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TWO BIFURCATION SETS ARISING FROM THE BETA TRANSFORMATION WITH A HOLE AT 0

SIMON BAKER AND DERONG KONG

ABSTRACT. Given $\beta \in (1, 2]$, the $\beta$-transformation $T_\beta : x \mapsto \beta x \mod 1$ on the circle $[0, 1)$ with a hole $[0, t)$ was investigated by Kalle et al. (2019). They described the set-valued bifurcation set

$$E_\beta := \{ t \in [0, 1) : K_\beta(t) \neq K_\beta(t) \forall t' > t \},$$

where $K_\beta(t) := \{ x \in [0, 1) : T_n^\beta(x) \geq t \forall n \geq 0 \}$ is the survivor set. In this paper we investigate the dimension bifurcation set

$$B_\beta := \{ t \in [0, 1) : \dim H K_\beta(t) \neq \dim H K_\beta(t) \forall t' > t \},$$

where $\dim H$ denotes the Hausdorff dimension. We show that if $\beta \in (1, 2]$ is a multinacci number then the two bifurcation sets $B_\beta$ and $E_\beta$ coincide. Moreover we give a complete characterization of these two sets. As a corollary of our main result we prove that for $\beta$ a multinacci number we have $\dim H (E_\beta \cap [t, 1)) = \dim H K_\beta(t)$ for any $t \in [0, 1)$. This confirms a conjecture of Kalle et al. for $\beta$ a multinacci number.

1. INTRODUCTION

Given $\beta \in (1, 2]$, the $\beta$-transformation $T_\beta$ on the circle $\mathbb{R}/\mathbb{Z} \sim [0, 1)$ is defined by

$$T_\beta : [0, 1) \to [0, 1); \quad x \mapsto \beta x \mod 1.$$

Following the pioneering work of Rényi [11] and Parry [9] there has been a great interest in the study of $T_\beta$. In general, the system $\Phi_\beta = ([0, 1), T_\beta)$ does not admit a Markov partition (cf. [12]), this makes describing the dynamics of $\Phi_\beta$ more challenging.

When $\beta = 2$, Urbański considered in [14, 15] the open dynamical system under the doubling map $T_2$ with a hole at zero. More precisely, for $t \in [0, 1)$ let

$$K_2(t) := \{ x \in [0, 1) : T_n^{2t}(x) \geq t \forall n \geq 0 \}.$$

Here we use a slightly different definition of $K_2(t)$ from that by Urbański. By [14, Theorem 1 and Corollary 1] it follows that the dimension function $t \mapsto \eta_2(t) := \dim_H K_2(t)$ is a Devil’s staircase on $[0, 1)$, that is (i) $\eta_2$ is decreasing and continuous on $[0, 1)$; (ii) $\eta_2$ is locally constant almost everywhere on $[0, 1)$; and (iii) $\eta_2$ is not constant on $[0, 1)$. Here and throughout the paper $\dim_H$ denotes the Hausdorff dimension. Moreover, Urbański investigated the bifurcation sets

$$E_2 := \{ t \in [0, 1) : K_2(t) \neq K_2(t) \forall t' > t \} \quad \text{and} \quad B_2 := \{ t \in [0, 1) : \eta_2(t') \neq \eta_2(t) \forall t' > t \}.$$

Clearly, $B_2 \subseteq E_2$. It can be easily deduced from the proof of Theorem 1 in [14] that $B_2 = E_2$, and its topological closure $\overline{B}_2$ is a Cantor set, i.e., a non-empty compact set that has neither

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isolated nor interior points. Furthermore, the following local dimension property was shown to hold: \( \lim_{r \to 0} \dim_H(\mathcal{E}_2 \cap (t-r, t+r)) = \eta_2(t) \) for all \( t \in \mathcal{E}_2 \). Recently, Carminati and Tiozzo in [1] showed that the local Hölder exponent of the dimension function \( \eta_2 \) at any \( t \in \mathcal{E}_2 \) equals \( \eta_2(t) \).

Inspired by the work of Urbański [14, 15], Kalle et al. in [6] considered the analogous problem for the \( \beta \)-transformation with a hole \([0,t)\). More precisely, for \( t \in [0,1) \) they investigated the survivor set

\[
K_\beta(t) := \{ x \in [0,1) : T^n_\beta(x) \geq t \ \forall \ n \geq 0 \},
\]

and showed that the dimension function \( t \mapsto \dim_H K_\beta(t) \) is also a Devil’s staircase on \([0,1)\). Furthermore, they characterized the **set-valued bifurcation set**

\[
\mathcal{E}_\beta := \{ t \in [0,1) : K_\beta(t') \neq K_\beta(t) \ \forall \ t' > t \},
\]

and proved that \( \mathcal{E}_\beta \) is a Lebesgue null set of full Hausdorff dimension for any \( \beta \in (1,2) \). Note that the bifurcation set \( \mathcal{E}_\beta \) defined here coincides with the set

\[
E_\beta^n := \{ t \in [0,1) : T^n_\beta(t) \geq t \ \forall n \geq 0 \}
\]

in [6]. Interestingly, they showed that \( \mathcal{E}_\beta \) contains infinitely many isolated points for Lebesgue almost every \( \beta \in (1,2) \). This is in contrast to the case where \( \beta = 2 \) and \( \mathcal{E}_\beta \) has no isolated points. For \( \beta \)-transformation with an arbitrary hole we refer to the work of Clark [2]. We also mention that the study of bifurcation sets plays an important role in one-dimensional dynamics (cf. [5]).

Since for each \( \beta \in (1,2) \) the dimension function \( \eta_\beta : t \mapsto \dim_H K_\beta(t) \) is a Devil’s staircase, it is natural to consider the **dimension bifurcation set**

\[
\mathcal{B}_\beta := \{ t \in [0,1) : \eta_\beta(t') \neq \eta_\beta(t) \ \forall t' > t \}.
\]

This set records those \( t \) for which the dimension function \( \eta_\beta \) has a ‘change’ within any right neighborhood. Since \( \eta_\beta \) is continuous, \( \mathcal{B}_\beta \) cannot have isolated points. On the other hand, the set-valued bifurcation set \( \mathcal{E}_\beta \) contains (infinitely many) isolated points for Lebesgue almost every \( \beta \in (1,2) \). So in general we cannot expect the coincidence of the two bifurcation sets \( \mathcal{B}_\beta \) and \( \mathcal{E}_\beta \). That being said, in this paper we show that if \( \beta \) is a multinacci number, i.e., the unique root in \((1,2)\) of the equation

\[
x^{m+1} = x^m + x^{m-1} + \cdots + x + 1
\]

for some \( m \in \mathbb{N} \), then the two bifurcation sets indeed coincide. Importantly, if \( \beta \) is a multinacci number then its quasi-greedy expansion of \( 1 \) is of the form \((1^n0)^\infty\). This property will be useful in our analysis. Here for \( \beta \in (1,2) \) the quasi-greedy \( \beta \)-expansion \( \delta(\beta) = \delta_1(\beta)\delta_2(\beta)\ldots \) of \( 1 \) is the lexicographically largest zero-one sequence not ending with an infinite string of zeros and satisfying \( 1 = \sum_{i=1}^\infty \delta_i(\beta)/\beta^i \) (see Section 2 for more details). Furthermore, throughout the paper we will use lexicographical order \( \prec, \preceq, \succ, \succeq \) and \( \preceq \) between sequences and words.

When \( \beta \in (1,2) \) is a multinacci number, the following result for the set-valued bifurcation set \( \mathcal{E}_\beta \) was established in [6, Theorems C and D]. We record it here for later use.

**Theorem 1.1** ([6]). Let \( \beta \in (1,2) \) be a multinacci number. Then the topological closure \( \bar{\mathcal{E}_\beta} \) is a Cantor set. Furthermore, \( \max \mathcal{E}_\beta = 1 - 1/\beta \).

In order to give a complete description of the dimension bifurcation set \( \mathcal{B}_\beta \) we introduce a class of basic intervals.
Definition 1.2. Let $\beta \in (1, 2]$. A word $s_1 \ldots s_m$ is called $\beta$-Lyndon if
\[ s_{i+1} \ldots s_m > s_1 \ldots s_{m-i} \quad \forall \ 1 \leq i < m, \ \text{and} \ \sigma^n((s_1 \ldots s_m)^\infty) < \delta(\beta) \quad \forall \ n \geq 0. \]
Accordingly, an interval $[t_L, t_R] \subset [0, 1)$ is called a $\beta$-Lyndon interval if there exists a $\beta$-Lyndon word $s_1 \ldots s_m$ such that
\[ t_L = \sum_{i=1}^m \frac{s_i}{\beta^i} \quad \text{and} \quad t_R = \frac{\beta^m}{\beta^m - 1} \cdot t_L. \]

Here we mention that in Definition 1.2 the left endpoint $t_L = (s_1 \ldots s_m 0^\infty)_\beta$ has a finite $\beta$-expansion and the right endpoint $t_R = ((s_1 \ldots s_m)^\infty)_\beta$ has a periodic $\beta$-expansion, see Section 2 for more explanations.

We will show that the $\beta$-Lyndon intervals are pairwise disjoint for all $\beta \in (1, 2]$, and when $\beta$ is multinacci they cover the interval $[0, 1 - 1/\beta]$ up to a Lebesgue null set. The latter statement can be seen as a consequence of our main result for the coincidence of the two bifurcation sets, which we state below.

Theorem 1. Let $\beta \in (1, 2]$ be a multinacci number. Then
\[
\mathcal{B}_\beta = \mathcal{E}_\beta = \left[\frac{0}{\beta}, 1 - \frac{1}{\beta}\right) \setminus \bigcup [t_L, t_R)
\]
\[= \left\{ t \in [0, 1) : \lim_{r \to 0} \dim_H(\mathcal{B}_\beta \cap (t, t + r)) = \dim_H K_\beta(t) > 0 \right\}, \]
where the union is taken over all pairwise disjoint $\beta$-Lyndon intervals.

By Theorem 1 it follows that the topological closure $[t_L, t_R]$ of each $\beta$-Lyndon interval is indeed a maximal interval where the dimension function $\eta_\beta$ is constant. As a corollary of Theorem 1 we confirm a conjecture of [6] for $\beta$ a multinacci number.

Corollary 2. If $\beta \in (1, 2]$ is a multinacci number, then
\[ \dim_H(\mathcal{E}_\beta \cap [t, 1]) = \dim_H K_\beta(t) \quad \forall \ t \in [0, 1). \]

The rest of the paper is organized as follows. In Section 2 we recall some properties from symbolic dynamics and the dimension formula for the survivor set $K_\beta(t)$. The proof of Theorem 1 and Corollary 2 will be given in Section 3. In Section 4 we make some remarks and point out that the method of proof for Theorem 1 can be applied to some other special values of $\beta \in (1, 2]$.

2. Preliminaries and $\beta$-Lyndon intervals

Given $\beta \in (1, 2]$, for each $x \in I_\beta := [0, 1/(\beta - 1)]$ there exists a sequence $(d_i) = d_1d_2 \ldots \in \{0, 1\}^\infty$ such that
\[ x = \sum_{i=1}^\infty \frac{d_i}{\beta^i} = ((d_i))_\beta. \]
The sequence $(d_i)$ is called a $\beta$-expansion of $x$. Sidorov [13] showed that for $\beta \in (1, 2)$ Lebesgue almost every $x \in I_\beta$ has a continuum of $\beta$-expansions. This is rather different from the case when $\beta = 2$ where every number in $I_2 = [0, 1]$ has a unique dyadic expansion except
for countably many points that have precisely two expansions. Given $x \in I_\beta$, among all of its $\beta$-expansions let

$$b(x,\beta) = (b_i(x,\beta))$$

be the greedy $\beta$-expansion of $x$, i.e., the lexicographically largest $\beta$-expansion of $x$. Such a sequence always exists and is generated by the orbit of $x$ under the map $T_\beta$. Let $\sigma$ be the left-shift on $\{0,1\}^\mathbb{N}$ defined by $\sigma((c_i)) = (c_{i+1})$. Then $b(T_\beta(x),\beta) = \sigma(b(x,\beta))$ for any $x \in [0,1)$. Similarly, for $x \in (0,1/(\beta-1)]$ let

$$a(x,\beta) = (a_i(x,\beta))$$

be the quasi-greedy $\beta$-expansion of $x$ (cf. [3]), which is the lexicographically largest $\beta$-expansion of $x$ not ending with $0^\infty$. Here for a word $c$ we denote by $c^{\infty} := cc\cdot \cdot \cdot$ the periodic sequence with periodic block $c$. Throughout the paper we will use the lexicographic order between sequences and words in the usual way. For example, for two sequences $(c_i), (d_i) \in \{0,1\}^\mathbb{N}$ we write $(c_i) < (d_i)$ if $c_1 < d_1$, or there exists $n > 1$ such that $c_1 \ldots c_n = d_1 \ldots d_{n-1}$ and $c_n < d_n$. Furthermore, for two words $c, d$ we say $c < d$ if $c0^\infty < d0^\infty$.

For $\beta \in (1,2]$ recall that

$$\delta(\beta) = \delta_1(\beta)\delta_2(\beta) \ldots$$

is the quasi-greedy $\beta$-expansion of 1, i.e., $\delta(\beta) = a(1,\beta)$. The following lexicographic characterizations of $\delta(\beta)$ and the greedy expansion $b(x,\beta)$ are essentially due to Parry [9] (see also [4]).

**Lemma 2.1.** (i) The map $\beta \mapsto \delta(\beta)$ is a strictly increasing bijection from $(1,2]$ onto the set of sequences $(\delta_i) \in \{0,1\}^\mathbb{N}$ not ending with $0^\infty$ and satisfying

$$\sigma^n((\delta_i)) \leq (\delta_i) \quad \forall \ n \geq 0.$$

(ii) Let $\beta \in (1,2]$. Then the map $x \mapsto b(x,\beta)$ is a strictly increasing bijection from $[0,1)$ onto the set of all sequences $(b_i) \in \{0,1\}^\mathbb{N}$ satisfying

$$\sigma^n((b_i)) < \delta(\beta) \quad \forall \ n \geq 0.$$

(iii) For any $\beta \in (1,2]$ the sequence $b(1,\beta) = (b_i)$ satisfies $\sigma^n((b_i)) \leq \delta(\beta) \quad \forall \ n \geq 1$.

For $\beta \in (1,2]$ let $[t_L,t_R)$ be a $\beta$-Lyndon interval generated by a $\beta$-Lyndon word $s_1 \ldots s_m$. Then by Definition 1.2 and Lemma 2.1 (ii) it follows that

$$b(t_L,\beta) = s_1 \ldots s_m0^\infty \quad \text{and} \quad b(t_R,\beta) = (s_1 \ldots s_m)\infty.$$

**Lemma 2.2.** For any $\beta \in (1,2]$ the $\beta$-Lyndon intervals are pairwise disjoint.

**Proof.** Let $[t'_L,t'_R)$ be two $\beta$-Lyndon intervals generated by the $\beta$-Lyndon words $s_1 \ldots s'_p$ and $s'_1 \ldots s'_q$, respectively. Suppose on the contrary that $[t_L,t_R) \cap [t'_L,t'_R) \neq \emptyset$. Without loss of generality we assume $t_L < t'_L < t_R$. Then by Definition 1.2 and Lemma 2.1(ii) it follows that

$$s_1 \ldots s_p0^\infty \prec s'_1 \ldots s'_q0^\infty \prec (s_1 \ldots s_p)^\infty.$$

This implies

$$q > p, \quad s'_1 \ldots s'_p = s_1 \ldots s_p \quad \text{and} \quad s'_{p+1} \ldots s'_q0^\infty \prec (s_1 \ldots s_p)^\infty.$$

Write $q = Np + r$ with $N \geq 1$ and $0 < r \leq p$. So, either there exists $1 \leq k < N$ such that

$$s'_{p+1} \ldots s'_{kp} = (s_1 \ldots s_p)^{k-1} \quad \text{and} \quad s'_{kp+1} \ldots s'_{(k+1)p} \prec s_1 \ldots s_p.$$
or
\[ s'_{p+1} \ldots s'_{Np} = (s_1 \ldots s_p)^{N-1} \] and \[ s'_{Np+1} \ldots s'_{q} \preceq s_1 \ldots s_{q-Np}. \]

Using \( s'_1 \ldots s'_p = s_1 \ldots s_p \) we conclude in both cases that
\[ s'_{j+1} \ldots s'_{q} \preceq s'_1 \ldots s'_{q-j} \]
for some \( j \in \{p, p+1, \ldots, q - 1\} \).

This is not possible by the definition of a \( \beta \)-Lyndon word. \( \square \)

To describe the Hausdorff dimension of the survivor set
\[ K_\beta(t) = \{ x \in [0, 1) : T^n_\beta(x) \geq t \ \forall n \geq 0 \}, \]
we recall from [8, Chapter 4] the definition of topological entropy for a symbolic set. For a set \( X \subset \{0, 1\}^\mathbb{N} \), its topological entropy is defined to be
\[ h(X) = \liminf_{n \to \infty} \frac{\log \# B_n(X)}{n}, \]
where \( B_n(X) \) is the set of all length \( n \) prefixes of sequences from \( X \).

The following characterization of the set-valued bifurcation set \( E_\beta \) was implicitly given in [14] (see also [6, Proposition 2.3]). Furthermore, the Hausdorff dimension of \( K_\beta(t) \) was implicitly given by Raith in [10], and was recently explicitly presented in [6, Equation (2.6)].

**Proposition 2.3.** (i) Let \( \beta \in (1, 2] \). Then
\[ E_\beta = \{ t \in [0, 1) : T^n_\beta(t) \geq t \ \forall n \geq 0 \}. \]

(ii) Let \( \beta \in (1, 2] \) and \( t \in [0, 1) \). Then the Hausdorff dimension of \( K_\beta(t) \) is given by
\[ \dim_H K_\beta(t) = \frac{h(\bar{K}_\beta(t))}{\log \beta}, \]
where \( \bar{K}_\beta(t) := \{ (x_i) \in \{0, 1\}^\mathbb{N} : b(t, \beta) \preceq \sigma^n((x_i)) \preceq \delta(\beta) \ \forall n \geq 0 \} \). Furthermore, the dimension function \( \eta_\beta : t \mapsto \dim_H K_\beta(t) \) is a Devil's staircase, i.e., \( \eta_\beta \) is a non-constant, decreasing and continuous function which is locally constant almost everywhere in \( [0, 1) \).

3. PROOF OF THEOREM 1

In this section we will prove Theorem 1. First we show that the dimension bifurcation set \( B_\beta \) coincides with the set-valued bifurcation set \( E_\beta \), we then derive a complete characterization of these sets via the \( \beta \)-Lyndon intervals. The proof heavily relies upon the transitivity of the symbolic survivor set \( \bar{K}_\beta(t) \) (see Lemma 3.2 below).

**Proposition 3.1.** Let \( \beta \in (1, 2) \) be a multinacci number. Then
\[ B_\beta = E_\beta = [0, 1 - \frac{1}{\beta}) \setminus \bigcup [t_L, t_R), \]
where the union is taken over all \( \beta \)-Lyndon intervals.

Observe by Lemma 2.2 that the \( \beta \)-Lyndon intervals are pairwise disjoint. In fact the closed \( \beta \)-Lyndon intervals \( \{[t_L, t_R]\} \) are also pairwise disjoint. So by Proposition 3.1 it follows that each closed \( \beta \)-Lyndon interval is a maximal interval where the dimension function \( \eta_\beta \) is constant.
The proof of Proposition 3.1 will be split into several lemmas. We fix a multinacci number
\( \beta \in \{1, 2\} \) with \( \delta(\beta) = (1^{m}0)^{\infty} \) for some \( m \geq 1 \). In view of Proposition 2.3 it is necessary to investigate the symbolic survivor set
\[
\tilde{K}_\beta(t) = \left\{ (x_i) \in \{0, 1\}^\mathbb{N} : b(t, \beta) \leq \sigma^n((x_i)) \leq \delta(\beta) \forall n \geq 0 \right\}.
\]

**Lemma 3.2.** Let \( \beta \in \{1, 2\} \) with \( \delta(\beta) = (1^{m}0)^{\infty} \), and let \([t_L, t_R) \subset [0, 1 - 1/\beta)\) be a \( \beta \)-Lyndon interval. Then the set-valued map \( t \mapsto \tilde{K}_\beta(t) \) is constant on \([t_L, t_R)\), and the set \(\tilde{K}_\beta(t_R)\) is a transitive subshift of finite type.

**Proof.** Suppose \([t_L, t_R)\) is a \( \beta \)-Lyndon interval generated by \( s_1 \ldots s_p \). First we claim that
\[
(3.1) \quad \sigma^n((x_i)) \succ s_1 \ldots s_p 0^\infty \forall n \geq 0 \iff \sigma^n((x_i)) \preceq (s_1 \ldots s_p)^{\infty} \forall n \geq 0.
\]
Since \((s_1 \ldots s_p)^{\infty} \succ s_1 \ldots s_p 0^\infty\), the implication \( \preceq \) in (3.1) is obvious. For the reverse implication we assume \( \sigma^n((x_i)) \prec (s_1 \ldots s_p)^{\infty} \) for some \( n \geq 0 \). Then there exists \( \ell \geq 0 \) such that
\[
x_{n+1} \ldots x_{n+p} = (s_1 \ldots s_p)^{\ell} \quad \text{and} \quad x_{n+p+1} \ldots x_{n+p+(\ell+1)p} \prec s_1 \ldots s_p.
\]
This yields \( \sigma^{n+p}(x_i) \prec s_1 \ldots s_p 0^\infty \), completing the proof of \( \Rightarrow \) in (3.1).

Take \( t \in [t_L, t_R) \). Then by Lemma 2.1(ii) it follows that
\[
\tilde{K}_\beta(t_R) \subseteq \tilde{K}_\beta(t) \subseteq \tilde{K}_\beta(t_L).
\]
Observe that \( \delta(\beta) = (1^{m}0)^{\infty} \) for some \( m \in \mathbb{N} \). Then
\[
(3.2) \quad \tilde{K}_\beta(t_L) = \left\{ (x_i) : s_1 \ldots s_p 0^\infty \preceq \sigma^n((x_i)) \preceq (1^{m}0)^{\infty} \forall n \geq 0 \right\}
= \left\{ (x_i) : (s_1 \ldots s_p)^{\infty} \preceq \sigma^n((x_i)) \preceq (1^{m}0)^{\infty} \forall n \geq 0 \right\} = \tilde{K}_\beta(t_R).
\]
So, the set-valued map \( t \mapsto \tilde{K}_\beta(t) \) is constant on \([t_L, t_R)\). Furthermore, \(\tilde{K}_\beta(t_R)\) is a subshift of finite type with the set of forbidden blocks given by
\[
\mathcal{F} = \left\{ c_1 \ldots c_k \in \{0, 1\}^k : c_1 \ldots c_k 0^\infty \prec s_1 \ldots s_p 0^\infty \quad \text{or} \quad c_1 \ldots c_k 0^\infty \succ (1^{m}0)^{\infty} \right\},
\]
where \( k = \max \{p, m + 1\} \). It remains to prove the transitivity of \(\tilde{K}_\beta(t_R)\).

Since \([t_L, t_R) \subset [0, 1 - 1/\beta)\), by Lemma 2.1 (ii) it follows that \( b(t_R, \beta) < b(1 - 1/\beta, \beta) \), which gives
\[
(3.3) \quad (s_1 \ldots s_p)^{\infty} \prec 01^{m}0^{\infty}.
\]
Arbitrarily fix an admissible word \( \varepsilon = \varepsilon_1 \ldots \varepsilon_k \) and an admissible sequence \( \gamma = \gamma_1 \gamma_2 \ldots \) in \(\tilde{K}_\beta(t_R)\). We will construct a word \( \nu \) such that \( \varepsilon \nu \gamma \in \tilde{K}_\beta(t_R) \). Observe that \( \sigma^n((s_1 \ldots s_p)^{\infty}) \prec (1^{m}0)^{\infty} \) for all \( n \geq 0 \). Thus, there exists a large integer \( N \) such that
\[
(3.4) \quad \sigma^n((s_1 \ldots s_p)^{\infty}) \prec (1^{m}0)^{N}0^{\infty} \quad \text{for all} \quad n \geq 0.
\]
Denote by \( (\delta_i) := \delta(\beta) = (1^{m}0)^{\infty} \). Note that \( \varepsilon_{i+1} \ldots \varepsilon_k \preceq \delta_1 \ldots \delta_{k-1} \) for all \( 0 \leq i < k \). Let \( i_0 \in \{0, 1, \ldots, k - 1\} \) be the smallest index such that
\[
\varepsilon_{i_0+1} \ldots \varepsilon_k = \delta_1 \ldots \delta_{k-i_0}.
\]
If such an index \( i_0 \) does not exist, then we put \( i_0 = k \). In either case there exists a word \( \mu \) such that \( \varepsilon \mu = \varepsilon_1 \ldots \varepsilon_{i_0} (1^{m}0)^{N} \). Since \( \gamma \preceq (1^{m}0)^{\infty} \), there exists \( q \in \{0, 1, \ldots, m\} \) such that
γ begins with γ₁...γ_{q+1} = 1^q0. We emphasize here that if q = 0 then γ begins with digit 0. Now we claim that
\[ \varepsilon \mu 1^{m-q} \gamma = \varepsilon_1...\varepsilon_{i_0} (1^{m}0)^{N+1} \gamma_{q+2} \gamma_{q+3} \ldots \in \bar{K}_\beta(t_R), \]
or equivalently,
\[
(3.5) \quad (s_1 \ldots s_p) \bowtie \sigma^n(\varepsilon \mu 1^{m-q} \gamma) \bowtie (1^{m}0)^{\infty} \quad \text{for all } n \geq 0.
\]

First we prove the second inequality in (3.5). By the definition of \( i_0 \) it follows that
\[ \sigma^n(\varepsilon \mu 1^{m-q} \gamma) \bowtie \delta(\beta) = (1^{m}0)^{\infty} \]
holds for all \( 0 \leq n < i_0 \). Furthermore, since γ ∈ \( \bar{K}_\beta(t_R) \), the second inequality in (3.5) also holds for \( n \geq |\varepsilon| + |\mu| + m - q \). Here for a word \( c \) we denote its length by |c|. For the remaining \( n \) we observe that \( \sigma^{i_0}(\varepsilon \mu 1^{m-q} \gamma) = (1^{m}0)^{N+1} \gamma_{q+2} \gamma_{q+3} \ldots \)
and \( \gamma_{q+2} \gamma_{q+3} \ldots \in \bar{K}_\beta(t_R) \). So it is easy to verify that
\[ \sigma^n(\varepsilon \mu 1^{m-q} \gamma) \bowtie (1^{m}0)^{\infty} \quad \text{for all } i_0 \leq n < |\varepsilon| + |\mu| + m - q. \]

This proves the second inequality in (3.5).

For the first inequality in (3.5) we observe that \( \varepsilon \mu 1^{m-q} \gamma = \varepsilon_1...\varepsilon_{i_0} (1^{m}0)^{N} \gamma_{q+2} \gamma_{q+3} \ldots \)
and \( \gamma_{q+2} \gamma_{q+3} \ldots \in \bar{K}_\beta(t_R) \). Then by (3.3) it follows that
\[ \sigma^n(\varepsilon \mu 1^{m-q} \gamma) \bowtie (s_1 \ldots s_p)^{\infty} \quad \text{for all } n \geq i_0. \]
If \( i_0 = 0 \), then we are done. Otherwise, we take \( 0 \leq n < i_0 \). Since \( \varepsilon_1...\varepsilon_{i_0} \) is an admissible word in \( \bar{K}_\beta(t_R) \), we have
\[ \varepsilon_{n+1}...\varepsilon_{i_0} \bowtie t_1...t_{i_0-n}, \]
where \( (t_i) := (s_1 \ldots s_p)^{\infty} \). The first inequality in (3.5) now holds by (3.4), which tells us that
\[ (1^{m}0)^{N} \gamma_{q+2} \gamma_{q+3} \ldots \bowtie t_{i_0-n+1}t_{i_0-n+2} \ldots. \]
This completes the proof of our claim.

Since ε and γ are chosen arbitrarily, it follows that \( \bar{K}_\beta(t_R) \) is transitive. \( \Box \)

Remark 3.3. \begin{itemize}
\item The fact that \( \bar{K}_\beta(t_R) \) is a subshift of finite type can also be deduced from [7].
\item The proof of Lemma 3.2 can be adjusted to prove the more general case with \( \beta > 2 \)
with \( \delta(\beta) = (M^m k)^{\infty} \), where \( M = \lceil \beta \rceil - 1 \) and \( k \in \{0, 1, \ldots, M-1 \} \). The transitivity property of \( \bar{K}_\beta(t_R) \) holds only for \( t_R \) sufficiently close to 0.
\end{itemize}

To prove the coincidence of \( \mathcal{B}_\beta \) and \( \mathcal{E}_\beta \) we still need the following inequalities.

Lemma 3.4. Let \( (t_1 \ldots t_N)^{\infty} \in \{0, 1\}^N \) be a periodic sequence with period \( N \geq 2 \). If
\[ \sigma^n((t_1 \ldots t_N)^{\infty}) \bowtie (t_1 \ldots t_N)^{\infty} \quad \forall \ n \geq 0, \]
then
\[ t_{j+1} \ldots t_N > t_1 \ldots t_{N-j} \quad \forall \ 1 \leq j < N. \]

Proof. Note that \( N \geq 2 \) is the period of \( (t_1 \ldots t_N)^{\infty} \), and
\[ \sigma^n((t_1 \ldots t_N)^{\infty}) \bowtie (t_1 \ldots t_N)^{\infty} \quad \forall \ n \geq 0. \]
Then \( t_1 = 0 \) and \( t_N = 1 \). Taking the reflection on both sides of (3.6) it follows that
\[ \sigma^n((\overline{t_1 \ldots t_N})^{\infty}) \bowtie (t_1 \ldots t_N)^{\infty} \quad \text{for all } n \geq 0. \]
Here for a word $c_1 \ldots c_k \in \{0,1\}^k$ its reflection is defined by $\overline{c_1 \ldots c_k} := (1-c_1)(1-c_2) \ldots (1-c_k)$. By Lemma 2.1(i) it follows that $(t_1 \ldots t_N)\infty$ is the quasi-greedy expansion of 1 for some base $\beta' \in (1,2]$, i.e., $\delta(\beta') = (t_1 \ldots t_N)\infty$. Since $N$ is the period of the sequence $\delta(\beta')$, the greedy $\beta'$-expansion of 1 is given by

$$b(1,\beta') = \overline{t_1 \ldots t_{N-1}}10^\infty.$$ 

So, by Lemma 2.1 (iii) it follows that

$$t_{j+1} \ldots t_N < \overline{t_{j+1} \ldots t_{N-1}}1 \leq \overline{t_1 \ldots t_{N-j}} \text{ for all } 1 \leq j < N.$$

Then the lemma follows by taking the reflection in the above equation. \qed

Now we prove the coincidence of the two bifurcation sets.

**Lemma 3.5.** Let $\beta \in (1,2]$ with $\delta(\beta) = (1^m0)\infty$. Then $\mathcal{E}_\beta = \mathcal{B}_\beta$.

**Proof.** By the definition of the two bifurcation sets it is easy to see that $\mathcal{B}_\beta \subset \mathcal{E}_\beta$. So in the following we prove $\mathcal{E}_\beta \subset \mathcal{B}_\beta$.

Let $t \in \mathcal{E}_\beta$ with its greedy $\beta$-expansion $b(t,\beta) = (t_i)$. Then by Theorem 1.1 we have $t \leq 1 - 1/\beta < 1/\beta$. This gives $t_1 = 0$. By Lemmas 2.1 (ii) and Proposition 2.3 (i) it follows that

$$\sigma^n((t_i)) \succ (t_i) \text{ for all } n \geq 0.$$

Let $N \geq 1$ be the smallest index such that $\sigma^n((t_i)) = (t_i)$. If such an integer $N$ does not exist, then we set $N = \infty$. In the following we will prove $t \in \mathcal{B}_\beta$ by considering the following two cases: (I) $N < \infty$; and (II) $N = \infty$.

Case (I). $N < \infty$. We claim that $t_1 \ldots t_N$ is a $\beta$-Lyndon word. If $N = 1$, then $(t_i) = t_1^\infty = 0^\infty$. It is easy to check that $t_1 = 0$ is a $\beta$-Lyndon word. In the following we assume $N \geq 2$. Since $\sigma^N((t_i)) = (t_i)$, we have $(t_i) = (t_1 \ldots t_N)\infty$. Note that $(t_i)$ is the greedy $\beta$-expansion of $t$. Then by Lemma 2.1 (ii) it follows that

$$\sigma^n((t_1 \ldots t_N)\infty) \succ \delta(\beta) \text{ for all } n \geq 0.$$

Note that $\sigma^n((t_1 \ldots t_N)\infty) \succ (t_1 \ldots t_N)\infty$. Then by Lemma 3.4 and the definition of $N$, it follows that

$$t_{j+1} \ldots t_N > t_1 \ldots t_{N-j} \text{ for all } 1 \leq j < N.$$

So by Definition 1.2 we establish the claim.

Hence, $t = ((t_1 \ldots t_N)\infty)\beta = t_R$ is the right endpoint of a $\beta$-Lyndon interval generated by $t_1 \ldots t_N$. By Lemma 3.2 it follows that $\tilde{K}_\beta(t)$ is a transitive subshift of finite type. Observe that for any $t' > t$ we have

$$\tilde{K}_\beta(t') \subset \tilde{K}_\beta(t) \text{ and } (t_1 \ldots t_N)\infty \in \tilde{K}_\beta(t) \setminus \tilde{K}_\beta(t').$$

Recall by [8, Corollary 4.4.9] that for any transitive subshift of finite type, any proper subshift has strictly smaller topological entropy. Therefore,

$$h(\tilde{K}_\beta(t')) < h(\tilde{K}_\beta(t)) \text{ for any } t' > t.$$

By Proposition 2.3 (ii) this yields $\eta_\beta(t') < \eta_\beta(t)$ for any $t' > t$. So $t \in \mathcal{B}_\beta$.

Case (II). $N = \infty$. Then $\sigma^n((t_i)) \succ (t_i)$ for all $n \geq 1$. So $(t_i)$ is not periodic. Observe that $(t_i)$ begins with digit 0, and

$$\sigma^n((t_i)) \prec (1^m0)\infty \text{ for all } n \geq 0.$$
So there exists a subsequence \((m_k)\) of positive integers such that for any \(k \geq 1\) we have \(t_{m_k} = 0\), and the word \(t_1 \ldots t_{m_k} := t_1 \ldots t_{m_k-1}1\) does not contain \(m+1\) consecutive ones. Then by noting \(t_1 = 0\) it follows that
\[
\sigma^n((t_1 \ldots t_{m_k})^\infty) \prec (1^m0)^\infty \quad \forall \ n \geq 0.
\]
Since \(\sigma^n((t_i)) \succeq (t_i)\) for all \(n \geq 0\), by Definition 1.2 it follows that \(t_1 \ldots t_{m_k}\) is a \(\beta\)-Lyndon word for any \(k \geq 1\). Let \(s_k := ((t_1 \ldots t_{m_k})^\infty)_\beta\). Then \(s_k\) is the right endpoint of a \(\beta\)-Lyndon interval generated by \(t_1 \ldots t_{m_k}\). Furthermore, \(s_k\) strictly decreases to \(t = ((t_i))_\beta\) as \(k \to \infty\).

So, for any \(t' > t\) we can find \(k\) such that \(s_k \in (t, t')\). By the same arguments as in the proof of Case (I) for \(s_k\) we conclude that
\[
\eta_\beta(t') < \eta_\beta(s_k) \leq \eta_\beta(t).
\]
So \(t \in \mathcal{B}_\beta\), completing the proof. \(\square\)

Finally, we describe the bifurcation sets via the \(\beta\)-Lyndon intervals.

**Lemma 3.6.** Let \(\beta \in (1, 2]\) with \(\delta(\beta) = (1^m0)^\infty\). Then
\[
\left[0, 1 - \frac{1}{\beta}\right) \setminus \bigcup [t_L, t_R) \subset \mathcal{E}_\beta.
\]

**Proof.** Take \(t \in [0, 1 - 1/\beta) \setminus \mathcal{E}_\beta\) with its greedy \(\beta\)-expansion \((t_i)\). Then \(t_1 = 0\). Since \(t \notin \mathcal{E}_\beta\), by Proposition 2.3 (i) there exists a smallest positive integer \(N\) such that \(T_\beta^N(t) < t\), which implies
\[
(3.7) \quad t_{N+1}t_{N+2} \ldots \prec (t_i).
\]
We claim that \(t_1 \ldots t_N\) is a \(\beta\)-Lyndon word. Clearly, if \(N = 1\) then \(t_1 = 0\) is a \(\beta\)-Lyndon word. In the following we assume \(N \geq 2\). By Definition 1.2 it suffices to prove
\[
(3.8) \quad t_{j+1} \ldots t_N > t_1 \ldots t_{N-j} \quad \text{for all } 1 \leq j < N,
\]
and
\[
(3.9) \quad \sigma^n((t_1 \ldots t_N)^\infty) \prec (1^m0)^\infty \quad \text{for all } n \geq 0.
\]

First we prove (3.8). By the definition of \(N\) in (3.7) it follows that
\[
(3.10) \quad t_{j+1}t_{j+2} \ldots \succ (t_i) \quad \text{for all } 1 \leq j < N,
\]
which implies \(t_{j+1} \ldots t_N \succ t_1 \ldots t_{N-j}\) for all \(1 \leq j < N\). Suppose \(t_{j+1} \ldots t_N = t_1 \ldots t_{N-j}\) for some \(j \in \{1, 2, \ldots, N-1\}\). Applying (3.7) and then (3.10) it follows that
\[
t_{j+1}t_{j+2} \ldots = t_1 \ldots t_{N-j}t_{N+1}t_{N+2} \ldots \prec t_1 \ldots t_{N-j}t_1t_2 \ldots \prec (t_i),
\]
leading to a contradiction with the minimality of \(N\). This proves (3.8).

To prove (3.9) we observe that \(\delta(\beta) = (1^m0)^\infty\) and \((t_i)\) is the greedy \(\beta\)-expansion of \(t\). Then by Lemma 2.1 (ii) it follows that \(t_1 \ldots t_N\) cannot contain \(m+1\) consecutive ones. Since \(t_1 = 0\), we have
\[
\sigma^n((t_1 \ldots t_N)^\infty) \prec (1^m0)^\infty \quad \text{for all } n \geq 0.
\]
So to prove (3.9) it remains to prove that \(\sigma^n((t_1 \ldots t_N)^\infty) \neq (1^m0)^\infty\) for any \(n \geq 0\). Suppose the equality \(\sigma^n((t_1 \ldots t_N)^\infty) = (1^m0)^\infty\) holds for some \(n \geq 0\). Then by using \(t_1 = 0\) it follows that
\[
t_1 \ldots t_{m+1} = 01^m.
\]
This implies \( b(t, \beta) = (t_i) \not\in 01^\infty \) = \( b(1 - 1/\beta, \beta) \). By Lemma 2.1 (ii) we have \( t \geq 1 - 1/\beta \), leading to a contradiction. This establishes (3.9).

By the claim there exists a \( \beta \)-Lyndon interval \([t_L, t_R]\) generated by \( t_1 \ldots t_N \). Furthermore, by (3.7) it follows that

\[
(t_i) = t_1 \ldots t_N t_{N+1} t_{N+2} \ldots < t_1 \ldots t_N t_1 t_2 \ldots = (t_1 \ldots t_N)^2 t_{N+1} t_{N+2} \ldots < (t_1 \ldots t_N)^3 t_{N+1} t_{N+2} \ldots \
\cdots \
\leq (t_1 \ldots t_N)\infty.
\]

Therefore, \( t_1 \ldots t_N 0^\infty \not\leq (t_i) < (t_1 \ldots t_N)^\infty \), which gives \( t \in [t_L, t_R] \) by Lemma 2.1 (ii). This completes the proof. \(\square\)

**Proof of Proposition 3.1.** By Lemmas 3.5 and 3.6 it suffices to prove

\[
\mathcal{B}_\beta \subseteq \left[ 0, 1 - \frac{1}{\beta} \right) \setminus \bigcup [t_L, t_R).
\]

Note by Lemma 3.5 and Theorem 1.1 that \( \mathcal{B}_\beta = \mathcal{E}_\beta \subseteq [0, 1 - 1/\beta] \). In fact we have \( \mathcal{E}_\beta \subseteq [0, 1 - 1/\beta] \). Observe that \( b(1 - 1/\beta, \beta) = 01^\infty 0^\infty \). Then \( T_\beta^{m+1} (1 - 1/\beta) < 1 - 1/\beta \). By Proposition 2.3 (i) this implies \( 1 - 1/\beta \not\in \mathcal{E}_\beta \). Hence, \( \mathcal{E}_\beta \subseteq \mathcal{B}_\beta \subseteq [0, 1 - 1/\beta] \).

In the following it remains to prove \( \mathcal{B}_\beta \cap \bigcup [t_L, t_R) = \emptyset \). Take a \( \beta \)-Lyndon interval \([t_L, t_R]\). If \( t \in [t_L, t_R) \), then by (3.2) it follows that

\[
\tilde{K}_\beta(t) = \tilde{K}_\beta(t_L) = \tilde{K}_\beta(t_R),
\]

which gives \( \eta_\beta(t') = \eta_\beta(t) = \eta_\beta(t_L) \) for all \( t' \in (t, t_R) \). So, \( t \not\in \mathcal{B}_\beta \). \(\square\)

As a consequence of Proposition 3.1 and Theorem 1.1 it follows that for \( \beta \in (1, 2] \) a multinacci number the \( \beta \)-Lyndon intervals cover \([0, 1 - 1/\beta] \) up to a Lebesgue null set.

**Corollary 3.7.** Let \( \beta \in (1, 2] \) be a multinacci number.

(i) The union of all \( \beta \)-Lyndon intervals covers \([0, 1 - 1/\beta] \) up to a Lebesgue null set. Furthermore, for any \( t \in \mathcal{B}_\beta \) and any \( r > 0 \) the interval \((t, t + r)\) contains infinitely many \( \beta \)-Lyndon intervals.

(ii) \( \eta_\beta(t) > 0 \) if and only if \( t < 1 - 1/\beta \).

**Proof.** Note that \( \mathcal{E}_\beta \) is a Lebesgue null set which, by Theorem 1.1, has no isolated points. Then (i) follows from Proposition 3.1 which tells us that \( \bigcup [t_L, t_R) = [0, 1 - 1/\beta] \setminus \mathcal{E}_\beta \). For (ii) it can be deduced from Proposition 3.1 and Theorem 1.1 that \( \sup \mathcal{B}_\beta = 1 - 1/\beta \) and \( 1 - 1/\beta \not\in \mathcal{B}_\beta \). \(\square\)

Now we turn to investigate the local dimension of the bifurcation set \( \mathcal{B}_\beta \).

**Lemma 3.8.** Let \( \beta \in (1, 2] \) with \( \delta(\beta) = (1^m 0)^\infty \). Then

\[
\lim_{r \to 0} \dim_H (\mathcal{B}_\beta \cap (t, t + r)) = \dim_H \tilde{K}_\beta(t) > 0 \quad \forall \ t \in \mathcal{B}_\beta.
\]
Proof. Take $t \in \mathcal{B}_\beta$. By Proposition 3.1 we have $t < 1 - 1/\beta$, and then by Corollary 3.7 (ii) it gives $\eta_\beta(t) = \dim_H K_\beta(t) > 0$. Note by Proposition 3.1 and Proposition 2.3 (i) that
$$
\mathcal{B}_\beta \cap (t, t + r) = \mathcal{E}_\beta \cap (t, t + r) \subseteq K_\beta(t) \quad \text{for any } r > 0.
$$
Then $\lim_{r \to 0} \dim_H (\mathcal{B}_\beta \cap (t, t + r)) \leq \eta_\beta(t)$. So it remains to prove
$$
\lim_{r \to 0} \dim_H (\mathcal{B}_\beta \cap (t, t + r)) \geq \eta_\beta(t).
$$
We prove this now by considering the following two cases: (I) $t = t_R$ is the right endpoint of a $\beta$-Lyndon interval; (II) $t \in [0, 1 - 1/\beta) \setminus \bigcup [t_L, t_R]$.

Case (I). Suppose $t = t_R$ is the right endpoint of a $\beta$-Lyndon interval. Let $(t_i) = (t_1 \ldots t_p)^\infty$ be the greedy $\beta$-expansion of $t_R$. Note that $t_R \in \mathcal{B}_\beta$. Then by Corollary 3.7 (i) there exists a sequence $(t_R^{(n)}) \subseteq \mathcal{B}_\beta$ such that each $t_R^{(n)}$ is a right endpoint of a $\beta$-Lyndon interval and $t_R^{(n)} \searrow t_R$ as $n \to \infty$. Fix $r > 0$. Then we can find a large integer $N$ satisfying
$$
t_R^{(n)} \in (t_R, t_R + r) \quad \text{for all } n \geq N.
$$
Furthermore, since $b(t_R, \beta) = (t_1 \ldots t_p)^\infty$, by Lemma 2.1 (ii) it follows that for each $n \geq N$ there exists an integer $k_n$ such that the greedy $\beta$-expansion $b(t_R^{(n)}, \beta)$ of $t_R^{(n)}$ satisfies
$$
b(t_R^{(n)}, \beta) > (t_1 \ldots t_p)^{k_n} 1^\infty.
$$
Observe by Proposition 3.1 and Proposition 2.3 (i) that
$$
\mathcal{B}_\beta = \mathcal{E}_\beta = \{(s_i) : (s_i) \preceq \sigma^n((s_i)) < (1m0)^\infty \forall n \geq 0 \}.
$$
So by using $t_R \in \mathcal{B}_\beta$, (3.12) and Lemma 2.1 (ii) it follows that for any $n \geq N$,
$$
\{(t_1 \ldots t_p)^{k_n} x_1 x_2 \ldots) : x_1 \ldots x_p = t_1 \ldots t_p, (x_i) \in \mathcal{K}_\beta(t_R^{(n)})\}
\subseteq \mathcal{B}_\beta \cap [t_R, t_R^{(n)}]
\subseteq \mathcal{B}_\beta \cap [t_R, t_R + r).
$$
Note by Lemma 3.2 that $\mathcal{K}_\beta(t_R^{(n)})$ is a transitive subshift of finite type. Then by (3.13) it follows that
$$
\dim_H (\mathcal{B}_\beta \cap (t_R, t_R + r)) \geq \dim_H K_\beta(t_R^{(n)}) = \eta_\beta(t_R^{(n)}) \quad \text{for all } n \geq N.
$$
Letting $n \to \infty$ and by the continuity of $\eta_\beta$ (see Proposition 2.3 (ii)) we obtain that
$$
\dim_H (\mathcal{B}_\beta \cap (t_R, t_R + r)) \geq \eta_\beta(t_R).
$$
Since $r > 0$ was given arbitrary, letting $r \to 0$ we conclude that
$$
\lim_{r \to 0} \dim_H (\mathcal{B}_\beta \cap (t_R, t_R + r)) \geq \eta_\beta(t_R).
$$

Case (II). $t \in [0, 1 - 1/\beta) \setminus \bigcup [t_L, t_R]$. Then by Corollary 3.7 (i) there exists a sequence $(t_R^{(k)})$ such that each $t_R^{(k)}$ is the right endpoint of a $\beta$-Lyndon interval, and $t_R^{(k)} \searrow t$ as $k \to \infty$. So, for any $r > 0$ there exists a sufficiently large integer $k$ such that $t_R^{(k)} \in (t, t + r)$. By (3.14) with $t_R$ replaced by $t_R^{(k)}$ it follows that for any $\varepsilon > 0$ there exists $r_k > 0$ such that $(t_R^{(k)} <= t_R^{(k)} + r_k) \subseteq (t, t + r)$ and
$$
\dim_H (\mathcal{B}_\beta \cap (t, t + r)) \geq \dim_H (\mathcal{B}_\beta \cap (t_R^{(k)}, t_R^{(k)} + r_k)) \geq \eta_\beta(t_R^{(k)}) - \varepsilon.
$$
Letting $r \to 0$, and then $t^{(k)}_R \to t$, we conclude by the continuity of $\eta_\beta$ that
\[
\lim_{{r \to 0}} \dim_H(\mathcal{B}_\beta \cap (t, t + r)) \geq \eta_\beta(t) - \varepsilon.
\]
Since $\varepsilon > 0$ was arbitrary, we obtain $\lim_{{r \to 0}} \dim_H(\mathcal{B}_\beta \cap (t, t + r)) \geq \eta_\beta(t)$. This, together with (3.14), proves (3.11).

\[\square\]

**Proof of Theorem 1.** Let $\beta \in (1, 2)$ with $\delta(\beta) = (1^m0)^\infty$. By Lemma 2.2, Proposition 3.1 and Lemma 3.8 it suffices to prove

\[
(3.15) \quad \left\{ t \in [0, 1) : \lim_{{r \to 0}} \dim_H(\mathcal{B}_\beta \cap (t, t + r)) = \eta_\beta(t) > 0 \right\} \subset \mathcal{B}_\beta.
\]

Take $t \in [0, 1) \setminus \mathcal{B}_\beta$. Then by Proposition 3.1 we have $t \in [1 - 1/\beta, 1)$ or $t \in [t_L, t_R)$ for some $\beta$-Lyndon interval. If $t \geq 1 - 1/\beta$, then $\eta_\beta(t) = 0$ by Corollary 3.7 (ii). If $t \in [t_L, t_R)$, then by Proposition 3.1 there exists $r > 0$ such that $\mathcal{B}_\beta \cap (t, t + r) = \emptyset$. This completes the proof.

\[\square\]

**Proof of Corollary 2.** Note by Proposition 3.1 that $\mathcal{E}_\beta \subset [0, 1 - 1/\beta)$. So if $t \geq 1 - 1/\beta$, then clearly the result holds by Corollary 3.7 (ii). Now let $t \in [0, 1 - 1/\beta)$. Observe by Proposition 2.3 (i) that $\mathcal{E}_\beta \cap [t, 1] \subset K_\beta(t)$. So it suffices to prove

\[
(3.16) \quad \dim_H(\mathcal{E}_\beta \cap [t, 1]) \geq \dim_H K_\beta(t).
\]

If $t \in [0, 1 - 1/\beta) \setminus [t_L, t_R)$, then (3.16) follows by Lemma 3.8. If $t \in [t_L, t_R)$, then we still have (3.16) by using Lemma 3.8 that

\[
\dim_H(\mathcal{E}_\beta \cap [t, 1]) \geq \dim_H(\mathcal{E}_\beta \cap [t, 1]) \geq \dim_H(\mathcal{E}_\beta \cap [t_R, 1]) = \dim_H K_\beta(t_R) = \dim_H K_\beta(t),
\]

where the last equality holds by (3.2).

\[\square\]

4. **Final remarks**

The main results obtained in this paper can be easily modified to study the following analogous bifurcation sets:

\[
\mathcal{E}_\beta = \left\{ t \in [0, 1) : K_\beta(t') ≠ K_\beta(t) \forall t' ≠ t \right\},
\]

\[
\mathcal{B}_\beta = \left\{ t \in [0, 1) : \dim_H K_\beta(t') ≠ \dim_H K_\beta(t) \forall t' ≠ t \right\}.
\]

If $\beta \in (1, 2]$ is a multinacci number, one can show that

\[
\mathcal{B}_\beta = \mathcal{E}_\beta = \left[ 0, 1 - \frac{1}{\beta} \right) \cup [t_L, t_R]
\]

\[
= \left\{ t \in [0, 1) : \lim_{{r \to 0}} \dim_H(\mathcal{E}_\beta \cap (t - r, t)) = \dim_H(\mathcal{E}_\beta \cap (t, t + 0)) = \dim_H K_\beta(t) > 0 \right\},
\]

where the union is taken over all pairwise disjoint closed $\beta$-Lyndon intervals.

Observe that the main result Theorem 1 holds under the assumption that $\beta \in (1, 2]$ is a multinacci number, i.e., $\delta(\beta) = (1^m0)^\infty$ for some $m \in \mathbb{N}$. The method used in this paper can be adapted to show that Theorem 1 still holds for $\beta \in (1, 2]$ with $\delta(\beta) = (10^m)^\infty$. It is worth mentioning that in [6] Kalle et al. considered a general Farey word base $\beta$, i.e., $\delta(\beta) = (s_1…s_p)^\infty$ with $s_p, s_{p-1}…s_2 s_1$ a non-degenerate Farey word. They showed that for a general Farey word base $\beta \in (1, 2)$, the set-valued bifurcation set $\mathcal{E}_\beta$ has no isolated points and Theorem 1.1 holds. We finish by posing the following conjecture.
Conjecture 4.1. Let $\beta \in (1, 2]$. Then $\mathcal{B}_\beta = \mathcal{E}_\beta$ if and only if $\mathcal{E}_\beta$ has no isolated points.

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