb{N}_0$, hence $f^{m+an}(U) \subseteq O$. It follows that $f^k(U) \cap W = \emptyset$ for all $k \geq m$, this contradicts Lemma 2.3. \square

4. Even continuity. Even continuity, as defined by Kelley [19, p. 234], is a weaker concept than that of topological equicontinuity. If X and Y are topological spaces we say a collection of maps \mathcal{F} from X to Y is *evenly continuous* at an ordered pair $(x, y) \in X \times Y$ if for any $O \in \mathcal{N}_y$ there exist neighbourhoods $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that, for any $f \in \mathcal{F}$, if $f(x) \in V$ then $f(U) \subseteq O$; when this is the case we refer to (x, y) as an *even continuity pair*. We say \mathcal{F} is *evenly continuous* at a point $x \in X$ if it is evenly continuous at (x, y) for all $y \in Y$. We say the collection is *evenly continuous* if it is evenly continuous at every $x \in X$. We remark that when Y is a compact Hausdorff space the notions of topological equicontinuity, even continuity and equicontinuity coincide (see [19, Theorem 7.23]). Finally, we observe that if a family is evenly continuous (resp. topological equicontinuous) then each member of that family is necessarily continuous [12, pp. 162].

Given a dynamical system (X, f) , we denote the collection of even continuity pairs and the collection of even continuity points by $\text{EvP}(X, f) \subseteq X \times X$ and

$\text{Ev}(X, f) \subseteq X$ respectively. Note that in this case, if we consider the above definitions, we have $Y = X$ and $\mathcal{F} = \{f^n \mid n \in \mathbb{N}\}$. By definition it follows that (X, f) is evenly continuous precisely when $\text{EvP}(X, f) = X \times X$. For $(x, y) \in \text{EvP}(X, f)$, we refer to the condition

$$\forall O \in \mathcal{N}_y \exists U \in \mathcal{N}_x \exists V \in \mathcal{N}_y : \forall n \in \mathbb{N}, f^n(x) \in V \implies f^n(U) \subseteq O, \quad (9)$$

as the *even continuity condition* for x and y . We say that U and V , as in Equation 9, satisfy the even continuity condition for x , y and O .

Remark 6. Clearly every equicontinuity pair is an even continuity pair.

As pointed out by others (e.g. [28]), the converse to Remark 6 is not true in general. The following example demonstrates this.

Example 3. For each $n \in \mathbb{N}$, let X_n be the finite word 10^n and take $\mathbf{x} = X_1 X_2 X_3 \dots$. For each $n \in \mathbb{N}$, let $\mathbf{z}_n = X_1 X_2 \dots X_n 0^\infty$. Let $\mathbf{y} = 0^\infty$. Take

$$Y := \{0^m \mathbf{z}_n, 0^m \mathbf{x}, 0^\infty \mid n, m \in \mathbb{N}\},$$

and let

$$X := \overline{\{\sigma^k(\mathbf{y}) \mid \mathbf{y} \in Y, k \in \mathbb{N}_0\}},$$

where the closure is taken with regard to the full shift $\Sigma^{\mathbb{N}_0}$.

Note that,

$$\omega(\mathbf{x}) = \{0^\infty, 0^n 10^\infty \mid n \in \mathbb{N}_0\},$$

and for each $i \in \mathbb{N}$,

$$\omega(\mathbf{z}_i) = \{0^\infty\}.$$

Considering the shift system (X, σ) , it is easy to see that $(\mathbf{x}, 0^\infty)$ is a non-trivial even continuity pair in (X, σ) (i.e. it is an even continuity pair and $0^\infty \in \omega(\mathbf{x})$). Furthermore, it is not an equicontinuity pair; arbitrarily close to \mathbf{x} are points that map onto 0^∞ , which is a fixed point, but \mathbf{x} itself is not pre-periodic. To show this explicitly, take $O = [0] \in \mathbb{N}_{0^\infty}$. Picking $U \in \mathcal{N}_x$, there exists $N \in \mathbb{N}$ such that $\sigma^k(U) \ni 0^\infty$ for all $k > N$; in particular, for any $V \in \mathcal{N}_{0^\infty}$, $\sigma^k(U) \cap V \neq \emptyset$ for all $k \geq N$. But there exists $k \geq N$ such that $\sigma^k(\mathbf{x}) \in [1]$, hence $(\mathbf{x}, 0^\infty) \notin \text{EqP}(X, \sigma)$.

Proposition 4. *Let (X, f) be a dynamical system where X Hausdorff space. Let $x, y \in X$ and suppose $y \notin \omega(x)$. Then $(x, y) \in \text{EvP}(X, f)$.*

Proof. Let $O \in \mathcal{N}_y$ be given. Since $y \notin \omega(x)$ there exist $V \in \mathcal{N}_y$ and $N \in \mathbb{N}$ such that $f^n(x) \notin V$ for all $n > N$.

By Corollary 2, there exist $U' \in \mathcal{N}_x$ and $V' \in \mathcal{N}_y$ such that, for any $k \in \{1, \dots, N\}$, if $f^k(U') \cap V' \neq \emptyset$ then $f^k(U') \subseteq O$; without loss of generality $V' \subseteq V \cap O$. In particular this means that, for all $k \in \{1, \dots, N\}$, if $f^k(x) \in V'$ then $f^k(U') \subseteq O$. Then, since $f^n(x) \notin V'$ for all $n > N$, U' and V' satisfy the even continuity condition for x , y and O . As $O \in \mathcal{N}_y$ was picked arbitrarily the result follows. \square

Remark 7. If X is a Hausdorff space, putting together propositions 2 and 4, we have, for a pair $x, y \in X$, the following:

- If $y \notin \omega(x)$ then $(x, y) \in \text{EvP}(X, f)$.
- If $y \notin \Omega(x)$ then $(x, y) \in \text{EqP}(X, f)$.

Definition 4.1. If (X, f) is a dynamical system, where X is a Hausdorff space. We say $(x, y) \in X \times X$ is a *trivial even continuity pair* if $y \notin \omega(x)$.

Remark 8. Proposition 4 tells us that a trivial even continuity pair is indeed an even continuity pair. We emphasise that, by definition, if $(x, y) \in \text{EvP}(X, f)$ then either they are a trivial even continuity pair or $y \in \omega(x)$. Finally, it is worth observing that, by Lemma 3.4 and Remark 6, a non-trivial equicontinuity pair is also a non-trivial even continuity pair.

The statement $(x, y) \notin \text{EvP}(X, f)$, for $x, y \in X$, means precisely

$$\exists O \in \mathcal{N}_y : \forall U \in \mathcal{N}_x \forall V \in \mathcal{N}_y \exists n \in \mathbb{N} : f^n(x) \in V \text{ and } f^n(U) \not\subseteq O. \quad (10)$$

We shall refer to a neighbourhood such as O in equation 10 as an *even-splitting neighbourhood* of y with regard to x . It is straightforward to see that every even-splitting neighbourhood of y with regard to x is also a splitting neighbourhood of y with regard to x . Notice that, by Proposition 4, if $(x, y) \notin \text{EvP}(X, f)$ then $y \in \omega(x)$, so we have:

Corollary 8. *Let (X, f) be a dynamical system where X Hausdorff space. If x has no even continuity partners, then x is a transitive point.*

The proof of lemmas 4.2 and 4.3 are very similar to that of lemmas 3.7 and 3.8 respectively and are thereby omitted.

Lemma 4.2. *Let (X, f) be a dynamical system, where X is a Hausdorff space. If $(x, y) \in \text{EvP}(X, f)$, f is open at y and there is a neighbourhood base for y , $\mathcal{B}_y \subseteq \mathcal{N}_y$, such that $f^{-1}(f(O \cap \text{Orb}(x))) = O \cap \text{Orb}(x)$ for all $O \in \mathcal{B}_y$, then $(x, f(y)) \in \text{EvP}(X, f)$.*

Corollary 9. *Let (X, f) be a dynamical system, where X is a Hausdorff space. If f is a homeomorphism and $(x, y) \in \text{EvP}(X, f)$ then $(x, f(y)) \in \text{EvP}(X, f)$.*

Proof. Immediate from Lemma 4.2. □

Lemma 4.3. *Let (X, f) be a dynamical system, where X is Hausdorff space. Suppose $(x, y) \notin \text{EvP}(X, f)$ and let O be an even-splitting neighbourhood of y with regard to x . Then, for any pair of neighbourhoods U and V of x and y respectively, the set of natural numbers n for which $f^n(x) \in V$ and $f^n(U) \not\subseteq O$ is infinite.*

Lemma 4.4. *Let (X, f) be a dynamical system, where X is Hausdorff space. Let $x, y \in X$. If $(x, y) \notin \text{EvP}(X, f)$ and O is an even-splitting neighbourhood of y with regard to x then, for any $n \in \mathbb{N}$, $(f^n(x), y) \notin \text{EvP}(X, f)$ and O is an even-splitting neighbourhood of y with regard to $f^n(x)$.*

Proof. Let $U \in \mathcal{N}_{f^n(x)}$ and $V \in \mathcal{N}_y$. Then $W = f^{-n}(U) \in \mathcal{N}_x$. By Lemma 4.3 the set

$$A = \{k \in \mathbb{N} \mid f^k(x) \in V \text{ and } f^k(W) \not\subseteq O\},$$

is infinite. Taking $m > n$ with $m \in A$ gives the result. □

Remark 9. We emphasise the contrapositive of Lemma 4.4: Let (X, f) be a dynamical system, where X is Hausdorff space. Suppose $(x, y) \in \text{EvP}(X, f)$ and $x \in \text{Orb}(z)$. Then $(z, y) \in \text{EvP}(X, f)$.

Proposition 5. *Let X be a Hausdorff space. If (X, f) is a dynamical system and there exists a point $x \in X$ with no even continuity partners, then x is a transitive point and (X, f) has no equicontinuity pairs.*

Proof. Let $x \in X$ be a point with no even continuity partners. By Corollary 8, x is a transitive point.

Let $y, z \in X$ be picked arbitrarily. Let O be an even-splitting neighbourhood of z with regard to x . Let $U \in \mathcal{N}_y$ and $V \in \mathcal{N}_z$. As x is transitive there exists $n \in \mathbb{N}$ such that $f^n(x) \in U$. By Lemma 4.4, $(f^n(x), z)$ is not an even continuity pair and O is an even-splitting neighbourhood of z with regard to $f^n(x)$. It follows that there exists $m \in \mathbb{N}$ such that $f^m(U) \cap V \neq \emptyset$ and $f^m(U) \not\subseteq O$; hence $(y, z) \notin \text{EqP}(X, f)$. \square

At the end of the previous section, we asked, in Question 1, whether there exists a transitive system with an equicontinuity pair but no point of equicontinuity. We now answer, in the positive, an analogous question with regard to even continuity pairs.

Theorem 4.5. *There exists a transitive system (X, f) with a non-trivial even continuity pair but no point of even continuity: Furthermore, there is such a system which is additionally Auslander-Yorke chaotic, densely and strongly Li-Yorke chaotic, but not Devaney chaotic, whilst having no equicontinuity pairs.*

Due to the length and technical nature of the proof of Theorem 4.5, we leave it until the end of the paper (Section 8).

Remark 10. Devaney [11, pp. 50] defined chaos as a topologically transitive, sensitive system with a dense set of periodic points. This last property means that, “in the midst of random behaviour, we nevertheless have an element of regularity.” The construction in the proof of Theorem 4.5 shows that a system which is, in some sense, extremely chaotic (it is not only sensitive but expansive, whilst having only two periodic points) can still exhibit some element of regularity: the even continuity pair $(x, 0^\infty)$ provides some regularity associated with x . When x moves close to 0^∞ , everything from a certain neighbourhood of x also moves close to 0^∞ .

The following corollary is an immediate consequence of Theorem 4.5.

Corollary 10. *The notions of equicontinuity pair and even continuity pair, in general, remain distinct for transitive dynamical systems.*

The last result in this section is a variation on Proposition 3; it gives us some information about the types of pairs which cannot be even continuity pairs in transitive systems.

Proposition 6. *Suppose (X, f) is a transitive dynamical system where X is an infinite Hausdorff space. If $x \in X$ is an eventually periodic point then (x, y) is not a non-trivial even continuity pair for any $y \in X$.*

The proof of Proposition 6 is very similar to that of Proposition 3 and is thereby omitted.

5. Equicontinuity, transitivity and splitting. A subset $N = \{n_1, n_2, n_3, \dots\} \subseteq \mathbb{N}$, where $n_1 < n_2 < n_3 \dots$ is said to be *syndetic* if there exists $l \in \mathbb{N}$ such that $n_{i+1} - n_i \leq l$; such an l is called a *bound of the gaps*. A subset is called *thick* if it contains arbitrarily long strings without gaps. A subset is called *cofinite* if its complement is finite. Using this, a dynamical system (X, f) is said to be

1. Syndetically (resp. thickly) transitive if $N(U, V)$ is syndetic (resp. thick) for any nonempty open U and V .

2. Syndetically (resp. thickly / resp. cofinitely) sensitive if there exists a symmetric $D \in \mathcal{D}$ such that, for any nonempty open $U \subseteq X$, the set $N_D(U)$ is syndetic (resp. thick / resp. cofinite).
3. Strong mixing if $N(U, V)$ is cofinite for any nonempty open U and V .

In this section we investigate the link between topological equicontinuity, transitivity and sensitivity. Trivially, if a dynamical system has an equicontinuity point then it is not sensitive. If we restrict our attention to compact metric systems, adding the condition of transitivity is enough to give a partial converse; a transitive map with no equicontinuity points is sensitive [1]. The proof provided by Akin *et al* does not rely on the space being metrizable; with only minor adjustments the result generalises to give the following.

Theorem 5.1. [1] *Let (X, f) be a dynamical system, where X is a compact Hausdorff space. If there exists a transitive point and $\text{Eq}(X, f) = \emptyset$ then (X, f) is sensitive.*

If X is a compact metric space, and (X, f) a transitive dynamical system, then there exists a transitive point (since X is non-meagre and has a countable π -base). By Theorems 5.1 and 3.6 it follows that, for a compact metric system, no equicontinuity pairs implies both transitivity and sensitivity.

Corollary 11. *Let X be a compact Hausdorff space which yields a countable π -base. If $\text{EqP}(X, f) = \emptyset$ then the system is both transitive and sensitive.*

Proof. By Theorem 3.6 the system is transitive. Since compact Hausdorff spaces are non-meagre there exists a transitive point. Applying Theorem 5.1 completes the result. \square

Proposition 7. *Let X be a Hausdorff space and (X, f) a dynamical system. If X is nonmeagre with a countable π -base, then $\text{EqP}(X, f) = \emptyset$ if and only if there exists a transitive point $x \in X$ with no equicontinuity partners.*

Proof. Assume the latter and let x be such a transitive point. Suppose that $(a, b) \in \text{EqP}(X, f)$, for some $a, b \in X$. Then (a, b) is a non-trivial equicontinuity pair by Theorem 3.6. As x is a transitive point $a \in \omega(x)$. It follows from Corollary 5 that $(x, b) \in \text{EqP}(X, f)$, a contradiction.

Now suppose the former. By Corollary 11 the system is transitive, which entails the existence of a transitive point as X is nonmeagre with a countable π -base. \square

We now turn our attention to examining sufficient conditions for $\text{EqP}(X, f) = \emptyset$. One obvious such condition is the following.

Proposition 8. *Let (X, f) be a dynamical system, where X is a Hausdorff uniform space. Suppose there exists a symmetric $D \in \mathcal{D}$ such that for any nonempty open sets U and V , $N(U, V) \cap N_D(U) \neq \emptyset$, then there are no equicontinuity pairs.*

Note that, when the hypothesis of this proposition occurs, it is equivalent to being able to move the existential quantifier to the front of the statement stating $\text{EqP}(X, f) = \emptyset$. To be clear, $\text{EqP}(X, f) = \emptyset$ means,

$$\forall x, y \in X \exists D \in \mathcal{D} : \forall U \in \mathcal{N}_x \forall V \in \mathcal{N}_y, N(U, V) \cap N_D(U) \neq \emptyset.$$

whilst the hypothesis states,

$$\exists D \in \mathcal{D} : \forall x, y \in X \forall U \in \mathcal{N}_x \forall V \in \mathcal{N}_y, N(U, V) \cap N_D(U) \neq \emptyset.$$

For any pair of sets $U, V \subseteq X$, we define $N_D(U, V) := N(U, V) \cap N_D(U)$; if X is a metric space and $\delta > 0$ we similarly define $N_\delta(U, V) := N(U, V) \cap N_\delta(U)$. Such a set is extremely relevant in an applied setting, where small rounding errors mean that a different point than the one intended might be being tracked. This set tells us precisely when U meets V whilst also expanding to at least diameter δ . The importance of such a set leads us to give the following definition.

Definition 5.2. Let (X, f) be a dynamical system, where X is a Hausdorff uniform space. We say that (X, f) experiences *splitting* if there is a symmetric $D \in \mathcal{D}$ such that for any pair of nonempty open sets U and V we have $N_D(U, V) \neq \emptyset$. Such a D is called a splitting entourage for (X, f) .

In similar fashion, if X is a metric space we say the system (X, f) has *splitting* if there exists $\delta > 0$ such that for any pair of nonempty open sets U and V we have $N_\delta(U, V) \neq \emptyset$. Thus a system has splitting when every nonempty open set ‘hits’ every other such set whilst simultaneously being pulled apart to diameter at least δ . Proposition 8 then states that any splitting system has no equicontinuity pairs. To take a purely topological approach, as we did in the previous section, if X is a Hausdorff space we say the system has *Hausdorff splitting* if there exists a finite open cover \mathcal{U} such that for any pair of nonempty open sets U and V we have $N_{\mathcal{U}}(U, V) \neq \emptyset$.

Remark 11. In this section we mainly deal with the case when the phase space is a Hausdorff uniform space. We observe, however, that each of the results in this section have analogous versions where the space is T_3 . These analogous versions are precisely the ‘natural’ ones that one might expect: they refer to a finite open cover \mathcal{U} instead of an entourage D and to the sets $N_{\mathcal{U}}(U)$ and $N_{\mathcal{U}}(U, V)$ instead of $N_D(U)$ and $N_D(U, V)$ respectively. For the sake of space we do not include these versions here.

The following lemma is analogous to several previously stated.

Lemma 5.3. *If (X, f) is a Hausdorff uniform system with splitting, with splitting entourage D , then for any nonempty open pair U and V , $N_D(U, V)$ is infinite.*

Proof. Suppose $N_D(U, V)$ is finite. Since (X, f) has splitting, with splitting entourage D , $N_D(U, V) \neq \emptyset$. Let $k \in \mathbb{N}$ be the greatest element of $N_D(U, V)$. Let $W \subseteq U \cap f^{-k}(V)$ be open such that, for any $i \in \{1, \dots, k\}$ and any $x, y \in W$, $(f^i(x), f^i(y)) \in D$. As $N_D(W, V) \neq \emptyset$ and $W \subseteq U$ we have a contradiction and the result follows. \square

Corollary 12. *Let X be a Hausdorff uniform space with at least two points. If (X, f) is weakly mixing, then (X, f) experiences splitting.*

Proof. Suppose (X, f) exhibits weak mixing. Let \mathcal{D} be a compatible uniformity for X and $E \in \mathcal{D}$ be a symmetric entourage such that, for any $x \in X$, we have $E[x] \neq X$. Let $D \in \mathcal{D}$ be symmetric such that $3D \subseteq E$. Let U and V be nonempty open sets. Let $x \in V$ and pick $y \in X$ such that $(x, y) \notin E$. By weak mixing, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap (D[x] \cap V) \neq \emptyset$ and $f^n(U) \cap D[y] \neq \emptyset$. Let $u \in f^n(U) \cap (D[x] \cap V)$ and $u' \in f^n(U) \cap D[y]$; by symmetry $(x, u) \in D$ and $(u', y) \in D$. If $(u, u') \in D$ then $(x, y) \in 3D \subseteq E$, a contradiction. \square

We remark that, with regard to a system (X, f) , since topologically exact implies strong mixing which implies weak mixing, exactness and strong mixing are each sufficient for a system to have splitting.¹

Clearly we also have the following result.

Proposition 9. *Let (X, f) be a dynamical system, with X a Hausdorff uniform space. Let P and Q be properties of subsets \mathbb{N} such that if A and B , subsets of \mathbb{N} , have P and Q respectively, then $A \cap B \neq \emptyset$. Then if (X, f) is P -transitive (by which we mean for any pair of nonempty open sets U and V , $N(U, V)$ has property P) and Q -sensitive (by which we mean there exists a symmetric entourage $D \in \mathcal{D}$ such that for any nonempty open set U , $N_D(U)$ has property Q), then it experiences splitting.*

For example, if (X, f) is syndetically transitive and thickly sensitive it follows that it has splitting. Also, since transitivity implies $N(U, V)$ is infinite for any nonempty open pair U and V , we have that a transitive system which is cofinitely sensitive has splitting; in particular any transitive map on $[0, 1]$ has splitting.²

It turns out that any Devaney chaotic system on a compact space has splitting, and consequently has no equicontinuity pairs. We will see that this follows as a corollary to Theorem 5.4.

Theorem 5.4. *Let (X, f) a syndetically transitive dynamical system, where X is a compact Hausdorff space. If there are two distinct minimal sets then there exists a symmetric entourage $D \in \mathcal{D}$ such that for any nonempty open pair U and V , $N_D(U, V)$ is syndetic; i.e. the system experiences syndetic splitting.*

(NB. The proof below mimics Moothathu's [26, Theorem 1] proof that a non-minimal syndetically transitive system has syndetic sensitivity for metric systems.)

Proof. Let M_1 and M_2 be distinct minimal sets; it follows that $M_1 \cap M_2 = \emptyset$. Let $x \in M_1$ and $y \in M_2$; so $\overline{\text{Orb}(x)} = M_1$ and $\overline{\text{Orb}(y)} = M_2$. Let $D \in \mathcal{D}$ be symmetric such that, for any $z_1 \in M_1$ and any $z_2 \in M_2$, $(z_1, z_2) \notin 8D$. Now let U and V be nonempty open sets and take $z \in V$; without loss of generality $V \subseteq D[z]$. Suppose there is $p \in M_1$ and $q \in M_2$ such that $(p, z) \in 4D$ and $(z, q) \in 4D$; then $(p, q) \in 8D$, contradicting our choice of D . Without loss of generality we may thereby assume $(p, z) \notin 4D$ for any $p \in M_1$. Let l_1 be a bound of the gaps for $N(U, V)$. Let $W \ni x$ be open such that if $w \in W$ then $(f^i(w), f^i(x)) \in D$ for all $i \in \{0, 1, \dots, l_1\}$; W exists by continuity. By construction, for any $w \in W$, any $v \in V$ and any $i \in \{0, 1, \dots, l_1\}$ we have $(f^i(w), v) \notin 2D$. Let l_2 be a bound of the gaps for $N(U, W)$. It can now be verified that $N(U, V) \cap N(U, W)$ is itself syndetic, with $l_1 + l_2$ a bound of the gaps. Since $N_D(U, V) \supseteq N(U, V) \cap N(U, W)$ the result follows. \square

The following corollaries follow from Theorem 5.4 and Proposition 8.

Corollary 13. *Let (X, f) be a syndetically transitive dynamical system, where X is a compact Hausdorff space. If there are two distinct minimal sets then there are no equicontinuity pairs.*

Corollary 14. *Let (X, f) be a non-minimal transitive system with a dense set of minimal points, where X is a compact Hausdorff space. Then (X, f) has syndetic splitting.*

¹The system (X, f) is topologically exact if, for any nonempty open set U there exists $n \in \mathbb{N}_0$ such that $f^n(U) = X$.

²Any such map is cofinitely sensitive (see [26]).

Proof. Moothathu [26] shows that a transitive system with a dense set of minimal points is syndetically transitive. If the system is non-minimal but the set of minimal points is dense, there exist multiple minimal sets. \square

Corollary 15. *Let (X, f) be a Devaney chaotic dynamical system where X is a compact Hausdorff space. Then (X, f) has syndetic splitting.*

Proof. This follows from Corollary 14. \square

Corollary 16. *Let (X, f) exhibit shadowing and chain transitivity, where X is a compact Hausdorff space. If there are two distinct minimal sets then (X, f) has syndetic splitting.*

Proof. Li [22] shows that a non-minimal compact metric system with shadowing and chain transitivity is syndetically transitive; this result generalises easily to compact Hausdorff systems. The result follows from Theorem 5.4. \square

Question 2. For a dynamical system (X, f) , where X is a Hausdorff uniform space, is splitting distinct from Auslander-Yorke chaos?

We asked previously (Question 1), whether or not a transitive system can have an equicontinuity pair (x, y) without the system being equicontinuous at x . A more restrictive question is the following: Is it possible for a transitive point to have an equicontinuity partner but not be an equicontinuity point? This itself is related to Question 2. Indeed, if there exists a compact Hausdorff system (X, f) , with a transitive point $x \notin \text{Eq}(X, f)$ and a point $y \in X$ with $(x, y) \in \text{EqP}(X, f)$, then it would follow that splitting is not equivalent to Auslander-Yorke chaos; such a system would be both transitive and, since there would be no equicontinuity points by Theorem 3.10, sensitive (Theorem 5.1). However, for any entourage $D \in \mathcal{D}$, there would exist neighbourhoods $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that $N_D(U, V) = \emptyset$, hence the system would not have splitting.

6. Other results connecting transitivity and sensitivity. For a property P which may be held by a subset of the natural numbers (for example, the property of being infinite), we denote by \mathcal{F}_P the set of all subsets of \mathbb{N} which satisfy P . (Thus $\mathcal{F}_P \subseteq \mathcal{P}(\mathbb{N})$.) For such a property, we say a dynamical system is P -transitive if, for all nonempty open sets U and V , $N(U, V) \in \mathcal{F}_P$. In similar fashion, for a uniform dynamical system with uniformity \mathcal{D} , we say the system is P -sensitive if there exists $D \in \mathcal{D}$ such that, for any nonempty open set U , $N_D(U) \in \mathcal{F}_P$.

For the results in this section, let P be a property that may be exhibited by certain subsets of \mathbb{N} , such that the following conditions hold:

1. The collection \mathcal{F}_P is *hereditary upward*, i.e. if $A \in \mathcal{F}_P$ and $B \supseteq A$ then $B \in \mathcal{F}_P$. (This means \mathcal{F}_P forms a *Furstenberg family*.)
2. The collection \mathcal{F}_P is *full*, i.e. if $A \in \mathcal{F}_P$ then for any $k \in \mathbb{N}$, $A \setminus \{0, 1, 2, \dots, k\} \in \mathcal{F}_P$.
3. The collection \mathcal{F}_P is *proper*, i.e. $\emptyset \notin \mathcal{F}_P$.

Now let Q be a property which induces the *dual family* of \mathcal{F}_P , i.e., $\mathcal{F}_Q = \{A \in \mathcal{P}(\mathbb{N}) \mid \forall B \in \mathcal{F}_P, A \cap B \neq \emptyset\}$. It is an easy exercise to show that \mathcal{F}_Q is both hereditary upward, full and proper.

Examples of such pairs of properties are:

1. $P = \text{syndetic}$, $Q = \text{thick}$.
2. $P = \text{infinite}$, $Q = \text{cofinite}$.

3. $P = IP^*$, $Q = IP$.³

The proof given by Moothathu for [26, Proposition 3] (which links syndetic transitivity with syndetic sensitivity for metric systems) can be mimicked almost exactly to give the following sufficient condition for a system being P -sensitive.

Proposition 10. *Let (X, f) be a dynamical system where X is a Hausdorff uniform space. Suppose that for any nonempty open pair U and V , $N(U, V) \in \mathcal{F}_P$. Suppose there exists a set $A \in \mathcal{F}_Q$ and a closed symmetric $D \in \mathcal{D}$ such that for any $n \in A$ there exists $x \in X$ with $(x, f^n(x)) \notin D$. Then for any nonempty open U , $N_D(U) \in \mathcal{F}_P$; so (X, f) is P -sensitive, with sensitivity entourage D .*

Proof. Let $U \subseteq X$ be nonempty open. Since $N(U, U) \in \mathcal{F}_P$, and $A \in \mathcal{F}_Q$, there exists $n \in N(U, U) \cap A$. Define $W := U \cap f^{-n}(U)$; then W is nonempty open. Since $n \in A$ there exists $x \in X$ with $(x, f^n(x)) \notin D$. Let $V \ni x$ be open such that if $y \in V$ then $(y, f^n(y)) \notin D$; such a set exists because D is closed and f is continuous. Now consider the P -set $N(W, V)$; we will show $N(W, V) \subseteq N_D(U)$, from which it will follow that $N_D(U) \in \mathcal{F}_P$. Let $m \in N(W, V)$ and let $w \in W \subseteq U$ be such that $f^m(w) \in V$. Then $(f^m(w), f^{m+n}(w)) \notin D$ by our choice of V . Let $u = f^n(w) \in U$; then $(f^m(w), f^m(u)) \notin D$, so $m \in N_D(U)$. \square

This result yields a corollary relating to recurrence, which generalises [1, Theorem 3.6] and [32, Theorem 7]. (NB. In contrast to these results, we do not assume f to be uniformly continuous.)

Corollary 17. *Let X be a Hausdorff uniform space. A topologically transitive system (X, f) is either sensitive or uniformly rigid.*

Proof. By Lemma 2.3, since (X, f) is transitive, $N(U, V)$ is infinite for any pair of nonempty open sets U and V . If (X, f) is not uniformly rigid then there exists $D \in \mathcal{D}$ such that for any $n \in \mathbb{N}$ there exists $x \in X$ such that $(x, f^n(x)) \notin D$. Without loss of generality we may assume D to be closed and symmetric. The result follows by applying Proposition 10. \square

It follows immediately from Corollary 17, that a transitive Hausdorff uniform system with a non-recurrent point is sensitive. This, in turn, gives us the following corollary.

Corollary 18. *Let (X, f) be a transitive system where X is a Hausdorff uniform space. If $\bigcup_{x \in X} \omega(x) \neq X$ then the system is sensitive.*

Proof. The result follows immediately from Corollary 17. (Of course, for a system with a transitive point this is vacuous.) \square

The following is a corollary to Proposition 10; it is the natural generalisation of [26, Corollary 2].

Corollary 19. *Let (X, f) be a P -transitive system where X is a Hausdorff uniform space. Suppose that there exists two distinct points $x, y \in X$ and a Q -set $A = \{n_k \mid k \in \mathbb{N}\}$ such that for any symmetric $D \in \mathcal{D}$, there exists $l \in \mathbb{N}$ such that $(f^{n_k}(x), f^{n_k}(y)) \in D$ for all $k \geq l$. Then f is P -sensitive.*

³Recall, a set of natural numbers is called an IP -set if it contains all finite sums of some subsequence of itself (when viewing the set as a sequence). An IP^* -set is one which has non-empty intersection with any IP -set. See [6] for more information.

Proof. Choose $D \in \mathcal{D}$ such that $3D \not\supseteq (x, y)$. Now let $l \in \mathbb{N}$ be such that $(f^{n_k}(x), f^{n_k}(y)) \in D$ for all $k \geq l$. Then, for any $k \geq l$, either $(x, f^{n_k}(x)) \notin D$ or $(y, f^{n_k}(y)) \notin D$. Indeed, suppose that both are in D . Then by the triangle inequality, used twice, $(x, y) \in 3D$; a contradiction. Now take $B = \{n_k \mid k \geq l\}$; B is a Q -set, therefore by Proposition 10 we are done. \square

Corollary 20. *Let (X, f) be a non-injective dynamical system where X is a Hausdorff uniform space.*

1. *If (X, f) is P -transitive then it is P -sensitive.*
2. *If (X, f) is syndetically transitive then it is syndetically sensitive.*
3. *If (X, f) is thickly transitive then it is thickly sensitive.*
4. *If (X, f) is transitive then it is sensitive⁴, so that a transitive map that is not sensitive is a homeomorphism (see [10, pp .335]).*

7. Eventual sensitivity. The following definition was motivated by the following thought: sensitivity means that, no matter where you start, there are two points arbitrarily close to each other and to that starting location which will move far apart as time progresses; a universal ‘far’. Clearly this is extremely relevant in an applied setting; rounding errors mean a computer will not, generally, track true orbits. But what if every point moves arbitrarily close to another point that it will then move away from? What if a computer starts with a true orbit and tracks it accurately, but then the point moves close to another point which will end up going in completely the other direction? - these two points may be so close together that the computer cannot differentiate between them; it may start tracking the wrong orbit and give an extremely inaccurate prediction of the future.

Definition 7.1. We say a metric dynamical system (X, f) is *eventually sensitive* if there exists $\delta > 0$ such that for any $x \in X$ and any $\epsilon > 0$ there exist $n, k \in \mathbb{N}_0$ and $y \in B_\epsilon(f^n(x))$ such that $d(f^{n+k}(x), f^k(y)) \geq \delta$. We refer to such a δ as an eventual-sensitivity constant.

If X is a compact Hausdorff space, we say that (X, f) is eventually sensitive if there exists $D \in \mathcal{D}$ such that for any $x \in X$ and any $E \in \mathcal{D}$ there exist $n, k \in \mathbb{N}_0$ and $y \in E[f^n(x)]$ such that $(f^{n+k}(x), f^k(y)) \notin D$. We refer to such a D as an eventual-sensitivity entourage.

Clearly a system which is sensitive is also eventually sensitive; just take $n = 0$ in the above definition. The variable n is something that needs to be taken into account in an applied setting (and clearly it may depend on one’s starting point); if the least such n is large, then the computer may provide an accurate model of the reasonably distant future. However, if the least such n is small, or 0 as in the case of sensitivity, the orbit the computer is attempting to track may quickly diverge from what the computer predicts. The example below is an example of an eventually sensitive but non-sensitive system.

⁴This result holds because saying for all nonempty open pairs $N(U, V) \neq \emptyset$ is equivalent to saying $N(U, V)$ is infinite for all such pairs.

Example 4. Let $X = [0, 1]$. Define a map $f: X \rightarrow X$ by

$$f(x) = \begin{cases} 2x & \text{if } x \in [0, 1/4], \\ 1 - 2x & \text{if } x \in (1/4, 1/2], \\ 10x/3 - 5/3 & \text{if } x \in (1/2, 3/5], \\ 1/3 & \text{if } x \in (3/5, 4/5], \\ 10x/3 - 7/3 & \text{if } x \in (4/5, 1]. \end{cases}$$

Then $f: X \rightarrow X$, depicted in Figure 1, is a continuous surjection which is eventually sensitive but not sensitive.

The point $3/4$ has a neighbourhood on which the map is constant, so that f is not sensitive. However, it is eventually sensitive. To see this, notice that every point in $[0, 1)$ is eventually mapped into $[0, 1/2]$, where the map is simply a copy of the tent map, which is sensitive (indeed, it is cofinitely so). On the other hand, 1 is a fixed point $f(1) = 1$, which is of a fixed distance $1/2$ from the interval $[0, 1/2]$.

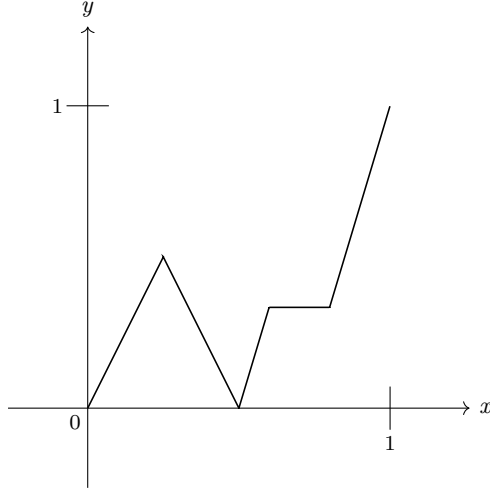


FIGURE 1. A non-sensitive, eventually-sensitive system

For transitive dynamical systems we prove the following dichotomy.

Theorem 7.2. (*Generalised Auslander-Yorke dichotomy II*) *Let X be a compact Hausdorff space. A transitive dynamical system (X, f) is either equicontinuous or eventually sensitive. Specifically, it is eventually sensitive if and only if it is not equicontinuous.*

Proof. Suppose first that the system is not equicontinuous. Suppose the system has a dense set of minimal points. If the system is minimal then it is sensitive (see [4, Corollary 2] or Corollary 7) and the result follows. If it is non-minimal then it is sensitive (see [1, Theorem 2.5]) and therefore eventually sensitive. Now suppose the set of minimal points M is not dense in X . Let $q \in X$ and $D \in \mathcal{D}$ be symmetric such that $3D[q] \cap \overline{M} = \emptyset$. Let $z \in X$ be picked arbitrarily and let $E \in \mathcal{D}$ be given; without loss of generality $E \subset D$. Let $m \in \omega(z)$ be minimal. Then there exists $n \in \mathbb{N}$ such that $m \in E[f^n(z)]$. By transitivity, there exists $k \in \mathbb{N}$ such that $f^k(E[f^n(z)]) \cap D[q] \neq \emptyset$. Let $y \in E[f^n(z)]$ be such that $f^k(y) \in D[q]$. Then

$(f^k(y), f^k(m)) \notin 2D$ as $f^k(m)$ is minimal. Then either $(f^{n+k}(z), f^k(y)) \notin D$ or $(f^{n+k}(z), f^k(m)) \notin D$. Therefore (X, f) is eventually sensitive.

Now suppose that the system is eventually sensitive; let $D \in \mathcal{D}$ by an eventual-sensitivity entourage. Assume the system is equicontinuous. Since X is compact the system is uniformly equicontinuous. Let D_0 be such that for any $x, y \in X$ if $(x, y) \in D_0$ then for any $n \in \mathbb{N}$, $(f^n(x), f^n(y)) \in D$. Let $x \in X$ be given. By eventual sensitivity there exist $n, k \in \mathbb{N}_0$ and $y \in D_0[f^n(x)]$ such that $(f^{n+k}(x), f^k(y)) \notin D$; this contradicts our assumption that the system is equicontinuous. \square

We conclude this section with a theorem which is simply a collation of Auslander-Yorke type results for transitive dynamical systems on compact spaces. The results it collates are: Corollary 17, Corollary 20 (which is [10, Corollary 7.1.12]), Theorem 7.2, [1, Theorem 2.4] and [4, Corollary 2].

Theorem 7.3. *Let X be a compact Hausdorff space and $f: X \rightarrow X$ be a continuous function. If f is transitive then exactly one of the following holds:*

1. f is sensitive.
2. f is a non-sensitive, uniformly rigid homeomorphism, and exactly one of the following holds:
 - (a) There exists a transitive point and either
 - (i) f is equicontinuous and minimal; or,
 - (ii) f is eventually sensitive and $\text{Eq}(X, f) = \text{Trans}(f) \neq \emptyset$.
 - (b) There is no transitive point, so $\text{Eq}(X, f) = \emptyset$, and f is eventually sensitive.

We close with the following question suggested by an anonymous referee, which we believe to be open:

Question 3. Does there exist a transitive compact Hausdorff system which is neither sensitive nor has any transitive points? Equivalently, is statement 2b) in Theorem 7.3 redundant?

8. Proof of Theorem 4.5. Recursively define the finite words C_n as follows. Let $C_0 := 10$ and, for all $n \geq 1$, take

$$C_n := 1^{8^n |C_0 C_1 \dots C_{n-1}|} 0^{2^n |C_0 C_1 \dots C_{n-1}|}.$$

For each $n \geq 1$ define

$$Q_n := 0^{8^n |C_0 C_1 \dots C_{n-1}|} 0^{2^n |C_0 C_1 \dots C_{n-1}|}.$$

Let $W_0 := C_0 Q_1$ and for each $n \geq 1$ let $W_n := W_0 W_1 \dots W_{n-1} C_0 C_1 \dots C_n Q_{n+1}$ (so $W_1 = W_0 C_0 C_1 Q_2$, $W_2 = W_0 W_1 C_0 C_1 C_2 Q_3$ and so on).

The first $8^n |C_0 C_1 \dots C_{n-1}|$ symbols of C_n will be referred to as the 1-part of C_n . Similarly, the last $2^n |C_0 C_1 \dots C_{n-1}|$ symbols of C_n will be referred to as the 0-part of C_n . We will refer to the word $C_0 \dots C_n Q_{n+1}$ as the *closing segment* of W_n .

Remark 12. For any $n \in \mathbb{N}$, $|C_n| = |Q_n|$. We emphasise that Q_n consists solely of 0's.

To prove Theorem 4.5 we will first need to prove the following lemma concerning the length of various words in our system.

Lemma 8.1. *For any $n \in \mathbb{N}_0$,*

$$6 \left(8^{n+1} |C_{n+1}| \right) \geq |W_0 W_1 \dots W_n C_0 \dots C_{n+1}| + 2 |W_n|. \quad (11)$$

Proof. Let $P(n)$ be the statement

$$6 \left(8^{n+1} |C_{n+1}| \right) \geq |W_0 W_1 \dots W_n C_0 \dots C_{n+1}| + 2|W_n|.$$

Case when $n = 0$. Then $6(8^1|C_1|) = 960$ whilst $|W_0 C_0 C_1| + 2|W_0| = 88$. Hence $P(0)$ holds.

Assume that $P(n)$ is true for all $n \leq k$ for some $k \in \mathbb{N}_0$. Will will prove $P(k+1)$ holds. For $P(k+1)$:

$$\begin{aligned} \text{RHS} &= |W_0 W_1 \dots W_k W_{k+1} C_0 \dots C_{k+1} C_{k+2}| + 2|W_{k+1}| \\ &= |W_0 W_1 \dots W_k C_0 \dots C_{k+1}| + 3|W_{k+1}| + |C_{k+2}| \\ &= 4|W_0 W_1 \dots W_k C_0 \dots C_{k+1}| + 4|Q_{k+2}| \quad \text{as } |Q_{k+2}| = |C_{k+2}| \\ &= 4|W_0 W_1 \dots W_k C_0 \dots C_{k+1}| + 4 \left(8^{k+2} |C_0 \dots C_{k+1}| \right) \\ &\quad + 4 \left(2^{k+2} |C_0 \dots C_{k+1}| \right) \\ &\leq 4|W_0 W_1 \dots W_k C_0 \dots C_{k+1}| \\ &\quad + 4 \left(8^{k+2} |W_0 W_1 \dots W_k C_0 \dots C_{k+1}| \right) \\ &\quad + 4 \left(2^{k+2} |W_0 W_1 \dots W_k C_0 \dots C_{k+1}| \right) \\ &\leq 6 \left(8^{k+2} |W_0 W_1 \dots W_k C_0 \dots C_{k+1}| \right) \quad \text{as } 2(8^{k+2}) \geq 4 + 4(2^{k+2}) \\ &\leq 6 \left(8^{k+2} (6(8^{k+1} |C_{k+1}|)) \right) \quad \text{by the induction hypothesis} \\ &\leq 6 \left(8^{k+2} (8^{k+2} |C_{k+1}|) \right) \\ &\leq 6 \left(8^{k+2} |C_{k+2}| \right) \quad \text{by definition} \\ &= \text{LHS.} \quad \square \end{aligned}$$

Remark 13. The length of the 1-part of C_{n+2} is $8^{n+2}|C_0 \dots C_{n+1}|$. Notice that,

$$8^{n+2}|C_0 \dots C_{n+1}| \geq 6 \left(8^{n+1} |C_{n+1}| \right) + 2 \left(8^{n+1} |C_{n+1}| \right) > 6 \left(8^{n+1} |C_{n+1}| \right).$$

The final line is the LHS of Equation 11. By Lemma 8.1 this then means that the length of the 1-part of C_{n+1} is more than $2 \left(8^{n+1} |C_{n+1}| \right)$ greater than the length of $W_0 W_1 \dots W_n C_0 \dots C_{n+1}$ plus two times the length of W_n . This observation will prove important later.

Corollary 21. For any $n, k \in \mathbb{N}_0$,

$$6 \left(8^{n+1+k} |C_{n+1+k}| \right) \geq |W_0 W_1 \dots W_{n+k} C_0 \dots C_{n+1+k}| + \sum_{i=0}^{k-1} |W_{n+1+i}|. \quad (12)$$

Proof. Immediate from Lemma 8.1. \square

We now define a shift system (X, σ) as follows. Let $\mathbf{x} := C_0 C_1 C_2 C_3 \dots$ and $\mathbf{y} := W_0 W_1 W_2 W_3 \dots$. Using the shift map σ take

$$X = \overline{\text{Orb}_\sigma(\mathbf{x}) \cup \text{Orb}_\sigma(\mathbf{y}) \cup \{0^n \mathbf{x}, 0^n \mathbf{y} \mid n \in \mathbb{N}\}},$$

where the closure is taken with regard to the full shift $\Sigma^{\mathbb{N}_0}$.

The point \mathbf{y} is transitive in the system (X, σ) . Notice that $0^\infty, 10^\infty \in X$ since they are in $\omega(\mathbf{x})$. We will show through a sequence of lemmas that the system (X, σ) satisfies the conditions in the theorem, in particular we will show $(\mathbf{x}, 0^\infty)$ is a non-trivial even continuity pair but $(\mathbf{x}, 10^\infty) \notin \text{EvP}(X, \sigma)$.

When working with dynamical systems, it can be helpful to visualise the forward orbit of a point as how it moves through time. In proving our claim we will use language like, ‘the first time \mathbf{x} visits $U \subseteq X$ ’ or ‘when \mathbf{x} enters U for the first time.’ By such statements we mean, the least such $c \in \mathbb{N}_0$ such that $\sigma^c(\mathbf{x}) \in U$. In similar fashion, we may speak of points travelling through words. For example, ‘When \mathbf{x} enters the 0-part of C_1 for the first time, \mathbf{y} is travelling through W_0 for the first time; more specifically, \mathbf{y} is travelling through the Q_1 -part of Q_1C_0 for the first time.’ This means that, if t is such that $\sigma^t(\mathbf{x})$ is in the 0-part of C_1 (i.e. $[0^4]$) for the first time, then there exists a unique $a \leq t$ such that $\sigma^a(\mathbf{y}) \in W_0$ and $t - a < |W_0|$. Similarly there exists a unique $b \leq t$ such that $\sigma^b(\mathbf{y}) \in Q_1C_0$ and $t - b < |Q_1|$. In this particular example it can be seen that $t = 18$, $a = 0$ and $b = 2$.

We introduce the following, *first-hitting time*, notation. For $\mathbf{w} \in X$ and $A \subseteq X$ such that $N(\mathbf{w}, A) \neq \emptyset$,

$$\tau(\mathbf{w}, A) := \min N(\mathbf{w}, A).$$

For example, $\tau(\mathbf{x}, [C_2]) = 22$ whilst $\tau(\mathbf{y}, [Q_1C_0]) = 2$. This allows us to translate long-winded sentences such as ‘ \mathbf{y} enters $[Q_1C_0]$ for the first time before \mathbf{x} enters $[C_2]$ for the first time’ into an equation, in this example:

$$\tau(\mathbf{y}, [Q_1C_0]) < \tau(\mathbf{x}, [C_2]).$$

Lemma 8.2. $(\mathbf{x}, 10^\infty) \notin \text{EvP}(X, \sigma)$.

Proof. Let $O = [10]$; we claim this is an even-splitting neighbourhood of 10^∞ with regard to \mathbf{x} . Let U and V be neighbourhoods of \mathbf{x} and 10^∞ respectively. Without loss of generality write $U = [C_0C_1C_2 \dots C_m]$ and $V = [10^l]$, where $m \geq l \geq 1$. There exists a point $\mathbf{p} \in \text{Orb}(\mathbf{y})$ such that $\mathbf{p} \in [C_0C_1C_2 \dots C_mQ_{m+1}]$. Define $t := |C_0C_1C_2 \dots C_m|$ and note that $\sigma^t(\mathbf{x}) \in [C_{m+1}]$, $\sigma^t(\mathbf{p}) \in [Q_{m+1}]$. Let $k = 8^{m+1}|C_0C_1C_2 \dots C_m|$; this is the length of the 1-part of C_{m+1} . It follows that

$$\sigma^{t+k-1}(\mathbf{x}) \in V,$$

and

$$\sigma^{t+k-1}(\mathbf{p}) \in [0^{2^{m+1}}].$$

Hence $\sigma^{t+k-1}(\mathbf{p}) \notin O$. Since U and V were picked arbitrarily this means $(\mathbf{x}, 10^\infty) \notin \text{EvP}(X, \sigma)$. In particular $\mathbf{x} \notin \text{Ev}(X, \sigma)$. □

Lemma 8.3. *There are no points of even continuity.*

Proof. To see that $\text{Ev}(X, \sigma) = \emptyset$, note that, since X is compact, $\text{Ev}(X, \sigma) = \text{Eq}(X, \sigma)$ (see [19, Theorem 7.23]). But since (X, σ) is a shift space with no isolated points it is sensitive, hence $\text{Eq}(X, \sigma) = \emptyset$. □

We will now set about showing that $(\mathbf{x}, 0^\infty)$ is a non-trivial even continuity pair. To do this we will need the following lemma.

Lemma 8.4. *Let $n, a \in \mathbb{N}$ be such that $C_0 \dots C_n Q_{n+1}$ is an initial segment of $\mathbf{z} = \sigma^a(\mathbf{y})$. Then for any $k > n$,*

$$\tau(x, [C_k]) \leq \tau(z, [Q_k C_0]) \leq 6 \left(8^{k-1} |C_{k-1}| \right)$$

In words, the first inequality means that \mathbf{x} enters $[C_k]$ for the first time no later than \mathbf{z} enters $[Q_k C_0]$ for the first time - which itself has happened by *time* “ $6(8^{k-1}|C_{k-1}|)$ ” by the second inequality.

The final three inequalities emphasise that \mathbf{z} enters $[Q_k C_0]$ for the first time before \mathbf{x} enters the 0-part of C_k for the first time; in particular when \mathbf{z} enters $[Q_k C_0]$ for the first time \mathbf{x} still has to travel through at least $2(8^{k-1}|C_{k-1}|)$ more 1's in the 1-part of C_k before it enters the 0-part of C_k .

Proof. Let

$$n_0 = \max\{c \in \mathbb{N} \mid \exists b < a : \sigma^b(\mathbf{y}) \in [W_c]\}.$$

Note that n_0 is well defined and that $n_0 \geq n$. This means that \mathbf{z} is travelling through W_{n_0} for the first time.

Let $k > n$ be given. The first inequality follows immediately from the construction: The word $Q_k C_0$ appears in the sequence of \mathbf{z} for the first time only after the word $C_0 \dots C_{k-1}$. Similarly the word C_k appears in the sequence of \mathbf{x} for the first time exactly after the word $C_0 \dots C_{k-1}$. Observing that $\mathbf{x} = C_0 C_1 \dots C_k C_{k+1} \dots$ now gives the inequality, $\tau(\mathbf{x}, [C_k]) \leq \tau(\mathbf{z}, [Q_k C_0])$. It remains to show that the second inequality holds.

Let $\mathbf{z}' \in \text{Orb}(\mathbf{y})$ be the point at which \mathbf{y} first enters $C_0 \dots C_n Q_{n+1}$; i.e. $\mathbf{z}' = \sigma^m(\mathbf{y})$ where $m = \tau(\mathbf{y}, [C_0 \dots C_n Q_{n+1}])$. Note that \mathbf{z}' lies at the start of the closing segment of W_n . Indeed,

$$\mathbf{z}' = C_0 \dots C_n Q_{n+1} W_{n+1} W_{n+2} W_{n+3} \dots$$

It is not difficult to see that $\tau(\mathbf{z}, [Q_k C_0]) \leq \tau(\mathbf{z}', [Q_k C_0])$; it takes \mathbf{z}' at least as long to enter $[Q_k C_0]$ for the first time as it does for \mathbf{z} to enter $[Q_k C_0]$ for the first time. (Observe that the letters (counting multiplicities) appearing in \mathbf{z} before the first appearance $Q_k C_0$ can be written as a list of words (including multiplicities) which also appear in \mathbf{z}' (with multiplicities) before the first appearance of $Q_k C_0$ there. Hence the initial segment of \mathbf{z}' up to the first appearance of $Q_k C_0$ is longer than that of the initial segment of \mathbf{z} up to the first appearance of $Q_k C_0$. We know $k > n$. First suppose that $k > n + 1$. Then, by construction,

$$\begin{aligned} \tau(\mathbf{z}', [Q_k C_0]) &= |C_0 \dots C_n Q_{n+1}| + \left(\sum_{i=n+1}^{k-2} |W_i| \right) + |W_0 \dots W_{k-2} C_0 \dots C_{k-1}| \\ &\leq |W_0 \dots W_{k-2} C_0 \dots C_{k-1}| + 2|W_{k-2}| \\ &\leq 6(8^{k-1}|C_{k-1}|) \end{aligned} \quad \text{by Lemma 8.1.}$$

Since $\tau(\mathbf{z}', [Q_k C_0]) \leq 6(8^{k-1}|C_{k-1}|)$, and $\tau(\mathbf{z}, [Q_k C_0]) \leq \tau(\mathbf{z}', [Q_k C_0])$, we have that

$$\tau(\mathbf{z}, [Q_k C_0]) \leq 6(8^{k-1}|C_{k-1}|).$$

Now suppose that $k = n + 1$. Then by Lemma 8.1

$$\begin{aligned} \tau(\mathbf{z}', [Q_k C_0]) &= \tau(\mathbf{x}, [C_k]) \\ &= |C_0 \dots C_{k-1}| \\ &\leq 6(8^{k-1}|C_{k-1}|). \end{aligned} \quad \square$$

Corollary 22. *Let $n \in \mathbb{N}$ be such that $C_0 \dots C_n Q_{n+1}$ is an initial segment of $\mathbf{z} = \sigma^a(\mathbf{y})$ for some $a \in \mathbb{N}_0$. For any $k > n$, \mathbf{x} enters $[C_k]$ for the first time no later than \mathbf{z} enters $[Q_k C_0]$ for the first time. Additionally, \mathbf{z} enters $[Q_k C_0]$ for the first time before \mathbf{x} enters the 0-part of C_k for the first time. In symbols:*

$$\tau(\mathbf{x}, [C_k]) \leq \tau(\mathbf{z}, [Q_k C_0]) \leq \tau\left(\mathbf{x}, \left[0^{2^k |C_0 \dots C_{k-1}|}\right]\right).$$

Proof. By Lemma 8.4 it will suffice to show $6(8^{k-1}|C_{k-1}|) \leq \tau\left(\mathbf{x}, \left[0^{2^k |C_0 \dots C_{k-1}|}\right]\right)$.

Notice that \mathbf{x} has to travel through the 1-part of C_k before reaching $\left[0^{2^k |C_0 \dots C_{k-1}|}\right]$. The length of the 1-part of C_k is $8^k |C_0 \dots C_{k-1}| > 6(8^{k-1}|C_{k-1}|)$. \square

Corollary 23. *The ordered pair $(\mathbf{x}, 0^\infty)$ is a non-trivial even continuity pair.*

Proof. Since $0^\infty \in \omega(\mathbf{x})$, by definition $(\mathbf{x}, 0^\infty)$ is not a trivial even continuity pair. It thus suffices to show that $(\mathbf{x}, 0^\infty) \in \text{EvP}(X, \sigma)$, i.e.

$$\forall O \in \mathcal{N}_{0^\infty} \exists U \in \mathcal{N}_{\mathbf{x}} \exists V \in \mathcal{N}_{0^\infty} : \forall n \in \mathbb{N}, \sigma^n(\mathbf{x}) \in V \implies \sigma^n(U) \subseteq O.$$

Without loss of generality, let O be the basic open neighbourhood $[0^n]$ of 0^∞ . We claim $U = [C_0 C_1 \dots C_n]$ and $V = [0^n]$ satisfy the even continuity condition. Since $\text{Orb}(\mathbf{y})$ is dense and O, U and V are clopen, it suffices to consider only points in U which are elements of the orbit of \mathbf{y} . Let $\mathbf{z} \in \text{Orb}(\mathbf{y}) \cap U$. Then $\mathbf{z} \in [C_0 \dots C_m Q_{m+1}]$ for some $m \geq n$. Suppose $l \in \mathbb{N}$ is such that $\sigma^l(\mathbf{x}) \in V$.

Case 1: $l \geq \tau(\mathbf{x}, [C_{m+1}])$. Let $k \geq m+1$ be the greatest integer such that $l \geq \tau(\mathbf{x}, [C_k])$. It follows that at time l , \mathbf{x} is travelling through the 0-part of C_k for the first time, with at least n 0's left to travel through. Furthermore, since $\tau(\mathbf{x}, [C_k]) \leq \tau(\mathbf{z}, [Q_k C_0])$, and as $|Q_k| = |C_k|$, we have that \mathbf{x} finishes travelling through C_k before \mathbf{z} finishes travelling through the Q_k -part of $Q_k C_0$. This means that at time l there are at least as many 0's remaining in Q_k (recall, Q_k consists solely of 0's) for \mathbf{z} to travel through than there are 0's remaining in C_k for \mathbf{x} to travel through. Since there are at least n 0's left in C_k for \mathbf{x} still to travel through (as $\mathbf{x} \in V$), it follows that $\mathbf{z} \in [0^n] = O$.

Case 2: $l < \tau(\mathbf{x}, [C_{m+1}])$. The initial segments of \mathbf{x} and \mathbf{z} are identical up to and including the first occurrence of C_{m+1} . The word C_{m+1} begins with a '1', therefore $l \leq \tau(\mathbf{x}, [C_{m+1}]) - n$, because $\sigma^l(\mathbf{x}) \in V = [0^n]$. In particular, it follows that $\sigma^l(\mathbf{z}) \in [0^n] = O$. \square

We will now set about showing that $\text{EqP}(X, \sigma) = \emptyset$. This will be completed in Lemma 8.6. First we show that \mathbf{y} is not topologically equicontinuous with either one of the fixed points.

Lemma 8.5. *Neither $(\mathbf{y}, 0^\infty)$ nor $(\mathbf{y}, 1^\infty)$ is an equicontinuity pair.*

Proof. Recall that, to show that $(\mathbf{y}, \mathbf{p}) \notin \text{EqP}(X, \sigma)$, where $\mathbf{p} \in X$, we need to show that:

$$\exists O \in \mathcal{N}_{\mathbf{p}} : \forall U \in \mathcal{N}_{\mathbf{y}} \forall V \in \mathcal{N}_{\mathbf{p}} \exists n \in \mathbb{N} : \sigma^n(U) \cap V \neq \emptyset \text{ and } \sigma^n(U) \not\subseteq O.$$

Let $O = [0]$. We claim O is a splitting neighbourhood of 0^∞ with regard to \mathbf{y} . Let $U \in \mathcal{N}_{\mathbf{y}}$ and $V \in \mathcal{N}_{0^\infty}$ be given and let $[W_0 \dots W_n] \subseteq U$ and $[0^n] \subseteq V$. Let $m \in \mathbb{N}$ be such that 0^n appears as a subword of C_m ; notice that it follows that 0^n is a subword of both C_k and W_k for all $k \geq m$. Let $l = \max\{n+2, m+2\}$. Notice

that $2(8^{l-1}|C_{l-1}|) > n + 1$. Let $t = \tau(\mathbf{y}, [W_l])$ and write $\mathbf{z} = \sigma^t(\mathbf{y})$. It follows that $\mathbf{z} \in U$. It is worth comparing \mathbf{z} and \mathbf{y} side by side.

$$\mathbf{z} = W_0 W_1 \dots W_{l-1} C_0 C_1 \dots C_l Q_{l+1} W_{l+1} W_{l+2} \dots,$$

and

$$\mathbf{y} = W_0 W_1 \dots W_{l-1} W_0 W_1 \dots W_{l-1} C_0 C_1 \dots C_l Q_{l+1} W_{l+1} W_{l+2} \dots$$

Thus \mathbf{z} and \mathbf{y} share the same initial segment of $W_0 \dots W_{l-1}$. After this \mathbf{z} enters $[C_0 C_1 \dots C_l Q_{l+1}]$ for the first time whilst \mathbf{y} enters $[W_l]$ for the first time. By Lemma 8.1,

$$6 \left(8^{l-1} |C_{l-1}| \right) \geq |W_0 W_1 \dots W_{l-2} C_0 \dots C_{l-1}|. \quad (13)$$

In particular the length of the 1-part of C_l is greater than

$$|W_0 W_1 \dots W_{l-2} C_0 \dots C_{l-1}| + 2 \left(8^{l-1} |C_{l-1}| \right).$$

It follows that

$$\tau(\mathbf{y}, [Q_l C_0 C_1 \dots C_l]) \leq \tau\left(\mathbf{z}, \left[0^{2^l |C_0 \dots C_{l-1}|} Q_{l+1}\right]\right) - 2 \left(8^{l-1} |C_{l-1}| \right).$$

That is, \mathbf{y} enters $[Q_l C_0 C_1 \dots C_l]$ for the first time before \mathbf{z} enters the 0-part of $[C_l]$ for the first time; in particular when \mathbf{y} enters $[Q_l C_0 C_1 \dots C_l]$ for the first time \mathbf{z} still has to travel through at least $2 \left(8^{l-1} |C_{l-1}| \right)$ more 1's in the 1-part of C_l before it enters the 0-part of C_l . Since $\tau(\mathbf{z}, [C_l]) \leq \tau(\mathbf{y}, [Q_l C_0 C_1 \dots C_l])$ we get that

$$\sigma^{\tau(\mathbf{y}, [Q_l C_0 C_1 \dots C_l])}(\mathbf{y}) \in V$$

but

$$\sigma^{\tau(\mathbf{y}, [Q_l C_0 C_1 \dots C_l])}(\mathbf{z}) \in \left[\mathbf{1}^{2 \left(8^{l-1} |C_{l-1}| \right)} \right] \implies \sigma^{\tau(\mathbf{y}, [Q_l C_0 C_1 \dots C_l])}(\mathbf{z}) \notin O.$$

Hence $(\mathbf{y}, 0^\infty) \notin \text{EqP}(X, \sigma)$. (Indeed, we have actually shown the stronger claim that $(\mathbf{y}, 0^\infty) \notin \text{EvP}(X, \sigma)$.)

Now let $O = [1]$. We claim O is a splitting neighbourhood of 1^∞ with regard to \mathbf{y} . Let $U \in \mathcal{N}_{\mathbf{y}}$ and $V \in \mathcal{N}_{1^\infty}$. Let $[W_0 \dots W_n] \subseteq U$ and $[1^n] \subseteq V$. Let $m \in \mathbb{N}$ be such that 1^m appears as a subword of C_m ; notice that it follows that 1^m is a subword of both C_k and W_k for all $k \geq m$. Let $l = \max\{n + 2, m + 2\}$. Notice that $2(8^{l-1}|C_{l-1}|) > n + 1$. Let $t = \tau(\mathbf{y}, [W_l])$ and write $\mathbf{z} = \sigma^t(\mathbf{y})$. It follows that $\mathbf{z} \in U$. As before, \mathbf{z} and \mathbf{y} share the same initial segment of $W_0 \dots W_{l-1}$. After this \mathbf{z} enters $[C_0 C_1 \dots C_l Q_{l+1}]$ for the first time whilst \mathbf{y} enters $[W_l]$ for the first time. By an almost identical argument to the one we used in the previous paragraph, we know that

$$\sigma^{\tau(\mathbf{y}, [Q_l C_0 C_1 \dots C_l])}(\mathbf{z}) \in \left[\mathbf{1}^{2 \left(8^{l-1} |C_{l-1}| \right)} \right] \subseteq V.$$

However

$$\sigma^{\tau(\mathbf{y}, [Q_l C_0 C_1 \dots C_l])}(\mathbf{y}) \in [0] \subseteq X \setminus O.$$

Hence $(\mathbf{y}, 1^\infty) \notin \text{EqP}(X, \sigma)$. \square

Lemma 8.6. *The system (X, σ) has no equicontinuity pairs.*

Proof. By Remark 5 it will suffice to show that $(\mathbf{y}, \mathbf{p}) \notin \text{EqP}(X, \sigma)$ for any $\mathbf{p} \in X$. We need to show that, for any $p \in X$,

$$\exists O \in \mathcal{N}_{\mathbf{p}} : \forall U \in \mathcal{N}_{\mathbf{y}} \forall V \in \mathcal{N}_{\mathbf{p}} \exists n \in \mathbb{N} : \sigma^n(U) \cap V \neq \emptyset \text{ and } \sigma^n(U) \not\subseteq O.$$

Suppose that $(\mathbf{y}, \mathbf{p}) \in \text{EqP}(X, \sigma)$; write $\mathbf{p} = \mathbf{p}_0\mathbf{p}_1\mathbf{p}_2\dots$. By Lemma 8.5 we have that $\mathbf{p} \notin \{0^\infty, 1^\infty\}$. This means that there exist $i, j \in \mathbb{N}_0$ such that $\mathbf{p}_i = 0$ and $\mathbf{p}_j = 1$. Fix such an i and a j and take $k \geq \max\{i, j\}$. Let $O = [\mathbf{p}_0\mathbf{p}_1\dots\mathbf{p}_k]$. We claim O is a splitting neighbourhood of \mathbf{p} with regard to \mathbf{y} . Let $U \in \mathcal{N}_{\mathbf{y}}$ and $V \in \mathcal{N}_{\mathbf{p}}$ be given and let $[W_0\dots W_n] \subseteq U$ and $[\mathbf{p}_0\mathbf{p}_1\dots\mathbf{p}_n] \subseteq V$; without loss of generality $n \geq k$. Let $m \in \mathbb{N}$ be such that $\mathbf{p}_0\mathbf{p}_1\dots\mathbf{p}_n$ appears as a subword of W_m ; notice that it follows that $\mathbf{p}_0\mathbf{p}_1\dots\mathbf{p}_n$ is a subword of W_a for all $a \geq m$. Let $l \geq \max\{n+2, m+2\}$ be such that $2(8^{l-1}|C_{l-1}|) > n+1$. Let $t = \tau(\mathbf{y}, [W_l])$ and write $\mathbf{z} = \sigma^t(\mathbf{y})$. It follows that $\mathbf{z} \in U$. It is worth comparing \mathbf{z} and \mathbf{y} side by side.

$$\mathbf{z} = W_0W_1\dots W_{l-1}C_0C_1\dots C_lQ_{l+1}W_{l+1}W_{l+2}\dots,$$

and

$$\mathbf{y} = W_0W_1\dots W_{l-1}W_0W_1\dots W_{l-1}C_0C_1\dots C_lQ_{l+1}W_{l+1}W_{l+2}\dots$$

Notice \mathbf{z} and \mathbf{y} share the same initial segment given by $W_0\dots W_{l-1}$. After this \mathbf{z} enters $[C_0C_1\dots C_lQ_{l+1}]$ for the first time whilst \mathbf{y} enters $[W_l]$ for the first time. Notice that, for all $i \in \mathbb{N}_0$, $|W_i| \geq |Q_{i+1}| = |C_{i+1}|$. In addition $|W_0| \geq |C_0C_1|$. It follows that

$$\tau(\mathbf{z}, [C_l]) \leq \tau(\mathbf{y}, [W_{l-1}C_0\dots C_lQ_{l+1}]). \quad (14)$$

Observe,

$$\tau(\mathbf{y}, [Q_lC_0\dots C_lQ_{l+1}]) = |W_0\dots W_{l-1}| + |W_0\dots W_{l-2}| + |W_0\dots W_{l-2}C_0\dots C_{l-1}|.$$

Similarly observe

$$\tau(\mathbf{z}, [C_l]) = |W_0\dots W_{l-1}| + |C_0\dots C_{l-1}|.$$

Therefore,

$$\begin{aligned} \tau(\mathbf{y}, [Q_lC_0\dots C_lQ_{l+1}]) - \tau(\mathbf{z}, [C_l]) &= 2|W_0\dots W_{l-2}|, \\ &\leq |W_0W_1\dots W_{l-2}| + 2|W_{l-2}|, \\ &\leq 6\left(8^{l-1}|C_{l-1}|\right) \quad \text{by Lemma 8.1.} \end{aligned}$$

Thus

$$\tau(\mathbf{z}, [C_l]) + 6\left(8^{l-1}|C_{l-1}|\right) \geq \tau(\mathbf{y}, [Q_lC_0\dots C_lQ_{l+1}]). \quad (15)$$

Putting inequalities (14) and (15) together we obtain:

$$\begin{aligned} \tau(\mathbf{z}, [C_l]) &\leq \tau(\mathbf{y}, [W_{l-1}C_0\dots C_lQ_{l+1}]) \\ &\leq \tau(\mathbf{y}, [Q_lC_0\dots C_lQ_{l+1}]) \\ &\leq \tau(\mathbf{z}, [C_l]) + 6\left(8^{l-1}|C_{l-1}|\right) \\ &\leq \tau\left(\mathbf{z}, \left[0^{2^l|C_0\dots C_{l-1}}|Q_{l+1}\right]\right) - 2\left(8^{l-1}|C_{l-1}|\right). \end{aligned}$$

The final inequality follows because, by definition, the length of the 1-part of C_l is more than $8^l|C_{l-1}|$. It follows that, whilst \mathbf{y} enters $W_0W_1\dots W_{l-2}C_0\dots C_{l-1}$ for the second time, \mathbf{z} is travelling through the 1-part of C_l . When \mathbf{y} finishes travelling through $W_0W_1\dots W_{l-2}C_0\dots C_{l-1}$ for the second time (and enters $[Q_lC_0\dots C_lQ_{l+1}]$ for the first time), \mathbf{z} still has to travel through at least $2\left(8^{l-1}|C_{l-1}|\right)$ more 1's in

the 1-part of C_l before it enters the 0-part of C_l . Because $\mathbf{p}_0 \dots \mathbf{p}_n$ is a subword of W_{l-2} , which is a subword of $W_0 W_1 \dots W_{l-2} C_0 \dots C_{l-1}$, and since $[\mathbf{p}_0 \mathbf{p}_1 \dots \mathbf{p}_n] \subseteq V$ it follows that \mathbf{y} enters V whilst travelling through $W_0 W_1 \dots W_{l-2} C_0 \dots C_{l-1}$ for the second time. Take $c \in \mathbb{N}_0$ such that $\sigma^c(\mathbf{y}) \in V$ where $c > \tau(\mathbf{y}, [W_l])$ and $c < \tau(\mathbf{y}, [Q_l C_0 \dots C_l Q_{l+1}])$. Since $2(8^{l-1}|C_{l-1}|) > n + 1$ it follows that $\sigma^c(\mathbf{z}) \in [1^{n+1}]$. But the word inducing O (i.e. $\mathbf{p}_0 \dots \mathbf{p}_k$) contains at least one 0 and $n + 1 \geq k + 1$. Hence $\sigma^c(\mathbf{z}) \notin O$; in particular $\sigma^c(U) \cap V \neq \emptyset$ and $\sigma^c(U) \not\subseteq O$. \square

Lemma 8.7. *The dynamical system (X, σ) is Auslander-Yorke chaotic but not Devaney chaotic.*

Proof. The system is both transitive and sensitive, this means it is Auslander-Yorke chaotic. It may be verified that the only periodic points are 0^∞ and 1^∞ , hence the system is not Devaney chaotic. \square

Lemma 8.8. *The system (X, σ) is both strongly and densely Li-Yorke chaotic.*

Proof. Huang and Ye [18] show that a compact metric system without isolated points is both strongly and densely Li-Yorke chaotic if the system is transitive and there is a fixed point. (See also [10, Corollary 7.3.7].) Since our system satisfies these conditions the result follows. \square

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