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DOI:
10.1016/j.disc.2014.02.013

Citation for published version (Harvard):
Leader, I & Long, E 2014, 'Long geodesics in subgraphs of the cube', *Discrete Mathematics*, vol. 326, pp. 29-33. https://doi.org/10.1016/j.disc.2014.02.013

Discrete Mathematics 326 (2014) 29–33

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Long geodesics in subgraphs of the cube

Imre Leader [a], Eoin Long [b,*]

[a] Department of Pure Mathematics and Mathematical Statistics, Centre for Mathematical Sciences, Cambridge CB3 0WB, United Kingdom

[b] School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, United Kingdom

ARTICLE INFO

Article history:
Received 23 January 2013
Received in revised form 13 February 2014
Accepted 14 February 2014
Available online 18 March 2014

Keywords:
Geodesic
Hypercube
Antipodal colouring

ABSTRACT

We show that any subgraph of the hypercube Q_n of average degree d contains a geodesic of length d, where by geodesic we mean a shortest path in Q_n. This result, which is best possible, strengthens a theorem of Feder and Subi. It is also related to the 'antipodal colourings' conjecture of Norine.

1. Introduction

Given a graph G of average degree d, a classic result of Dirac [3] guarantees that G contains a path of length d. Moreover, for general graphs this is the best possible bound, as can be seen by taking G to be K_{d+1}, the complete graph on $d+1$ vertices.

The hypercube Q_n has vertex set $\{0, 1\}^n$ and two vertices $x, y \in Q_n$ are joined by an edge if they differ on a single coordinate. In [9] a similar question was considered for subgraphs of the hypercube Q_n. That is, given a subgraph G of Q_n of average degree d, how long a path must G contain? The main result was the following theorem.

Theorem 1.1 ([9]). *Every subgraph G of Q_n of minimum degree d contains a path of length $2^d - 1$.*

Combining Theorem 1.1 with the standard fact that any graph of average degree d contains a subgraph with minimum degree at least $d/2$, we see that any subgraph G of Q_n with average degree d contains a path of length at least $2^{d/2} - 1$.

In this paper we consider the analogous question for geodesics. A path in Q_n is a geodesic if it forms a shortest path in Q_n between its endpoints. Equivalently, a path is a geodesic if no two of its edges have the same direction, where an edge $xy \in E(Q_n)$ is said to have direction i when x and y differ in coordinate i. Given a subgraph G of Q_n of average degree d, how long a geodesic must G contain?

It is trivial to see that any such graph must contain a geodesic of length $d/2$. Indeed, taking a subgraph G' of G with minimal degree at least $d/2$ and starting from any vertex of G', we can greedily pick a geodesic of length $d/2$ by choosing a new edge direction at each step.

On the other hand the d-dimensional cube Q_d shows that, in general, we cannot find a geodesic of length greater than d in G. Our main result is that this upper bound is sharp.

Theorem 1.2. *Every subgraph G of Q_n of average degree d contains a geodesic of length at least d.*

* Corresponding author.
E-mail addresses: I.Leader@dpmms.cam.ac.uk (I. Leader), Eoin.Long@maths.ox.ac.uk, E.P.Long@qmul.ac.uk (E. Long).

http://dx.doi.org/10.1016/j.disc.2014.02.013
0012-365X/

Since the endpoints of the geodesic in G guaranteed by Theorem 1.2 are at Hamming distance at least d, Theorem 1.2 extends the following result of Feder and Subi [4].

Theorem 1.3 ([4]). *Every subgraph G of Q_n of average degree d contains two vertices at Hamming distance d apart.*

We remark that neither Theorem 1.2 nor Theorem 1.3 follow from isoperimetric considerations alone. Indeed, if G is a subgraph of Q_n of average degree d, by the edge isoperimetric inequality for the cube ([1,5,6,8]; see [2] for background) we have $|G| \geq 2^d$. However if n is large, a Hamming ball of small radius may have size larger than 2^d without containing a long geodesic.

While Theorem 1.2 implies Theorem 1.3, we have also given an alternate proof of Theorem 1.3 from a result of Katona [7] which we feel may be of interest. The proofs of Theorems 1.2 and 1.3 are given in Sections 2 and 3 respectively.

Finally, Feder and Subi's theorem was motivated by a conjecture of Norine [10] on antipodal colourings of the cube. In the last section of this short paper we discuss Theorem 1.2 in relation to Norine's conjecture.

Notation: our notation is standard. Given a graph G, let $|G|$ denote the number of vertices of G and let $E(G)$ denote the edge set of G. Given a path $P = x_0 \ldots x_l$, we say that P has length l and denote this by writing $|P| = l$. Given a path $P = x \cdots y$ and a vertex $z \notin V(P)$, we write Pyz to denote the path obtained by adjoining the edge yz to P. Given a set X, we write $\mathcal{P}(X)$ for its power set and $X^{(k)}$ for the set of subsets of X of size k. For $n \in \mathbb{N}$, let $[n] = \{1, \ldots, n\}$.

2. Proofs of Theorems 1.2 and 1.3

To prove Theorem 1.2 we will actually establish a stronger result. A path $P = x_1x_2 \ldots x_l$ in Q_n is an *increasing geodesic* if the directions of the edges x_ix_{i+1} increase with i. An increasing geodesic P *ends* at a vertex x if $x = x_l$. For any vertex $x \in G$ we let $L_G(x)$ denote an increasing geodesic in G of maximum length which ends at x. The key idea to the proof is to show that on average $|L_G(x)|$ is large. This allows us to simultaneously keep track of geodesics for all vertices of G, which is vital in the inductive proof below.

Theorem 2.1. *Let G be a subgraph of Q_n of average degree d. Then*

$$\sum_{v \in V(G)} |L_G(v)| \geq d|G|.$$

Proof. Write $S(G)$ for $\sum_{v \in V(G)} |L_G(v)|$. We will show that for any subgraph G of Q_n, we have $S(G) \geq 2|E(G)|$, by induction on $|E(G)|$. The base case $|E(G)| = 0$ is immediate. Assume the result holds by induction for all graphs with $|E(G)| - 1$ edges and that we wish to prove the result for G.

Pick an edge $e = xy$ of G with largest coordinate direction and look at the graph $G' = G - e$. By the induction hypothesis, we have

$$S(G') = \sum_{v \in V(G')} |L_{G'}(v)| \geq 2|E(G')| = 2(|E(G)| - 1).$$

Now clearly we must have $|L_G(v)| \geq |L_{G'}(v)|$ for all vertices $v \in G$. Furthermore, notice that the coordinate direction of e cannot appear on the increasing geodesics $L_{G'}(x)$ and $L_{G'}(y)$. Indeed, the edge of $L_{G'}(x)$ adjacent to x has direction less than e and as $L_{G'}(x)$ is an increasing geodesic, the directions of all edges in $L_{G'}(x)$ must be less than e. We now consider two cases.

Case I: $|L_{G'}(x)| = |L_{G'}(y)|$. Then the paths $L_{G'}(x)xy$ and $L_{G'}(y)yx$ are increasing geodesics in G ending at y and x respectively. Therefore $|L_G(x)| \geq |L_{G'}(x)| + 1$ and $|L_G(y)| \geq |L_{G'}(y)| + 1$ and $S(G) \geq S(G') + 2 \geq 2|E(G')| + 2 = 2|E(G)|$.

Case II: $|L_{G'}(x)| \neq |L_{G'}(y)|$. Without loss of generality assume that $|L_{G'}(x)| \geq |L_{G'}(y)| + 1$. Then $L_{G'}(x)xy$ is an increasing geodesic ending at y of length $|L_{G'}(x)| + 1 \geq |L_{G'}(y)| + 2$. Therefore $|L_G(y)| \geq |L_{G'}(y)| + 2$ and $S(G) \geq S(G') + 2 \geq 2|E(G')| + 2 = 2|E(G)|$.

This concludes the inductive step and the proof. □

Note that it is immediate from Theorem 2.1 that $|L_G(v)| \geq d$ for some $v \in V(G)$ and therefore, G contains an increasing geodesic of length at least d, as claimed in Theorem 1.2.

We now give a strengthening of Theorem 2.1, showing that G must actually contain *many* geodesics of length d. First note that for $d \in \mathbb{N}$, taking a disjoint union of subgraphs isomorphic to Q_d gives a graph G with average degree d and exactly $d!|G|/2$ geodesics of length d. Indeed, suppose $G = \cup_i G_i$ where G_i are disjoint and isomorphic to Q_d for all i. Then any vertex in G_i is a starting vertex for $d!$ geodesics of length d. This gives $\sum_i d!|G_i|/2 = d!|G|/2$ geodesics in total. The following result proves that we can in fact guarantee this many geodesics of length d for general subgraphs of Q_n.

Theorem 2.2. *If G is a subgraph of Q_n with average degree at least d, then G contains at least $d!|G|/2$ geodesics of length d.*

Proof. We first use Theorem 2.1 to prove the following claim: G contains at least $|G|$ increasing geodesics of length d. To see this, first remove an edge e from G if it lies in at least two increasing geodesics of length d. Now repeat this with $G \setminus \{e\}$ and so on until we end up at a subgraph G' of G in which all edges lie in at most one increasing geodesic of length d. Let

$|E(G)| = |E(G')| + a$. Note that, by our removal process, the a edges removed from G remove at least $2a$ increasing geodesics of length d. Therefore, if $a \geq |G|/2$, then G contains at least $|G|$ increasing geodesics of length d. If not, by Theorem 2.1, we have

$$\sum_{v \in V(G')} |L_{G'}(v)| \geq 2|E(G')| = 2|E(G)| - 2a \geq d|G| - 2a = (d-1)|G| + (|G| - 2a). \quad (1)$$

Now note that since no edge of G' is contained in more than one increasing geodesic of length d, G' does not contain any increasing geodesics of length $d+1$. Therefore $|L_{G'}(v)| \leq d$ for all $v \in G'$. By (1), this shows that $|L_{G'}(v)| = d$ for at least $|G| - 2a$ vertices $v \in G'$. Combining these with the increasing geodesics of length d containing edges from $G \setminus G'$, this shows that G contains at least $2a + (|G| - 2a) = |G|$ increasing geodesics of length d, as claimed.

Now suppose that G contain L geodesics of length d. We will show that $L \geq d!|G|/2$. To see this, pick an ordering σ of $\{1, \ldots, n\}$ uniformly at random and consider the geodesics of length d which are increasing with respect to this ordering (i.e. paths in which the edges have directions $\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_d)$ where $i_1 < i_2 < \cdots < i_d$). The probability that a fixed geodesic of length d appears as an increasing geodesic with respect to the ordering σ is exactly $2/d!$. Taking X to be the random variable which counts the number of increasing geodesics of length d in G (with respect to the ordering σ), this gives that

$$\mathbb{E}(X) = \frac{2L}{d!}.$$

But by the claim above, $X \geq |G|$ for each choice of σ. Therefore $L \geq d!|G|/2$, as required. □

3. A proof of Theorem 1.3 using set systems

In this section, we give an alternate proof of Theorem 1.3. Note that it is enough to prove this theorem for induced subgraphs of Q_n, since if the result fails for some graph G, it must also fail for the induced subgraph of Q_n on vertex set $V(G)$.

As in [4], the following compression operation allows us a further reduction. Here we view the vertices of Q_n as elements of $\mathcal{P}[n]$, the power set of $[n]$. Then two sets $A, B \in \mathcal{P}[n]$ are adjacent if $|A \triangle B| = 1$ where $A \triangle B = (A \setminus B) \cup (B \setminus A)$. Given $A \in \mathcal{P}[n]$ and $i \in \{1, \ldots, n\}$, let

$$C_i(A) = \begin{cases} A - i & \text{if } i \in A; \\ A & \text{if } i \notin A. \end{cases}$$

Given $\mathcal{A} \subset \mathcal{P}[n]$, $C_i(\mathcal{A}) := \{C_i(A) : A \in \mathcal{A}\} \cup \{A : C_i(A) \in \mathcal{A}\}$, the down compression of $\mathcal{A}$ in the i-direction. A family $\mathcal{A}$ is said to be a down-compressed if $C_i(\mathcal{A}) = \mathcal{A}$ for all $i \in [n]$. The following lemma shows that we may also assume that the vertex set $V(G)$ is a down-compressed.

Lemma 3.1. *Let G be an induced subgraph of Q_n on vertex set $\mathcal{A} \subset \mathcal{P}[n]$ and let $i \in \{1, \ldots, n\}$. Suppose that G has average degree at least d and all vertices A and B of G are at Hamming distance less than k. Then the same is true for the induced subgraph G' of Q_n with vertex set $C_i(\mathcal{A})$.*

Proof. Since $|G| = |G'|$ in both cases, to see that G' has average degree at least d it suffices to show that G' has at least as many edges as G. To see this, define a map $f : E(G) \to E(G')$ given by

$$f(AB) = \begin{cases} C_i(A)C_i(B) & \text{if } A \triangle B \neq \{i\} \text{ and } C_i(A)C_i(B) \notin E(G); \\ AB & \text{otherwise.} \end{cases}$$

Noting that f is an injection, it follows that G' has average degree at least d.

Suppose for contradiction that G' had two vertices A' and B' at Hamming distance at least k apart. Now it is easily seen that exactly one of A' and B' must contain i as otherwise any pair $A, B \in \mathcal{A}$ with $C_i(A) = A'$ and $C_i(B) = B'$ are at Hamming distance at least k apart. Assume that $i \in A'$, $i \notin B'$. Now $A' \in C_i(\mathcal{A})$ implies that $A' - i, A' \in \mathcal{A}$. Since $A' \in \mathcal{A}, B' \notin \mathcal{A}$ and we have $B' \in C_i(\mathcal{A}) \setminus \mathcal{A}$. This implies $B' \cup \{i\} \in \mathcal{A}$. But then $A' - i, B' \cup \{i\} \in \mathcal{A}$ are at Hamming distance at least k, a contradiction. □

Our alternate proof of Theorem 1.3 is based on a theorem of Katona. Given a set system $\mathcal{A} \subset [n]^{(k)}$, the *shadow* of $\mathcal{A}$ is the set

$$\partial(\mathcal{A}) := \{B \in [n]^{(k-1)} : B \subset A \text{ for some } A \in \mathcal{A}\}.$$

The set $\partial^{(l)}(\mathcal{A})$ is defined as $\partial^{(l)}(\mathcal{A}) := \partial(\cdots(\partial(\mathcal{A}))\cdots)$, where ∂ is applied l times.

While, in general, the shadow $\partial\mathcal{A}$ of $\mathcal{A} \subset \mathcal{P}[n]$ can be much smaller than $|\mathcal{A}|$, a result of Katona [7] shows that if $\mathcal{A}$ is also an intersecting family ($A \cap B \neq \emptyset$ for all $A, B \in \mathcal{A}$), then $|\partial(\mathcal{A})| \geq |\mathcal{A}|$. More generally, Katona also gave lower bounds on the size of $|\partial^{(l)}(\mathcal{A})|$ for t-intersecting families $\mathcal{A}$. We will need the following special case.

Theorem 3.2 (*Katona*)**.** *Let $k, t \in \mathbb{N}$. Suppose that $\mathcal{A} \subset [n]^{(k)}$ is t-intersecting. Then*

$$|\partial^{(t)}(\mathcal{A})| \geq |\mathcal{A}|.$$

Proof of Theorem 1.3. Suppose for contradiction the result is false and let $\mathcal{A}$ be the vertex set of G. Using Lemma 3.1 we may assume that $\mathcal{A}$ is down-compressed.

Let $\mathcal{A}^{(k)} = \mathcal{A} \cap [n]^{(k)}$ for all $k \in [n] \cup \{0\}$. Since $\mathcal{A}$ is down-compressed we must have $\mathcal{A}^{(k)} = \emptyset$ for all $k \geq d$. Also, since $\mathcal{A}$ is down-compressed, for each $A \in \mathcal{A}$, the number of neighbours of A which lie below A in G is $|A|$. Therefore

$$\sum_{k=0}^{\lceil d\rceil - 1} k|\mathcal{A}^{(k)}| = \sum_{A\in\mathcal{A}} |A| = \frac{d|\mathcal{A}|}{2}. \tag{2}$$

Furthermore, again by compression, for $k \geq d/2$, the set $\mathcal{A}^{(k)}$ does not contain two vertices A and B with $|A \cup B| \geq d$. Therefore, $\mathcal{A}^{(k)}$ must be $(2k - \lceil d\rceil + 1)$-intersecting. Applying Theorem 3.2 we therefore have

$$|\partial^{(2k-\lceil d\rceil+1)}(\mathcal{A}^{(k)})| \geq |\mathcal{A}^{(k)}|. \tag{3}$$

But as $\mathcal{A}$ is down-compressed

$$\partial^{(2k-\lceil d\rceil+1)}(\mathcal{A}^{(k)}) \subset \mathcal{A}^{(\lceil d\rceil-k-1)}.$$

We now pair the contributions from $\mathcal{A}^{(k)}$ and $\mathcal{A}^{(\lceil d\rceil-k-1)}$ to (2) together for all $k \geq (\lceil d\rceil - 1)/2$ using (3):

$$\begin{aligned} k|\mathcal{A}^{(k)}| + (\lceil d\rceil - k - 1)|\mathcal{A}^{(\lceil d\rceil-k-1)}| &= \left(\frac{\lceil d\rceil - 1}{2}\right)|\mathcal{A}^{(k)}| + \left(k - \frac{\lceil d\rceil - 1}{2}\right)|\mathcal{A}^{(k)}| \\ &\quad + \left(\frac{\lceil d\rceil - 1}{2}\right)|\mathcal{A}^{(\lceil d\rceil-k-1)}| + \left(\frac{\lceil d\rceil - 1}{2} - k\right)|\mathcal{A}^{(\lceil d\rceil-k-1)}| \\ &\leq \frac{\lceil d\rceil - 1}{2}\left(|\mathcal{A}^{(k)}| + |\mathcal{A}^{(\lceil d\rceil-k-1)}|\right). \end{aligned}$$

But summing over $k \geq (\lceil d\rceil - 1)/2$, this contradicts (2) above. This proves the theorem. □

4. Antipodal colourings

We now discuss the relation of Theorem 1.2 with Norine's conjecture (see [10]) mentioned in the Introduction. Given a vertex $x \in Q_n$, its *antipodal vertex* $x' \in Q_n$ is the unique vertex with all coordinate entries differing from those of x. Also, given an edge $e = xy$ of Q_n, its *antipodal edge* $e' = x'y'$ where x' is antipodal to x and y' is antipodal to y. Finally, a 2-colouring of the edges of Q_n is said to be *antipodal* if no two antipodal edges receive the same colour.

Conjecture 4.1 (Norine)**.** *For $n \geq 2$, any antipodal colouring of $E(Q_n)$ contains a monochromatic path between some pair of antipodal vertices.*

Note that this is not true for general 2-colourings of $E(Q_n)$, as can be seen by colouring all edges in directions $\{1, \ldots, n-1\}$ red and edges in direction n blue. In [4], Feder and Subi made the following conjecture for general 2-colourings of $E(Q_n)$:

Conjecture 4.2 (Feder–Subi)**.** *Every 2-colouring of $E(Q_n)$ contains a path between some pair of antipodal vertices which changes colour at most once.*

It is easily seen that if Conjecture 4.2 is true, it implies Norine's conjecture. Indeed, given an antipodal colouring of Q_n, take the path P guaranteed by Conjecture 4.2 between two antipodal vertices in Q_n. Combining P with its antipodal path P^A (consisting of all edges $x'y'$ where xy is an edge of P) then gives that some two antipodal vertices on the cycle PP^A must be joined by a monochromatic path.

In [4], Feder and Subi proved that every 2-colouring of $E(Q_n)$ contains a monochromatic path between two vertices at (Hamming) distance $\lceil n/2\rceil$. Using Theorem 1.2 in place of Theorem 1.3, the following shows that we can actually take this path to be a geodesic.

Corollary 4.3. *In every 2-colouring c of $E(Q_n)$ there exists a monochromatic geodesic of length $\lceil n/2\rceil$.*

Proof. Pick a monochromatic connected component C of the colouring with average degree at least $n/2$ and apply Theorem 1.2 to it. □

This suggests that in both of the conjectures above, one can additionally ask for the path between antipodal vertices to be a geodesic.

Conjecture 4.4. *The following statements hold:*

(A) *Every antipodal colouring c of $E(Q_n)$ contains a monochromatic geodesic between some pair of antipodal vertices.*
(B) *In every 2-colouring c of $E(Q_n)$, there is a geodesic between some pair of antipodal vertices which changes colour at most once.*

Unfortunately we were not able to settle either of these conjectures. In fact, surprisingly, we were not even able to establish that in every 2-colouring of $E(Q_n)$ some two antipodal vertices are joined by a path which changes colour $o(n)$ times. Is this true?

Question 4.5. *Is it true that for every* 2*-colouring of* $E(Q_n)$*, there exist two antipodal vertices* x *and* x' *that are joined by a path that changes colour* $o(n)$ *times?*

While we were not able to prove either statement (A) or statement (B), our final result shows that they are equivalent.

Proposition 4.6. *Statement* (A) *holds for all* n *if and only if statement* (B) *holds for all* n*.*

Proof. First assume that statement (A) is true and let c be a 2-colouring of $E(Q_n)$. View Q_n as the subcube of Q_{n+1} consisting of all 0–1 vectors of length $n+1$, $(x_1, x_2, \ldots, x_{n+1})$ with $x_{n+1} = 0$. Pick any antipodal colouring c' of $E(Q_{n+1})$ which agrees with c on $E(Q_n)$. Statement (A) now guarantees that c' has a monochromatic geodesic P between two antipodal vertices of Q_{n+1}. Let P^A denote the geodesic formed by the edges antipodal to P. Since c' is antipodal, P^A must also be monochromatic and of opposite colour to P. The restriction of the cycle PP^A to our original subcube Q_n now gives a geodesic between two antipodal vertices (in Q_n) which changes colour at most once, i.e. statement (B) is true.

Now assume that statement (B) is true and let c be an antipodal 2-colouring of $E(Q_n)$. Applying statement (B) to c we obtain a geodesic P between two antipodal vertices which changes colour at most once. Let $P = P_r P_b$ where P_r is a red geodesic and P_b is a blue geodesic. But since c is antipodal P_r^A is a blue geodesic and $P_b P_r^A$ is a blue geodesic between antipodal vertices, i.e. statement (A) is true. □

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