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TWO BIFURCATION SETS ARISING FROM THE BETA TRANSFORMATION WITH A HOLE AT 0

SIMON BAKER AND DERONG KONG

ABSTRACT. Given $\beta \in (1,2]$, the $\beta$-transformation $T_\beta : x \mapsto \beta x \pmod{1}$ on the circle $[0,1)$ with a hole $[0,t)$ was investigated by Kalle et al. (2019). They described the set-valued bifurcation set

$$E_\beta := \{ t \in [0,1) : K_\beta(t') \neq K_\beta(t) \ \forall t' > t \},$$

where $K_\beta(t) := \{ x \in [0,1) : T_n^\beta(x) \geq t \ \forall n \geq 0 \}$ is the survivor set. In this paper we investigate the dimension bifurcation set

$$B_\beta := \{ t \in [0,1) : \dim H K_\beta(t') \neq \dim H K_\beta(t) \ \forall t' > t \},$$

where $\dim H$ denotes the Hausdorff dimension. We show that if $\beta \in (1,2]$ is a multinacci number then the two bifurcation sets $B_\beta$ and $E_\beta$ coincide. Moreover we give a complete characterization of these two sets. As a corollary of our main result we prove that for $\beta$ a multinacci number we have $\dim H (E_\beta \cap [t,1)) = \dim H K_\beta(t)$ for any $t \in [0,1)$. This confirms a conjecture of Kalle et al. for $\beta$ a multinacci number.

1. Introduction

Given $\beta \in (1,2]$, the $\beta$-transformation $T_\beta$ on the circle $\mathbb{R}/\mathbb{Z} \sim [0,1)$ is defined by

$$T_\beta : [0,1) \to [0,1); \quad x \mapsto \beta x \pmod{1}.$$  

Following the pioneering work of Rényi [11] and Parry [9] there has been a great interest in the study of $T_\beta$. In general, the system $\Phi_\beta = ([0,1), T_\beta)$ does not admit a Markov partition (cf. [12]), this makes describing the dynamics of $\Phi_\beta$ more challenging.

When $\beta = 2$, Urbański considered in [14, 15] the open dynamical system under the doubling map $T_2$ with a hole at zero. More precisely, for $t \in (0,1)$ let

$$K_2(t) := \{ x \in [0,1) : T_n^2(x) \geq t \ \forall n \geq 0 \}.$$  

Here we use a slightly different definition of $K_2(t)$ from that by Urbański. By [14, Theorem 1 and Corollary 1] it follows that the dimension function $t \mapsto \eta_2(t) := \dim H K_2(t)$ is a Devil’s staircase on $[0,1)$, that is (i) $\eta_2$ is decreasing and continuous on $[0,1)$; (ii) $\eta_2$ is locally constant almost everywhere on $[0,1)$; and (iii) $\eta_2$ is not constant on $[0,1)$. Here and throughout the paper $\dim H$ denotes the Hausdorff dimension. Moreover, Urbański investigated the bifurcation sets

$$E_2 := \{ t \in [0,1) : K_2(t') \neq K_2(t) \ \forall t' > t \} \quad \text{and} \quad B_2 := \{ t \in [0,1) : \eta_2(t') \neq \eta_2(t) \ \forall t' > t \}.$$  

Clearly, $B_2 \subseteq E_2$. It can be easily deduced from the proof of Theorem 1 in [14] that $B_2 = E_2$, and its topological closure $\overline{B}_2$ is a Cantor set, i.e., a non-empty compact set that has neither
isolated nor interior points. Furthermore, the following local dimension property was shown to hold: \( \lim_{r \to 0} \dim_H(E_{\beta} \cap (t-r, t+r)) = \eta_2(t) \) for all \( t \in E_{\beta} \). Recently, Carminati and Tiozzo in [1] showed that the local Hölder exponent of the dimension function \( \eta_2 \) at any \( t \in E_{\beta} \) equals \( \eta_2(t) \).

Inspired by the work of Urbański [14, 15], Kalle et al. in [6] considered the analogous problem for the \( \beta \)-transformation with a hole \([0, t)\). More precisely, for \( t \in [0, 1) \) they investigated the survivor set

\[
K_{\beta}(t) := \{ x \in [0, 1) : T_{\beta}^n(x) \geq t \ \forall \ n \geq 0 \},
\]

and showed that the dimension function \( t \mapsto \dim_H K_{\beta}(t) \) is also a Devil’s staircase on \([0, 1)\). Furthermore, they characterized the set-valued bifurcation set

\[
E_{\beta} := \{ t \in [0, 1) : K_{\beta}(t') \neq K_{\beta}(t) \ \forall \ t' > t \},
\]

and proved that \( E_{\beta} \) is a Lebesgue null set of full Hausdorff dimension for any \( \beta \in (1, 2) \). Note that the bifurcation set \( E_{\beta} \) defined here coincides with the set

\[
E_{\beta}^n := \{ t \in [0, 1) : T_{\beta}^n(t) \geq t \ \forall n \geq 0 \}
\]

in [6]. Interestingly, they showed that \( E_{\beta} \) contains infinitely many isolated points for Lebesgue almost every \( \beta \in (1, 2) \). This is in contrast to the case where \( \beta = 2 \) and \( E_{\beta} \) has no isolated points. For \( \beta \)-transformation with an arbitrary hole we refer to the work of Clark [2]. We also mention that the study of bifurcation sets plays an important role in one-dimensional dynamics (cf. [5]).

Since for each \( \beta \in (1, 2) \) the dimension function \( \eta_{\beta} : t \mapsto \dim_H K_{\beta}(t) \) is a Devil’s staircase, it is natural to consider the dimension bifurcation set

\[
\mathcal{B}_{\beta} := \{ t \in [0, 1) : \eta_{\beta}(t') \neq \eta_{\beta}(t) \ \forall t' > t \} .
\]

This set records those \( t \) for which the dimension function \( \eta_{\beta} \) has a ‘change’ within any right neighborhood. Since \( \eta_{\beta} \) is continuous, \( \mathcal{B}_{\beta} \) cannot have isolated points. On the other hand, the set-valued bifurcation set \( E_{\beta} \) contains (infinitely many) isolated points for Lebesgue almost every \( \beta \in (1, 2) \). So in general we cannot expect the coincidence of the two bifurcation sets \( \mathcal{B}_{\beta} \) and \( E_{\beta} \). That being said, in this paper we show that if \( \beta \) is a multinacci number, i.e., the unique root in \((1, 2)\) of the equation

\[
x^m+1 = x^m + x^{m-1} + \cdots + x + 1
\]

for some \( m \in \mathbb{N} \), then the two bifurcation sets indeed coincide. Importantly, if \( \beta \) is a multinacci number then its quasi-greedy expansion of 1 is of the form \(((1^m0)^\infty)\). This property will be useful in our analysis. Here for \( \beta \in (1, 2) \) the quasi-greedy \( \beta \)-expansion \( \delta(\beta) = \delta_1(\beta)\delta_2(\beta)\ldots \) of 1 is the lexicographically largest zero-one sequence not ending with an infinite string of zeros and satisfying \( 1 = \sum_{i=1}^{\infty} \delta_i(\beta)/\beta^i \) (see Section 2 for more details). Furthermore, throughout the paper we will use lexicographical order ‘<’, ‘\leq’, ‘>’ and ‘\geq’ between sequences and words.

When \( \beta \in (1, 2) \) is a multinacci number, the following result for the set-valued bifurcation set \( E_{\beta} \) was established in [6, Theorems C and D]. We record it here for later use.

**Theorem 1.1** ([6]). Let \( \beta \in (1, 2) \) be a multinacci number. Then the topological closure \( \overline{E_{\beta}} \) is a Cantor set. Furthermore, \( \max E_{\beta} = 1 - 1/\beta \).

In order to give a complete description of the dimension bifurcation set \( \mathcal{B}_{\beta} \) we introduce a class of basic intervals.
Definition 1.2. Let $\beta \in (1, 2]$. A word $s_1 \ldots s_m$ is called $\beta$-Lyndon if

$$s_{i+1} \ldots s_m > s_1 \ldots s_{m-1} \quad \forall \ 1 \leq i < m, \quad \text{and} \quad \sigma^n((s_1 \ldots s_m)_{\infty}) < \delta(\beta) \quad \forall \ n \geq 0.$$ 

Accordingly, an interval $[t_L, t_R) \subset [0, 1)$ is called a $\beta$-Lyndon interval if there exists a $\beta$-Lyndon word $s_1 \ldots s_m$ such that

$$t_L = \sum_{i=1}^{m} \frac{s_i}{\beta^i} \quad \text{and} \quad t_R = \frac{\beta^m}{\beta^{m-1}} \cdot t_L.$$ 

Here we mention that in Definition 1.2 the left endpoint $t_L = ((s_1 \ldots s_m)_{\infty})_{\beta}$ has a finite $\beta$-expansion and the right endpoint $t_R = ((s_1 \ldots s_m)_{\infty})_{\beta}$ has a periodic $\beta$-expansion, see Section 2 for more explanations.

We will show that the $\beta$-Lyndon intervals are pairwise disjoint for all $\beta \in (1, 2]$, and when $\beta$ is multinacci they cover the interval $[0, 1 - 1/\beta)$ up to a Lebesgue null set. The latter statement can be seen as a consequence of our main result for the coincidence of the two bifurcation sets, which we state below.

Theorem 1. Let $\beta \in (1, 2]$ be a multinacci number. Then

$$\mathcal{B}_\beta = \mathcal{E}_\beta = \left[0, 1 - \frac{1}{\beta}\right) \setminus \bigcup_{t_L, t_R}$$

$$= \left\{ t \in [0, 1) : \lim_{r \to 0} \dim_H(\mathcal{B}_\beta \cap (t, t + r)) = \dim_H K_\beta(t) > 0 \right\},$$

where the union is taken over all pairwise disjoint $\beta$-Lyndon intervals.

By Theorem 1 it follows that the topological closure $[t_L, t_R]$ of each $\beta$-Lyndon interval is indeed a maximal interval where the dimension function $\eta_\beta$ is constant. As a corollary of Theorem 1 we confirm a conjecture of [6] for $\beta$ a multinacci number.

Corollary 2. If $\beta \in (1, 2]$ is a multinacci number, then

$$\dim_H(\mathcal{E}_\beta \cap [t, 1]) = \dim_H K_\beta(t) \quad \forall \ t \in [0, 1).$$

The rest of the paper is organized as follows. In Section 2 we recall some properties from symbolic dynamics and the dimension formula for the survivor set $K_\beta(t)$. The proof of Theorem 1 and Corollary 2 will be given in Section 3. In Section 4 we make some remarks and point out that the method of proof for Theorem 1 can be applied to some other special values of $\beta \in (1, 2]$.

2. Preliminaries and $\beta$-Lyndon intervals

Given $\beta \in (1, 2]$, for each $x \in I_\beta := [0, 1/(\beta - 1)]$ there exists a sequence $(d_i) = d_1d_2 \ldots \in \{0, 1\}^\mathbb{N}$ such that

$$x = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} = ((d_i))_{\beta}.$$ 

The sequence $(d_i)$ is called a $\beta$-expansion of $x$. Sidorov [13] showed that for $\beta \in (1, 2)$ Lebesgue almost every $x \in I_\beta$ has a continuum of $\beta$-expansions. This is rather different from the case when $\beta = 2$ where every number in $I_2 = [0, 1]$ has a unique dyadic expansion except
for countably many points that have precisely two expansions. Given \( x \in I_\beta \), among all of its \( \beta \)-expansions let
\[
b(x, \beta) = (b_i(x, \beta))
\]
be the greedy \( \beta \)-expansion of \( x \), i.e., the lexicographically largest \( \beta \)-expansion of \( x \). Such a sequence always exists and is generated by the orbit of \( x \) under the map \( T_\beta \). Let \( \sigma \) be the left-shift on \( \{0, 1\}^\mathbb{N} \) defined by \( \sigma((c_i)) = (c_{i+1}) \). Then \( b(T_\beta(x), \beta) = \sigma(b(x, \beta)) \) for any \( x \in [0, 1) \).

Similarly, for \( x \in (0, 1/(\beta - 1)] \) let
\[
a(x, \beta) = (a_i(x, \beta))
\]
be the quasi-greedy \( \beta \)-expansion of \( x \) (cf. [3]), which is the lexicographically largest \( \beta \)-expansion of \( x \) not ending with \( 0^\infty \). Here for a word \( c \) we denote by \( c^\infty := ccc \cdots \) the periodic sequence with periodic block \( c \). Throughout the paper we will use the lexicographic order between sequences and words in the usual way. For example, for two sequences \( (c_i), (d_i) \in \{0, 1\}^\mathbb{N} \) we write \( (c_i) < (d_i) \) if \( c_1 < d_1 \), or there exists \( n > 1 \) such that \( c_1 \ldots c_n = d_1 \ldots d_{n-1} \) and \( c_n < d_n \). Furthermore, for two words \( c, d \) we say \( c < d \) if \( c0^\infty < d0^\infty \).

For \( \beta \in (1, 2] \) recall that
\[
\delta(\beta) = \delta_1(\beta)\delta_2(\beta) \ldots
\]
is the quasi-greedy \( \beta \)-expansion of \( 1 \), i.e., \( \delta(\beta) = a(1, \beta) \). The following lexicographic characterizations of \( \delta(\beta) \) and the greedy expansion \( b(x, \beta) \) are essentially due to Parry [9] (see also [4]).

**Lemma 2.1.** (i) The map \( \beta \mapsto \delta(\beta) \) is a strictly increasing bijection from \( (1, 2] \) onto the set of sequences \( (\delta_i) \in \{0, 1\}^\mathbb{N} \) not ending with \( 0^\infty \) and satisfying
\[
\sigma^n((\delta_i)) \ll (\delta_i) \quad \forall \ n \geq 0.
\]
(ii) Let \( \beta \in (1, 2] \). Then the map \( x \mapsto b(x, \beta) \) is a strictly increasing bijection from \([0, 1) \) onto the set of all sequences \( (b_i) \in \{0, 1\}^\mathbb{N} \) satisfying
\[
\sigma^n((b_i)) < \delta(\beta) \quad \forall \ n \geq 0.
\]
(iii) For any \( \beta \in (1, 2] \) the sequence \( b(1, \beta) = (b_i) \) satisfies \( \sigma^n((b_i)) \ll \delta(\beta) \quad \forall \ n \geq 1 \).

For \( \beta \in (1, 2] \) let \( [t_L, t_R) \) be a \( \beta \)-Lyndon interval generated by a \( \beta \)-Lyndon word \( s_1 \ldots s_m \). Then by Definition 1.2 and Lemma 2.1 (ii) it follows that
\[
b(t_L, \beta) = s_1 \ldots s_m 0^\infty \quad \text{and} \quad b(t_R, \beta) = (s_1 \ldots s_m)^\infty.
\]

**Lemma 2.2.** For any \( \beta \in (1, 2] \) the \( \beta \)-Lyndon intervals are pairwise disjoint.

**Proof.** Let \( [t_L, t_R) \) and \( [t'_L, t'_R) \) be two \( \beta \)-Lyndon intervals generated by the \( \beta \)-Lyndon words \( s_1 \ldots s_p \) and \( s'_1 \ldots s'_q \), respectively. Suppose on the contrary that \( [t_L, t_R) \cap [t'_L, t'_R) \neq \emptyset \).

Without loss of generality we assume \( t_L < t'_L < t_R \). Then by Definition 1.2 and Lemma 2.1(ii) it follows that
\[
s_1 \ldots s_p 0^\infty \ll s'_1 \ldots s'_q 0^\infty \ll (s_1 \ldots s_p)^\infty.
\]

This implies
\[
q > p, \quad s'_1 \ldots s'_p = s_1 \ldots s_p \quad \text{and} \quad s'_{p+1} \ldots s'_q 0^\infty \ll (s_1 \ldots s_p)^\infty.
\]

Write \( q = Np + r \) with \( N \geq 1 \) and \( 0 < r \leq p \). So, either there exists \( 1 \leq k < N \) such that
\[
s'_{p+1} \ldots s'_{kp} = (s_1 \ldots s_p)^{k-1} \quad \text{and} \quad s'_{kp+1} \ldots s'_{(k+1)p} < s_1 \ldots s_p.
\]
or
\[ s'_{p+1} \ldots s'_{Np} = (s_1 \ldots s_p)^{N-1} \quad \text{and} \quad s'_{Np+1} \ldots s'_q \preceq s_1 \ldots s_{q-Np}. \]

Using \( s'_1 \ldots s'_p = s_1 \ldots s_p \), we conclude in both cases that
\[ s'_{j+1} \ldots s'_q \preceq s'_1 \ldots s'_{q-j} \quad \text{for some} \quad j \in \{p, p + 1, \ldots, q - 1\}. \]

This is not possible by the definition of a \( \beta \)-Lyndon word. \( \square \)

To describe the Hausdorff dimension of the survivor set
\[ K_\beta(t) = \{ x \in [0, 1) : T_n^\beta(x) \geq t \quad \forall n \geq 0 \}, \]
we recall from [8, Chapter 4] the definition of topological entropy for a symbolic set. For a set \( X \subset \{0, 1\}^\mathbb{N} \), its topological entropy is defined to be
\[ h(X) = \liminf_{n \to \infty} \frac{\log \#B_n(X)}{n}, \]
where \( B_n(X) \) is the set of all length \( n \) prefixes of sequences from \( X \).

The following characterization of the set-valued bifurcation set \( E_\beta \) was implicitly given in [14] (see also [6, Proposition 2.3]). Furthermore, the Hausdorff dimension of \( K_\beta(t) \) was implicitly given by Raith in [10], and was recently explicitly presented in [6, Equation (2.6)].

**Proposition 2.3.**
(i) Let \( \beta \in (1, 2) \). Then
\[ E_\beta = \{ t \in [0, 1) : T_n^\beta(t) \geq t \quad \forall n \geq 0 \}. \]
(ii) Let \( \beta \in (1, 2) \) and \( t \in [0, 1) \). Then the Hausdorff dimension of \( K_\beta(t) \) is given by
\[ \dim_H K_\beta(t) = \frac{h(K_\beta(t))}{\log \beta}, \]
where \( K_\beta(t) := \{ (x_i) \in \{0, 1\}^\mathbb{N} : b(t, \beta) \preceq \sigma^n((x_i)) \preceq \delta(\beta) \quad \forall n \geq 0 \} \). Furthermore, the dimension function \( \eta_\beta : t \mapsto \dim_H K_\beta(t) \) is a Devil’s staircase, i.e., \( \eta_\beta \) is a non-constant, decreasing and continuous function which is locally constant almost everywhere in \([0, 1)\).

**3. Proof of Theorem 1**

In this section we will prove Theorem 1. First we show that the dimension bifurcation set \( B_\beta \) coincides with the set-valued bifurcation set \( E_\beta \), we then derive a complete characterization of these sets via the \( \beta \)-Lyndon intervals. The proof heavily relies upon the transitivity of the symbolic survivor set \( \tilde{K}_\beta(t) \) (see Lemma 3.2 below).

**Proposition 3.1.** Let \( \beta \in (1, 2) \) be a multinacci number. Then
\[ B_\beta = E_\beta = \left[ 0, 1 - \frac{1}{\beta} \right) \setminus \bigcup [t_L, t_R), \]
where the union is taken over all \( \beta \)-Lyndon intervals.

Observe by Lemma 2.2 that the \( \beta \)-Lyndon intervals are pairwise disjoint. In fact the closed \( \beta \)-Lyndon intervals \([t_L, t_R]\) are also pairwise disjoint. So by Proposition 3.1 it follows that each closed \( \beta \)-Lyndon interval is a maximal interval where the dimension function \( \eta_\beta \) is constant.
The proof of Proposition 3.1 will be split into several lemmas. We fix a multinacci number $\beta \in (1, 2)$ with $\delta(\beta) = (1^m0)^\infty$ for some $m \geq 1$. In view of Proposition 2.3 it is necessary to investigate the symbolic survivor set

$$\tilde{K}_\beta(t) = \left\{ (x_i) \in \{0, 1\}^\mathbb{N} : b(t, \beta) \preceq \sigma^n((x_i)) \preceq \delta(\beta) \forall n \geq 0 \right\}.$$  

**Lemma 3.2.** Let $\beta \in (1, 2)$ with $\delta(\beta) = (1^m0)^\infty$, and let $[t_L, t_R) \subset [0, 1 - 1/\beta)$ be a $\beta$-Lyndon interval. Then the set-valued map $t \mapsto \tilde{K}_\beta(t)$ is constant on $[t_L, t_R)$, and the set $\tilde{K}_\beta(t_R)$ is a transitive subshift of finite type.

**Proof.** Suppose $[t_L, t_R)$ is a $\beta$-Lyndon interval generated by $s_1 \ldots s_p$. First we claim that

$$\sigma^n((x_i)) \succ s_1 \ldots s_p0^\infty \forall n \geq 0 \iff \sigma^n((x_i)) \succ (s_1 \ldots s_p)^\infty \forall n \geq 0. \quad (3.1)$$

Since $(s_1 \ldots s_p)^\infty \succ s_1 \ldots s_p0^\infty$, the implication ‘$\implies$’ in (3.1) is obvious. For the reverse implication we assume $\sigma^n((x_i)) \prec (s_1 \ldots s_p)^\infty$ for some $n \geq 0$. Then there exists $\ell \geq 0$ such that

$$x_{n+1} \ldots x_{n+\ell p} = (s_1 \ldots s_p)^\ell \quad \text{and} \quad x_{n+\ell p+1} \ldots x_{n+(\ell+1)p} \prec s_1 \ldots s_p.$$  

This yields $\sigma^{n+\ell p}((x_i)) \prec s_1 \ldots s_p0^\infty$, completing the proof of ‘$\impliedby$’ in (3.1).

Take $t \in [t_L, t_R)$. Then by Lemma 2.1(ii) it follows that

$$\tilde{K}_\beta(t_R) \subseteq \tilde{K}_\beta(t) \subseteq \tilde{K}_\beta(t_L).$$  

Observe that $\delta(\beta) = (1^m0)^\infty$ for some $m \in \mathbb{N}$. Then

$$\tilde{K}_\beta(t_L) = \left\{ (x_i) : s_1 \ldots s_p0^\infty \preceq \sigma^n((x_i)) \preceq (1^m0)^\infty \forall n \geq 0 \right\}$$

$$= \left\{ (x_i) : (s_1 \ldots s_p)^\infty \preceq \sigma^n((x_i)) \preceq (1^m0)^\infty \forall n \geq 0 \right\} = \tilde{K}_\beta(t_R). \quad (3.2)$$

So, the set-valued map $t \mapsto \tilde{K}_\beta(t)$ is constant on $[t_L, t_R)$. Furthermore, $\tilde{K}_\beta(t_R)$ is a subshift of finite type with the set of forbidden blocks given by

$$\mathcal{F} = \left\{ c_1 \ldots c_k \in \{0, 1\}^k : c_1 \ldots c_k0^\infty \prec s_1 \ldots s_p0^\infty \text{ or } c_1 \ldots c_k0^\infty \succ (1^m0)^\infty \right\},$$

where $k = \max \{p, m + 1\}$. It remains to prove the transitivity of $\tilde{K}_\beta(t_R)$.

Since $[t_L, t_R) \subset [0, 1 - 1/\beta)$, by Lemma 2.1 (ii) it follows that $b(t_R, \beta) \prec b(1 - 1/\beta, \beta)$, which gives

$$\sigma^n((s_1 \ldots s_p)^\infty) \prec 0(1^m0)^\infty. \quad (3.3)$$

 Arbitrarily fix an admissible word $\varepsilon = \varepsilon_1 \ldots \varepsilon_k$ and an admissible sequence $\gamma = \gamma_1\gamma_2 \ldots \gamma_n$ in $\tilde{K}_\beta(t_R)$. We will construct a word $\nu$ such that $\varepsilon\nu\gamma \in \tilde{K}_\beta(t_R)$. Observe that $\sigma^n((s_1 \ldots s_p)^\infty) \prec (1^m0)^\infty$ for all $n \geq 0$. Thus, there exists a large integer $N$ such that

$$\sigma^n((s_1 \ldots s_p)^\infty) \prec (1^m0)^N0^\infty \quad \text{for all} \quad n \geq 0. \quad (3.4)$$

Denote by $(\delta_i) := \delta(\beta) = (1^m0)^\infty$. Note that $\varepsilon_{i+1} \ldots \varepsilon_k \preceq \delta_1 \ldots \delta_{k-1}$ for all $0 \leq i < k$. Let $i_0 \in \{0, 1, \ldots, k-1\}$ be the smallest index such that

$$\varepsilon_{i_0+1} \ldots \varepsilon_k = \delta_1 \ldots \delta_{k-i_0}.$$  

If such an index $i_0$ does not exist, then we put $i_0 = k$. In either case there exists a word $\mu$ such that $\varepsilon\mu = \varepsilon_1 \ldots \varepsilon_{i_0}(1^m0)^N$. Since $\gamma \succeq (1^m0)^\infty$, there exists $q \in \{0, 1, \ldots, m\}$ such that
\( \gamma \) begins with \( \gamma_1 \ldots \gamma_{q+1} = 1^0 \). We emphasize here that if \( q = 0 \) then \( \gamma \) begins with digit 0. Now we claim that
\[
\varepsilon \mu^{1-q} \gamma = \varepsilon_1 \ldots \varepsilon_{i_0} (1^0)^{N+1} \gamma_{q+2} \gamma_{q+3} \ldots \in \tilde{K}_\beta(t_R),
\]
or equivalently,
\[
(s_1 \ldots s_p)^\infty \prec \sigma^n(\varepsilon \mu^{1-q} \gamma) \prec (1^0)^\infty \quad \text{for all } n \geq 0.
\]

First we prove the second inequality in (3.5). By the definition of \( i_0 \) it follows that \( \sigma^n(\varepsilon \mu^{1-q} \gamma) < \delta(\beta) = (1^0)^\infty \) holds for all \( 0 \leq n < i_0 \). Furthermore, since \( \gamma \in \tilde{K}_\beta(t_R) \), the second inequality in (3.5) also holds for \( n \geq |\varepsilon| + |\mu| + m - q \). Here for a word \( c \) we denote its length by \(|c|\). For the remaining \( n \) we observe that \( \sigma^{i_0}(\varepsilon \mu^{1-q} \gamma) = (1^0)^{N+1} \gamma_{q+2} \gamma_{q+3} \ldots \) and \( \gamma_{q+2} \gamma_{q+3} \ldots \in \tilde{K}_\beta(t_R) \). So it is easy to verify that
\[
\sigma^n(\varepsilon \mu^{1-q} \gamma) \prec (1^0)^\infty \quad \text{for all } i_0 \leq n < |\varepsilon| + |\mu| + m - q.
\]

This proves the second inequality in (3.5).

For the first inequality in (3.5) we observe that \( \varepsilon \mu^{1-q} \gamma = \varepsilon_1 \ldots \varepsilon_{i_0} (1^0)^N \gamma_{q+1} \gamma_{q+2} \ldots \) and \( \gamma_{q+1} \gamma_{q+2} \ldots \in \tilde{K}_\beta(t_R) \). Then by (3.3) it follows that
\[
\sigma^n(\varepsilon \mu^{1-q} \gamma) \succ (s_1 \ldots s_p)^\infty \quad \text{for all } n \geq i_0.
\]
If \( i_0 = 0 \), then we are done. Otherwise, we take \( 0 \leq n < i_0 \). Since \( \varepsilon_1 \ldots \varepsilon_{i_0} \) is an admissible word in \( \tilde{K}_\beta(t_R) \), we have
\[
\varepsilon_{n+1} \ldots \varepsilon_{i_0} \succ t_1 \ldots t_{i_0-n},
\]
where \( (t_i) := (s_1 \ldots s_p)^\infty \). The first inequality in (3.5) now holds by (3.4), which tells us that
\[
(1^0)^{N} \gamma_{q+1} \gamma_{q+2} \ldots \succ t_{i_0-n+1} t_{i_0-n+2} \ldots
\]
This completes the proof of our claim.

Since \( \varepsilon \) and \( \gamma \) are chosen arbitrarily, it follows that \( \tilde{K}_\beta(t_R) \) is transitive. \( \square \)

Remark 3.3.
- The fact that \( \tilde{K}_\beta(t_R) \) is a subshift of finite type can also be deduced from [7].
- The proof of Lemma 3.2 can be adjusted to prove the more general case with \( \beta > 2 \) with \( \delta(\beta) = (M^m k)^\infty \), where \( M = \lceil \beta \rceil - 1 \) and \( k \in \{0,1,\ldots,M-1\} \). The transitivity property of \( \tilde{K}_\beta(t_R) \) holds only for \( t_R \) sufficiently close to 0.

To prove the coincidence of \( \Delta_\beta \) and \( \varepsilon_\beta \) we still need the following inequalities.

Lemma 3.4. Let \( (t_1 \ldots t_N)^\infty \in \{0,1\}^N \) be a periodic sequence with period \( N \geq 2 \). If
\[
\sigma^n((t_1 \ldots t_N)^\infty) \succ (t_1 \ldots t_N)^\infty \quad \forall \ n \geq 0,
\]
then
\[
t_{j+1} \ldots t_N \succ t_1 \ldots t_{N-j} \quad \forall \ 1 \leq j < N.
\]

Proof. Note that \( N \geq 2 \) is the period of \( (t_1 \ldots t_N)^\infty \), and
\[
\sigma^n((t_1 \ldots t_N)^\infty) \succ (t_1 \ldots t_N)^\infty \quad \forall \ n \geq 0.
\]
Then \( t_1 = 0 \) and \( t_N = 1 \). Taking the reflection on both sides of (3.6) it follows that
\[
\sigma^n((\overline{t_1} \ldots \overline{t_N})^\infty) \prec (t_1 \ldots t_N)^\infty \quad \forall \ n \geq 0.
\]
Here for a word $c_1 \ldots c_k \in \{0, 1\}^k$ its reflection is defined by $\overline{c_1 \ldots c_k} \coloneqq (1-c_1)(1-c_2)\ldots (1-c_k)$. By Lemma 2.1(i) it follows that $(t_1 \ldots t_N)\infty$ is the quasi-greedy expansion of 1 for some base $\beta' \in (1, 2]$, i.e., $\delta(\beta') = (t_1 \ldots t_N)\infty$. Since $N$ is the period of the sequence $\delta(\beta')$, the greedy $\beta'$-expansion of 1 is given by

$$b(1, \beta') = \overline{t_1 \ldots t_{N-1}} 10^\infty.$$ 

So, by Lemma 2.1 (iii) it follows that

$$\overline{t}_{j+1} \ldots t_N < \overline{t}_{j+1} \ldots t_{N-1} 1 \leq \overline{t}_1 \ldots t_{N-j} \quad \text{for all} \quad 1 \leq j < N.$$ 

Then the lemma follows by taking the reflection in the above equation.

Now we prove the coincidence of the two bifurcation sets.

**Lemma 3.5.** Let $\beta \in (1, 2)$ with $\delta(\beta) = (1^m0)\infty$. Then $\mathcal{E}_\beta = \mathcal{B}_\beta$.

**Proof.** By the definition of the two bifurcation sets it is easy to see that $\mathcal{B}_\beta \subset \mathcal{E}_\beta$. So in the following we prove $\mathcal{E}_\beta \subset \mathcal{B}_\beta$.

Let $t \in \mathcal{E}_\beta$ with its greedy $\beta$-expansion $b(t, \beta) = (t_i)$. Then by Theorem 1.1 we have $t \leq 1 - 1/\beta < 1/\beta$. This gives $t_1 = 0$. By Lemmas 2.1 (ii) and Proposition 2.3 (i) it follows that

$$\sigma^n((t_i)) \succ (t_i) \quad \text{for all} \quad n \geq 0.$$ 

Let $N \geq 1$ be the smallest index such that $\sigma^N((t_i)) = (t_i)$. If such an integer $N$ does not exist, then we set $N = \infty$. In the following we will prove $t \in \mathcal{B}_\beta$ by considering the following two cases: (I) $N < \infty$; and (II) $N = \infty$.

**Case (I).** $N < \infty$. We claim that $t_1 \ldots t_N$ is a $\beta$-Lyndon word. If $N = 1$, then $(t_i) = t_1^\infty = 0^\infty$. It is easy to check that $t_1 = 0$ is a $\beta$-Lyndon word. In the following we assume $N \geq 2$. Since $\sigma^N((t_i)) = (t_i)$, we have $(t_i) = (t_1 \ldots t_N)^\infty$. Note that $(t_i)$ is the greedy $\beta$-expansion of $t$. Then by Lemma 2.1 (ii) it follows that

$$\sigma^n((t_1 \ldots t_N)^\infty) \succ \delta(\beta) \quad \text{for all} \quad n \geq 0.$$ 

Note that $\sigma^n((t_1 \ldots t_N)^\infty) \succ (t_1 \ldots t_N)^\infty$. Then by Lemma 3.4 and the definition of $N$, it follows that

$$t_{j+1} \ldots t_N > t_1 \ldots t_{N-j} \quad \text{for all} \quad 1 \leq j < N.$$ 

So by Definition 1.2 we establish the claim.

Hence, $t = ((t_1 \ldots t_N)^\infty)_{\beta} = t_R$ is the right endpoint of a $\beta$-Lyndon interval generated by $t_1 \ldots t_N$. By Lemma 3.2 it follows that $\tilde{K}_\beta(t)$ is a transitive subshift of finite type. Observe that for any $t' > t$ we have

$$\tilde{K}_\beta(t') \subset \tilde{K}_\beta(t) \quad \text{and} \quad (t_1 \ldots t_N)^\infty \in \tilde{K}_\beta(t) \setminus \tilde{K}_\beta(t').$$ 

Recall by [8, Corollary 4.4.9] that for any transitive subshift of finite type, any proper subshift has strictly smaller topological entropy. Therefore,

$$h(\tilde{K}_\beta(t')) < h(\tilde{K}_\beta(t)) \quad \text{for any} \quad t' > t.$$ 

By Proposition 2.3 (ii) this yields $\eta_{\beta}(t') < \eta_{\beta}(t)$ for any $t' > t$. So $t \in \mathcal{B}_\beta$.

**Case (II).** $N = \infty$. Then $\sigma^n((t_i)) \succ (t_i)$ for all $n \geq 1$. So $(t_i)$ is not periodic. Observe that $(t_i)$ begins with digit 0, and

$$\sigma^n((t_i)) \prec (1^m0)^\infty \quad \text{for all} \quad n \geq 0.$$
So there exists a subsequence \((m_k)\) of positive integers such that for any \(k \geq 1\) we have \(t_{m_k} = 0\), and the word \(t_1\ldots t_{m_k}' := t_1\ldots t_{m_k-1}1\) does not contain \(m+1\) consecutive ones. Then by noting \(t_1 = 0\) it follows that

\[\sigma^n((t_1\ldots t_{m_k}')^\infty) \prec (1^m0)^\infty \quad \forall n \geq 0.\]

Since \(\sigma^n((t_i)) \succ (t_i)\) for all \(n \geq 0\), by Definition 1.2 it follows that \(t_1\ldots t_{m_k}'\) is a \(\beta\)-Lyndon word for any \(k \geq 1\). Let \(s_k := ((t_1\ldots t_{m_k}')^\infty)_\beta\). Then \(s_k\) is the right endpoint of a \(\beta\)-Lyndon interval generated by \(t_1\ldots t_{m_k}'\). Furthermore, \(s_k\) strictly decreases to \(t = ((t_i))_\beta\) as \(k \to \infty\).

So, for any \(t' > t\) we can find \(k\) such that \(s_k \in (t, t')\). By the same arguments as in the proof of Case (I) for \(s_k\) we conclude that

\[\eta_\beta(t') < \eta_\beta(s_k) \leq \eta_\beta(t).\]

So \(t \in \mathcal{B}_\beta\), completing the proof. \(\square\)

Finally, we describe the bifurcation sets via the \(\beta\)-Lyndon intervals.

**Lemma 3.6.** Let \(\beta \in (1, 2]\) with \(\delta(\beta) = (1^m0)^\infty\). Then

\[\left(0, 1 - \frac{1}{\beta}\right) \setminus \bigcup [t_L, t_R) \subset \mathcal{E}_\beta.\]

**Proof.** Take \(t \in \left[0, 1 - 1/\beta\right) \setminus \mathcal{E}_\beta\) with its greedy \(\beta\)-expansion \((t_i)\). Then \(t_1 = 0\). Since \(t \notin \mathcal{E}_\beta\), by Proposition 2.3 (i) there exists a smallest positive integer \(N\) such that \(T_N^\beta(t) < t\), which implies

\[(3.7) \quad t_{N+1}t_{N+2}\ldots \prec (t_i).\]

We claim that \(t_1\ldots t_N\) is a \(\beta\)-Lyndon word. Clearly, if \(N = 1\) then \(t_1 = 0\) is a \(\beta\)-Lyndon word. In the following we assume \(N \geq 2\). By Definition 1.2 it suffices to prove

\[(3.8) \quad t_{j+1}\ldots t_N > t_1\ldots t_{N-j}\text{ for all }1 \leq j < N,\]

and

\[(3.9) \quad \sigma^n((t_1\ldots t_N)^\infty) \prec (1^m0)^\infty \text{ for all }n \geq 0.\]

First we prove (3.8). By the definition of \(N\) in (3.7) it follows that

\[(3.10) \quad t_{j+1}t_jt_{j+2}\ldots \succ (t_i)\text{ for all }1 \leq j < N,\]

which implies \(t_{j+1}\ldots t_N \succ t_1\ldots t_{N-j}\) for all \(1 \leq j < N\). Suppose \(t_{j+1}\ldots t_N = t_1\ldots t_{N-j}\) for some \(j \in \{1, 2, \ldots, N - 1\}\). Applying (3.7) and then (3.10) it follows that

\[t_{j+1}t_{j+2}\ldots = t_1\ldots t_{N-j}t_{N+1}t_{N+2}\ldots \prec t_1\ldots t_{N-j}t_1t_2\ldots \prec (t_i),\]

leading to a contradiction with the minimality of \(N\). This proves (3.8).

To prove (3.9) we observe that \(\delta(\beta) = (1^m0)^\infty\) and \((t_i)\) is the greedy \(\beta\)-expansion of \(t\). Then by Lemma 2.1 (ii) it follows that \(t_1\ldots t_N\) cannot contain \(m+1\) consecutive ones. Since \(t_1 = 0\), we have

\[\sigma^n((t_1\ldots t_N)^\infty) \preceq (1^m0)^\infty \text{ for all }n \geq 0.\]

So to prove (3.9) it remains to prove that \(\sigma^n((t_1\ldots t_N)^\infty) \neq (1^m0)^\infty\text{ for any }n \geq 0\). Suppose the equality \(\sigma^n((t_1\ldots t_N)^\infty) = (1^m0)^\infty\) holds for some \(n \geq 0\). Then by using \(t_1 = 0\) it follows that

\[t_1\ldots t_{m+1} = 01^m.\]
This implies \(b(t, \beta) = (t_i) \equiv 01^m0^\infty = b(1 - 1/\beta, \beta)\). By Lemma 2.1 (ii) we have \(t \geq 1 - 1/\beta\), leading to a contradiction. This establishes (3.9).

By the claim there exists a \(\beta\)-Lyndon interval \([t_L, t_R]\) generated by \(t_1 \ldots t_N\). Furthermore, by (3.7) it follows that
\[
(t_i) = t_1 \ldots t_N t_{N+1} t_{N+2} \ldots < t_1 \ldots t_N t_1 t_2 \ldots = (t_1 \ldots t_N)^2 t_{N+1} t_{N+2} \ldots \\
< (t_1 \ldots t_N)^2 t_1 t_2 \ldots = (t_1 \ldots t_N)^3 t_{N+1} t_{N+2} \ldots \\
\ldots \\
\leq (t_1 \ldots t_N)\infty.
\]

Therefore, \(t_1 \ldots t_N 0^\infty \preceq (t_i) \prec (t_1 \ldots t_N)\infty\), which gives \(t \in [t_L, t_R]\) by Lemma 2.1 (ii). This completes the proof.

\[\square\]

**Proof of Proposition 3.1.** By Lemmas 3.5 and 3.6 it suffices to prove
\[
\mathcal{B}_\beta \subset \left[0, 1 - \frac{1}{\beta}\right) \setminus \bigcup [t_L, t_R).
\]

Note by Lemma 3.5 and Theorem 1.1 that \(\mathcal{B}_\beta = \mathcal{E}_\beta \subset [0, 1 - 1/\beta]\). In fact we have \(\mathcal{E}_\beta \subset [0, 1 - 1/\beta]\). Observe that \(b(1 - 1/\beta, \beta) = 01^m0^\infty\). Then \(T_{\beta}^{m+1}(1 - 1/\beta) < 1 - 1/\beta\). By Proposition 2.3 (i) this implies \(1 - 1/\beta \notin \mathcal{E}_\beta\). Hence, \(\mathcal{E}_\beta \subset [0, 1 - 1/\beta]\).

In the following it remains to prove \(\mathcal{B}_\beta \cap \bigcup [t_L, t_R] = \emptyset\). Take a \(\beta\)-Lyndon interval \([t_L, t_R]\). If \(t \in [t_L, t_R]\), then by (3.2) it follows that
\[
\bar{K}_\beta(t) = \bar{K}_\beta(t_L) = \bar{K}_\beta(t_R),
\]
which gives \(\eta_\beta(t') = \eta_\beta(t) = \eta_\beta(t_L)\) for all \(t' \in (t, t_R)\). So, \(t \notin \mathcal{B}_\beta\).

\[\square\]

As a consequence of Proposition 3.1 and Theorem 1.1 it follows that for \(\beta \in (1, 2]\) a multinacci number the \(\beta\)-Lyndon intervals cover \([0, 1 - 1/\beta]\) up to a Lebesgue null set.

**Corollary 3.7.** Let \(\beta \in (1, 2]\) be a multinacci number.

(i) The union of all \(\beta\)-Lyndon intervals covers \([0, 1 - 1/\beta]\) up to a Lebesgue null set. Furthermore, for any \(t \in \mathcal{B}_\beta\) and any \(r > 0\) the interval \((t, t + r)\) contains infinitely many \(\beta\)-Lyndon intervals.

(ii) \(\eta_\beta(t) > 0\) if and only if \(t < 1 - 1/\beta\).

**Proof.** Note that \(\mathcal{E}_\beta\) is a Lebesgue null set which, by Theorem 1.1, has no isolated points. Then (i) follows from Proposition 3.1 which tells us that \(\bigcup [t_L, t_R] = [0, 1 - 1/\beta] \setminus \mathcal{E}_\beta\). For (ii) it can be deduced from Proposition 3.1 and Theorem 1.1 that \(\sup \mathcal{B}_\beta = 1 - 1/\beta\) and \(1 - 1/\beta \notin \mathcal{B}_\beta\).

\[\square\]

Now we turn to investigate the local dimension of the bifurcation set \(\mathcal{B}_\beta\).

**Lemma 3.8.** Let \(\beta \in (1, 2]\) with \(\delta(\beta) = (1^m0)^\infty\). Then
\[
\lim_{r \to 0} \dim_H (\mathcal{B}_\beta \cap (t, t + r)) = \dim_H K_\beta(t) > 0 \quad \forall t \in \mathcal{B}_\beta.
\]
Proof. Take \( t \in \mathcal{B}_\beta \). By Proposition 3.1 we have \( t < 1 - 1/\beta \), and then by Corollary 3.7 (ii) it gives \( \eta_\beta(t) = \dim_H K_\beta(t) > 0 \). Note by Proposition 3.1 and Proposition 2.3 (i) that
\[
\mathcal{B}_\beta \cap (t, t + r) = \mathcal{E}_\beta \cap (t, t + r) \subseteq K_\beta(t)
\]
for any \( r > 0 \).

Then \( \lim_{r \to 0} \dim_H (\mathcal{B}_\beta \cap (t, t + r)) \leq \eta_\beta(t) \). So it remains to prove
\[
(3.11) \quad \lim_{r \to 0} \dim_H (\mathcal{B}_\beta \cap (t, t + r)) \geq \eta_\beta(t).
\]

We prove this now by considering the following two cases: (I) \( t = t_R \) is the right endpoint of a \( \beta \)-Lyndon interval; (II) \( t \in [0, 1 - 1/\beta) \setminus \bigcup [t_L, t_R] \).

Case (I). Suppose \( t = t_R \) is the right endpoint of a \( \beta \)-Lyndon interval. Let \( (t_i) = (t_1 \ldots t_p)^\infty \) be the greedy \( \beta \)-expansion of \( t_R \). Note that \( t_R \in \mathcal{B}_\beta \). Then by Corollary 3.7 (i) there exists a sequence \( (t_R^{(n)}) \subseteq \mathcal{B}_\beta \) such that each \( t_R^{(n)} \) is a right endpoint of a \( \beta \)-Lyndon interval and \( t_R^{(n)} \searrow t_R \) as \( n \to \infty \). Fix \( r > 0 \). Then we can find a large integer \( N \) satisfying
\[
t_R^{(n)} \in (t_R, t_R + r) \quad \text{for all } n \geq N.
\]
Furthermore, since \( b(t_R, \beta) = (t_1 \ldots t_p)^\infty \), by Lemma 2.1 (ii) it follows that for each \( n \geq N \) there exists an integer \( m \) such that the greedy \( \beta \)-expansion \( b(t_R^{(n)}, \beta) \) of \( t_R^{(n)} \) satisfies
\[
(3.12) \quad b(t_R^{(n)}, \beta) = (t_1 \ldots t_p)^{k_n 1} \infty.
\]

Observe by Proposition 3.1 and Proposition 2.3 (i) that
\[
\mathcal{B}_\beta = \mathcal{E}_\beta = \{((s_i))_\beta : (s_i) \preceq \sigma^n((s_i)) < (1^n 0)^\infty \forall n \geq 0 \}.
\]

So by using \( t_R \in \mathcal{B}_\beta \), (3.12) and Lemma 2.1 (ii) it follows that for any \( n \geq N \),
\[
\left\{ ((t_1 \ldots t_p)^{k_n x_1 x_2 \ldots})_\beta : x_1 \ldots x_p = t_1 \ldots t_p \right\} \subseteq \mathcal{B}_\beta \cap [t_R^{(n)}, t_R]
\]
\[
\subseteq \mathcal{B}_\beta \cap [t_R, t_R + r).
\]

Note by Lemma 3.2 that \( \bar{K}_\beta(t_R^{(n)}) \) is a transitive subshift of finite type. Then by (3.13) it follows that
\[
\dim_H (\mathcal{B}_\beta \cap (t_R, t_R + r)) \geq \dim_H \bar{K}_\beta(t_R^{(n)}) = \eta_\beta(t_R^{(n)}) \quad \text{for all } n \geq N.
\]
Letting \( n \to \infty \) and by the continuity of \( \eta_\beta \) (see Proposition 2.3 (ii)) we obtain that
\[
\dim_H (\mathcal{B}_\beta \cap (t_R, t_R + r)) \geq \eta_\beta(t_R).
\]

Since \( r > 0 \) was given arbitrary, letting \( r \to 0 \) we conclude that
\[
(3.14) \quad \lim_{r \to 0} \dim_H (\mathcal{B}_\beta \cap (t_R, t_R + r)) \geq \eta_\beta(t_R).
\]

Case (II). \( t \in [0, 1 - 1/\beta) \setminus \bigcup [t_L, t_R] \). Then by Corollary 3.7 (i) there exists a sequence \( (t_R^{(k)}) \) such that each \( t_R^{(k)} \) is the right endpoint of a \( \beta \)-Lyndon interval, and \( t_R^{(k)} \searrow t \) as \( k \to \infty \).

So, for any \( r > 0 \) there exists a sufficiently large integer \( k \) such that \( t_R^{(k)} \in (t, t + r) \). By (3.14) with \( t_R \) replaced by \( t_R^{(k)} \) it follows that for any \( \varepsilon > 0 \) there exists \( r_k > 0 \) such that \( (t_R^{(k)} + r_k) \subseteq (t, t + r) \) and
\[
\dim_H (\mathcal{B}_\beta \cap (t, t + r)) \geq \dim_H (\mathcal{B}_\beta \cap (t_R^{(k)} + r_k)) \geq \eta_\beta(t_R^{(k)}) - \varepsilon.
\]
Letting $r \to 0$, and then $t_R^{(k)} \to t$, we conclude by the continuity of $\eta_\beta$ that
\[
\lim_{r \to 0} \dim_H(\mathcal{B}_\beta \cap (t, t + r)) \geq \eta_\beta(t) - \varepsilon.
\]
Since $\varepsilon > 0$ was arbitrary, we obtain $\lim_{r \to 0} \dim_H(\mathcal{B}_\beta \cap (t, t + r)) \geq \eta_\beta(t)$. This, together with (3.14), proves (3.11).

\section*{Proof of Theorem 1}

Let $\beta \in (1, 2)$ with $\delta(\beta) = (1^n0)^\infty$. By Lemma 2.2, Proposition 3.1 and Lemma 3.8 it suffices to prove
\[
\{ t \in [0, 1) : \lim_{r \to 0} \dim_H(\mathcal{B}_\beta \cap (t, t + r)) = \eta_\beta(t) > 0 \} \subset \mathcal{B}_\beta.
\]

Take $t \in [0, 1) \setminus \mathcal{B}_\beta$. Then by Proposition 3.1 we have $t \in [1 - 1/\beta, 1)$ or $t \in [t_L, t_R)$ for some $\beta$-Lyndon interval. If $t \geq 1 - 1/\beta$, then $\eta_\beta(t) = 0$ by Corollary 3.7 (ii). If $t \in [t_L, t_R)$, then by Proposition 3.1 there exists $r > 0$ such that $\mathcal{B}_\beta \cap (t, t + r) = \emptyset$. This completes the proof.

\section*{Final remarks}

The main results obtained in this paper can be easily modified to study the following analogous bifurcation sets:
\[
\mathcal{E}_\beta' := \{ t \in [0, 1) : K_\beta(t') \neq K_\beta(t) \quad \forall t' \neq t \},
\]
\[
\mathcal{B}_\beta' := \{ t \in [0, 1) : \dim_H(\mathcal{E}_\beta') \neq \dim_H K_\beta(t) \quad \forall t' \neq t \}.
\]

If $\beta \in (1, 2]$ is a multinacci number, one can show that
\[
\mathcal{B}_\beta = \mathcal{E}_\beta' \cap [0, 1 - \frac{1}{\beta}) \cup [t_L, t_R]
\]
\[
= \{ t \in [0, 1) : \lim_{r \to 0} \dim_H(\mathcal{E}_\beta \cap (t - r, t)) = \lim_{r \to 0} \dim_H(\mathcal{E}_\beta \cap (t, t + r)) = \dim_H K_\beta(t) > 0 \},
\]
where the union is taken over all pairwise disjoint closed $\beta$-Lyndon intervals.

Observe that the main result Theorem 1 holds under the assumption that $\beta \in (1, 2]$ is a multinacci number, i.e., $\delta(\beta) = (1^n0)^\infty$ for some $m \in \mathbb{N}$. The method used in this paper can be adapted to show that Theorem 1 still holds for $\beta \in (1, 2]$ with $\delta(\beta) = (1^\infty m)^\infty$. It is worth mentioning that in [6] Kalle et al. considered a general Farey word base $\beta$, i.e., $\delta(\beta) = (s_1 \ldots s_p)^\infty$ with $s_p s_{p-1} \ldots s_2 s_1$ a non-degenerate Farey word. They showed that for a general Farey word base $\beta \in (1, 2)$, the set-valued bifurcation set $\mathcal{E}_\beta$ has no isolated points and Theorem 1.1 holds. We finish by posing the following conjecture.
**Conjecture 4.1.** Let $\beta \in (1, 2]$. Then $\mathcal{B}_\beta = \mathcal{E}_\beta$ if and only if $\mathcal{E}_\beta$ has no isolated points.

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