

# Recognition of Small Dimensional Representations of General Linear Groups

Magaard, Kay; O'Brien, EA; Seress, A

DOI:

[10.1017/S144678870800092X](https://doi.org/10.1017/S144678870800092X)

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*Document Version*

Publisher's PDF, also known as Version of record

*Citation for published version (Harvard):*

Magaard, K, O'Brien, EA & Seress, A 2008, 'Recognition of Small Dimensional Representations of General Linear Groups', *Journal of the Australian Mathematical Society*, vol. 85, no. 02, pp. 229-250.  
<https://doi.org/10.1017/S144678870800092X>

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## RECOGNITION OF SMALL DIMENSIONAL REPRESENTATIONS OF GENERAL LINEAR GROUPS

KAY MAGAARD, E. A. O'BRIEN<sup>✉</sup> and ÁKOS SERESS

(Received 4 February 2008; accepted 8 May 2008)

Communicated by Martin W. Liebeck

Dedicated to Cheryl Praeger for her sixtieth birthday

### Abstract

Let  $G$  be isomorphic to a group  $H$  satisfying  $\mathrm{SL}(d, q) \leq H \leq \mathrm{GL}(d, q)$  and let  $W$  be an irreducible  $\mathbf{F}_q G$ -module of dimension at most  $d^2$ . We present a Las Vegas polynomial-time algorithm which takes as input  $W$  and constructs a  $d$ -dimensional projective representation of  $G$ .

2000 *Mathematics subject classification*: primary 20C20, 20C40.

*Keywords and phrases*: matrix recognition project, symmetric square, alternating square adjoint representation.

### 1. Introduction

A major research topic over the past decade has been the development of efficient algorithms for the investigation of subgroups of  $\mathrm{GL}(d, \mathbf{F}_q)$  where  $\mathbf{F}_q$  is a finite field of size  $q = p^f$ . We refer to the recent survey [15] for background related to this work.

A particular focus is the development of algorithms to construct an isomorphism between an arbitrary representation of a classical group and its ‘standard’ (or natural) representation.

In 2001, Kantor and Seress [9] proved that there is a Las Vegas algorithm that, given as input an arbitrary permutation or (projective) matrix representation  $G$  of an almost simple classical group  $H$  of Lie type of known characteristic, constructs an isomorphism between  $G$  and the natural projective representation of  $H$ . Their algorithm also constructs a new ‘nice’ generating set  $S$  for  $G$  such that any element can be reached efficiently from  $S$  by a short *straight-line program*: an efficiently stored group word on  $S$  that evaluates to  $g$ . (For a formal definition and discussion of their significance, see [18, p. 10].)

In this paper, we present efficient algorithms to construct such an isomorphism for a *projective matrix representation* of degree at most  $d^2$  of the general linear groups

This work was supported in part by the NSA, the Marsden Fund of New Zealand, and the NSF.

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having natural module of dimension  $d$ . In the natural module, a ‘nice’ generating set can be constructed using the efficient algorithms of [5] or [11]. Hence this work supplements that of [9], providing fast polynomial time reduction for the most commonly occurring irreducible representations of general linear groups.

An additional motivation for our algorithm is the recent work of Ryba [17]. He presents a polynomial-time Las Vegas algorithm that, given as input an odd defining characteristic absolutely irreducible representation of a finite Chevalley group, constructs its action on the adjoint module. Since the adjoint module of  $\mathrm{GL}(d, q)$  has dimension at most  $d^2 - 1$ , a combination of Ryba’s algorithm and ours can be used to construct the action on the natural module.

A similar program has been carried out for the alternating and symmetric groups in [2] and [3]. The algorithm of [2] constructs an isomorphism between an arbitrary permutation or matrix representation of  $A_n$  or  $S_n$  and the natural permutation representation on  $n$  points; in [3] a specialized, faster algorithm does the same for the deleted permutation module, which is the smallest dimensional matrix representation of these groups.

## 2. Background and main result

We now consider in more detail our task. Let  $\mathrm{SL}(d, q) \leq H \leq \mathrm{GL}(d, q)$  with  $q = p^f$ . Let  $V$  denote the natural module of  $H$ , and  $V^*$  is its dual module. Define the Frobenius map  $\delta : \mathrm{GL}(d, q) \mapsto \mathrm{GL}(d, q)$  by  $(a_{i,j})^\delta = (a_{i,j}^p)$  for  $(a_{i,j}) \in \mathrm{GL}(d, q)$ .

Two representations  $\rho_1$  and  $\rho_2$  of  $H$  are *quasi-equivalent* if there exists  $\theta \in \mathrm{Aut}(H)$  such that  $\rho_1$  is equivalent to  $\theta\rho_2$ .

Let  $H$  act on an irreducible  $\mathbf{F}_q G$ -module  $W$  of dimension at most  $d^2$ . For a discussion of such irreducible representations, see [12]. In particular,  $W$  is quasi-equivalent to an irreducible section of  $V \otimes V^{\delta^e}$ , or  $V^* \otimes V^{\delta^e}$  where  $0 \leq e < f$ .

The irreducible sections of  $V \otimes V$  are the *symmetric* and *alternating* squares of  $V$  of dimension  $d(d+1)/2$  and  $d(d-1)/2$ , respectively.

Consider  $V^* \otimes V$  with basis  $\{e_i \otimes e_j \mid 1 \leq i, j \leq d\}$  and let

$$w := \sum_{i=1}^d e_i \otimes e_i, \quad U := \left\{ \sum_{i,j} \alpha_{i,j} e_i \otimes e_j \mid \sum_{i=1}^d \alpha_{i,i} = 0 \right\}, \quad W_1 := U \cap \langle w \rangle.$$

The *adjoint module* of  $V$  is  $W := U/W_1$ . If  $d \bmod p \equiv 0$  then  $W$  has dimension  $d^2 - 2$ , otherwise  $d^2 - 1$ .

The remaining irreducible representations of dimension at most  $d^2$  are  $V \otimes V^{\delta^e}$  and  $V^* \otimes V^{\delta^e}$  where  $0 < e < f$ .

Our principal goal is an algorithm that, given as input such an irreducible representation  $W$  of  $H$ , constructs a  $d$ -dimensional projective representation of  $H$ .

Our algorithm assumes that we can construct random elements of a finite group  $G$ . Following the notation of [18, p. 24], an algorithm constructs an  $\varepsilon$ -uniformly distributed random element  $x$  of  $G$  if  $(1 - \varepsilon)/|G| < \mathrm{Prob}(x = g) < (1 + \varepsilon)/|G|$  for

all  $g \in G$ ; if  $\varepsilon < 1/2$ , then the algorithm constructs *nearly uniformly distributed* random elements of  $G$ . Babai [1] presents a black-box Monte Carlo algorithm to construct such elements in polynomial time. Another, more practical, option is the product replacement algorithm of Celler *et al.* [6], which also runs in polynomial time (see [16]). For a discussion of both algorithms, we refer the reader to [18, pp. 26–30].

Let  $\xi$  denote the cost of constructing a nearly uniformly distributed random element in a group  $G$  and let  $\rho_r$  denote the cost of a field operation in a finite field  $\mathbf{F}_r$ . Our main result is the following.

**THEOREM 2.1.** *Let  $d \geq 2$  and let  $q = p^f$  be a prime power. Let  $\mathrm{SL}(d, q) \leq H \leq \mathrm{GL}(d, q)$  where  $H$  has natural module  $V$ . Suppose that  $H$  is given as  $G = \langle X \rangle$  acting irreducibly on an  $\mathbf{F}_q$ -vector space  $W$  of dimension  $n \leq d^2$ . Subject to Conjecture 4.9, given as input  $G$ , the value of  $d$ , and error probability  $\varepsilon > 0$ , there is a polynomial-time Las Vegas algorithm that, with probability at least  $1 - \varepsilon$ , sets up a data structure to construct the projective action of  $G$  on  $V$  in time  $O(\xi d^2 \log q \log(1/\varepsilon) + \rho_q d^9 \log^2 d \log^2 q)$ . The time requirement to evaluate the image of  $g \in G$  on  $V$  is  $O(\xi + \rho_q d^8 \log q)$ .*

We prove this theorem by exhibiting an algorithm with the stated complexity. We present the conjecture and evidence in its support in Section 4.3. Theorem 2.1 depends on Conjecture 4.9 *only* if  $W$  is the exterior square of  $V$ .

An  $n \times n$  matrix over  $\mathbf{F}_q$  requires  $\Theta(d^4 \log q)$  space, so the running time of the algorithm, in terms of the input length  $N$ , is  $O(\xi N + \rho_q N^{2.25} \log^2 N)$ . We use the conventional estimate of  $O(n^3)$  field operations for matrix multiplication; if it can be done in  $O(n^\omega)$  field operations for some constant  $\omega < 3$ , then our algorithm runs in  $O(\xi d^2 \log q + \rho_q d^{3+2\omega} \log^2 d \log^2 q)$  time.

In Section 3 we outline the basic algorithm common to all of the cases, and in Section 4 estimate the costs of common steps. We then study each representation in turn, and finally report briefly on an implementation of the algorithms in MAGMA [4].

### 3. The general strategy

Let  $q = p^f$  be a prime power and let  $r$  be a prime. Recall from [14] that  $r$  is a *primitive prime divisor* of  $q^d - 1$  if  $r \mid q^d - 1$  but  $r$  does not divide  $q^e - 1$  for  $e < d$ . We use the notation  $\mathrm{ppd}(q; d)$  to describe such an  $r$ .

By a theorem of Zsigmondy (see [14]),  $\mathrm{ppd}(q; d)$  primes exist for all  $q$  and  $d$  except when  $q = 2, d = 6$  and  $q$  is a Mersenne prime,  $d = 2$ . To cover these exceptional cases, we call 9 a  $\mathrm{ppd}(2; 6)$  prime and 4 a  $\mathrm{ppd}(q; 2)$  prime for Mersenne prime  $q$ .

Recall that  $H$  has natural module  $V$  and is given as  $G \leq \mathrm{GL}(W)$  where  $W = \mathbf{F}_q^n$ . Let  $s \in H$  and assume that  $|s|$  is divisible by some  $\mathrm{ppd}(q; d)$ . Hence  $s$  is a power of a Singer cycle and has  $d$  one-dimensional eigenspaces  $\langle e_1 \rangle, \langle e_2 \rangle, \dots, \langle e_d \rangle$  in  $V \otimes \mathbf{F}_{q^d}$ . Let  $\sigma = \delta^f$  be the Frobenius map of  $\mathrm{GL}(d, q^d)$  whose fixed points contain  $H$ . Thus  $\sigma$  centralizes  $\langle s \rangle$  and so  $\sigma$  transitively permutes the eigenspaces of  $s$  acting on  $V \otimes \mathbf{F}_{q^d}$ . Consequently, we can index the eigenspaces  $\langle e_i \rangle$  of  $s$  and choose the eigenvectors

$e_i$  within the eigenspaces in such a way that  $e_i^\sigma = e_{i+1}$  where the index is computed modulo  $d$ . If  $e_1 s = \lambda e_1$ , then  $e_i s = \lambda^{q^{i-1}} e_i$  for  $i \leq d$ .

Our goal is to write the action  $A$  of an arbitrary  $g \in G$  on  $V \otimes \mathbf{F}_{q^d}$ , in the basis  $e_1, \dots, e_d$ . If  $W$  is either the alternating or the symmetric square representation, then we also compute the base change matrix  $B$  between a particular  $\mathbf{F}_q$ -basis  $b_1, \dots, b_d$  and  $e_1, \dots, e_d$ . Hence, we also learn the action  $BAB^{-1}$  of  $g$  on the natural module  $V$ . In the other cases, the action of  $G$  in a suitable  $\mathbf{F}_q$ -basis is recovered using the algorithm of [8] which has complexity  $O(\rho_q d^5 \log^2 d)$ .

We now summarize the algorithm `Decompose` to construct the projective action of  $G$  on  $V$ .

- (1) Find, by random search,  $s \in G$  which satisfies the following:
  - if  $W$  is not the adjoint module, then  $s$  has  $n$  one-dimensional eigenspaces;
  - if  $W$  is the adjoint module, then  $s$  has  $d^2 - d$  one-dimensional eigenspaces and an  $n - (d^2 - d)$ -dimensional eigenspace for the eigenvalue 1;
  - $|s|$  is divisible by some  $\text{ppd}(q; d)$  prime.
- (2) Construct a basis  $\mathcal{B}_0$  consisting of eigenvectors for the action of  $s$  on  $W \otimes \mathbf{F}_{q^d}$ .
- (3) Label the elements of  $\mathcal{B}_0$  by ordered pairs  $(i, j)$  with  $1 \leq i, j \leq d$ . The labels are found by discovering a sufficient number of algebraic dependencies among the eigenvalues. This labelling must be commensurate with the basis  $e_1, \dots, e_d$  and consistent with the action of  $\sigma$ .
- (4) From the eigenspace labelled with  $(i, j)$ , compute the vector corresponding to  $e_i \otimes e_j$ .

Steps 1 to 4 create the data structure described in Theorem 2.1 and are applied *once*. To obtain the image of  $g \in G$ , we apply the following step.

- (5) First write  $g$  in the basis  $\mathcal{B}_0$ ; then compute the action of  $g$  on  $V \otimes \mathbf{F}_{q^d}$  in the basis  $e_1, e_2, \dots, e_d$ ; finally rewrite with respect to the basis  $b_1, b_2, \dots, b_d$  of the natural module  $V$ .

#### 4. The common steps

We first discuss costs associated with the extension field  $\mathbf{F}_{q^d}$ . Since step 1 is common to all representations, we next discuss it in detail and estimate its cost. To conclude this section, we discuss base change between the bases  $b_1, \dots, b_d$  of  $V$  and  $e_1, \dots, e_d$  of  $V \otimes \mathbf{F}_{q^d}$ , and consider properties of matrices written with respect to the latter basis.

**4.1. Construction of the extension field** Since we work in  $\mathbf{F}_{q^d}$ , as a preprocessing step we construct that field.

**LEMMA 4.1.** *The extension field  $\mathbf{F}_{q^d}$  can be constructed by a Las Vegas algorithm, in  $O(\rho_q d^3 \log^2 d \log q)$  time. The cost of a field operation in  $\mathbf{F}_{q^d}$  is  $O(\rho_q d^2 \log^2 d)$ . Taking a square root of an element of  $\mathbf{F}_{q^d}$  can be done in  $O(\rho_q d^3 \log^2 d \log q)$  time.*

**PROOF.** We construct  $\mathbf{F}_{q^d}$  as an extension of  $\mathbf{F}_q$ . To do this, we search for monic polynomials of degree  $d$  from the polynomial ring  $\mathbf{F}_q[x]$  until we find an irreducible  $f(x)$ . With probability at least  $1/(d + 1)$  (see [14]), a random polynomial of degree  $d$  defined over  $\mathbf{F}_q[x]$  is irreducible. That a polynomial of degree  $d$  is irreducible over  $\mathbf{F}_q[x]$  can be decided in  $O(\rho_q d^2 \log d \log \log d \log q)$  time by a Las Vegas algorithm (see [19, Theorem 14.14]).

The elements of  $\mathbf{F}_{q^d}$  are the residue classes of  $\mathbf{F}_q[x]$  modulo  $f(x)$ , and they can be represented by the polynomials of degree at most  $d - 1$ . We summarize the cost of field operations in  $\mathbf{F}_{q^d}$ .

- $O(\rho_q d)$  for addition and  $O(\rho_q d \log d \log \log d)$  for multiplication and division (see [19, Theorems 8.23 and 9.6]).
- $O(\rho_q d^2 \log d \log \log d)$  for taking inverses; we compute the inverse of  $a(x) \in \mathbf{F}_{q^d}$  by writing the greatest common divisor 1 of  $f(x)$  and  $a(x)$  in the form  $1 = f(x)g(x) + a(x)h(x)$  by a Euclidean algorithm of length at most  $d$ , and then taking  $h(x)$  as the inverse.
- Taking square roots of some  $c \in \mathbf{F}_{q^d}$  can be done by factorizing the polynomial  $x^2 - c$ , in time  $O(\rho_q d \log q^d) = O(\rho_q d^3 \log^2 d \log q)$  (see [19, Theorem 14.14]). □

We use the following result, obtained in [2, 4.6], as an application of [19, Algorithm 14.19].

**LEMMA 4.2.** *The distinct linear factors of some  $g(x) \in \mathbf{F}[x]$  of degree  $n$  can be computed by a Las Vegas algorithm, in  $O(\rho_{\mathbf{F}} n \log^2 n \log(n|\mathbf{F}|) \log \log n)$  time.*

**4.2. Step 1 of the general strategy: Finding  $s$**  We now discuss the search for random  $s \in G$  which satisfies the conditions associated with step 1 of Decompose. If an  $s$  satisfying the eigenvalue condition is found, then we check that  $|s|$  is divisible by a  $\text{ppd}(q; d)$  prime as follows. If  $(d, q)$  equals  $(6, 2)$  or  $(2, p)$  with  $p$  Mersenne, then define  $m := 21$  and  $m := p - 1$ , respectively; otherwise

$$m := q^d \prod_{j|d, j \neq d} \frac{d}{j} (q^j - 1). \tag{4.1}$$

Now  $|s|$  is divisible by a  $\text{ppd}(q; d)$  prime if and only if  $s^m \neq 1$ . We decide this by raising the eigenvalues of  $s$  to the  $m$ th power.

**LEMMA 4.3.** *We can decide if  $s \in G$  satisfies the conditions of step 1 by a Las Vegas algorithm in  $O(\rho_q d^6 \log q)$  time.*

**PROOF.** The characteristic polynomial  $c(x)$  of  $s$  can be computed using the algorithm of [10] in  $O(\rho_q n^3 \log n) = O(\rho_q d^6 \log d)$  time. By Lemma 4.2, the distinct linear factors of  $c(x)$  in  $\mathbf{F}_{q^d}$  are obtained by a Las Vegas algorithm, in  $O(\rho_q d n \log^2 n \log(nq^d) \log \log n) = O((\rho_q d^2 \log^2 d)(d^2 \log^2 d)(\log d + d \log q)(\log d)) = O(\rho_q d^5 \log^5 d \log q)$  time. If  $W$  is the adjoint module, then the 1-eigenspace of  $s$  can be computed in  $O(\rho_q d^6)$  time. Raising the eigenvalues of  $s$  to the

power  $m$  in (4.1) takes  $O(n\rho_{q^d} \log(q^{d^{3/2}})) = O(\rho_q d^{11/2} \log^2 d \log q)$  time, using the trivial upper estimate  $2\sqrt{d}$  for the number of divisors of  $d$ .  $\square$

We now derive a sufficient condition to identify a suitable element of  $H$ , and use this condition in Lemma 4.5 to estimate the number of random elements processed in our search.

**THEOREM 4.4.** *Suppose that  $(d, q) \neq (3, 4)$  and that the order of  $s \in H$  is a multiple of  $(q^d - 1)/(q - 1)$ . If  $W$  is not the adjoint module, then  $s$  has  $n$  distinct eigenvalues in  $\mathbf{F}_{q^d}$ . If  $W$  is the adjoint module, then 1 is an eigenvalue of  $s$  with eigenspace of dimension  $n - (d^2 - d)$ , and  $s$  has  $d^2 - d$  other eigenvalues.*

**PROOF.** Let  $\alpha$  be a primitive element of  $\mathbf{F}_{q^d}$ . If the order of  $s$  is a multiple of  $(q^d - 1)/(q - 1)$ , then the eigenvalues of  $s$  in  $V \otimes \mathbf{F}_{q^d}$  are  $\omega, \omega^q, \dots, \omega^{q^{d-1}}$ , where  $\omega = \alpha^k$  for some divisor  $k$  of  $q - 1$ .

If  $W$  is the symmetric square of  $V$ , then the eigenvalues of  $s$  in  $W \otimes \mathbf{F}_{q^d}$  are  $\omega^{q^{i-1}+q^{j-1}}$ , for  $1 \leq i \leq j \leq d$ . Suppose that  $\omega^{q^{i_1-1}+q^{j_1-1}} = \omega^{q^{i_2-1}+q^{j_2-1}}$  for some  $i_1 \leq j_1, i_2 \leq j_2$ . Then  $\alpha^{k(q^{i_1-1}+q^{j_1-1})} = \alpha^{k(q^{i_2-1}+q^{j_2-1})}$ . If the exponents on both sides are less than  $q^d - 1$ , then they must be equal. This implies that  $j_1 = j_2$ , and so  $i_1 = i_2$ . If  $k(q^{i_1-1} + q^{j_1-1}) \geq q^d - 1$ , then  $k = q - 1$  and  $i_1 = j_1 = d$ , so the only remaining possibility is  $2(q - 1)q^{d-1} = q^d - 1 + (q - 1)(q^{i_2-1} + q^{j_2-1})$ . This simplifies to  $q^{d-1} = q^{d-2} + q^{d-3} + \dots + q + 1 + q^{i_2-1} + q^{j_2-1}$ . If  $q = 2$ , then we further simplify to  $1 = q^{i_2-1} + q^{j_2-1}$ , which is a contradiction. If  $q \geq 3$ , then  $q = 3$  and  $i_2 = j_2 = 1$ , otherwise the right-hand side of the last equation is not divisible by  $q$ . But this also leads to a contradiction.

If  $W$  is the alternating square, then the eigenvalues of  $s$  in  $W \otimes \mathbf{F}_{q^d}$  are  $\omega^{q^{i-1}+q^{j-1}}$ , for  $1 \leq i < j \leq d$ . Since all occur as eigenvalues for the symmetric square, they are distinct.

If  $W$  is the adjoint module, then the eigenvalues of  $s$  in  $W \otimes \mathbf{F}_{q^d}$ , different from 1, are of the form  $\omega^{q^{i-1}-q^{j-1}}$ , for  $1 \leq i \leq j \leq d, i \neq j$ . If  $\omega^{q^{i_1-1}-q^{j_1-1}} = \omega^{q^{i_2-1}-q^{j_2-1}}$  for some  $i_1 \neq j_1, i_2 \neq j_2$ , then  $\alpha^{k(q^{i_1-1}-q^{j_1-1})} = \alpha^{k(q^{i_2-1}-q^{j_2-1})}$ . As in the symmetric square case, the only solution of this equation implies that  $j_2 \in \{i_2, j_1\}$ . Since  $j_2 \neq i_2$ , we must have  $j_2 = j_1$ , and so  $i_1 = i_2$ .

Now consider the case  $W = V \otimes V^\tau$ . If  $\tau = \delta^e$  and  $0 < e < f$ , then the eigenvalues of  $s$  in  $W \otimes \mathbf{F}_{q^d}$  are  $\omega^{q^{i-1}+p^e q^{j-1}}$ , for  $1 \leq i, j \leq d$ . Suppose that  $\omega^{q^{i_1-1}+p^e q^{j_1-1}} = \omega^{q^{i_2-1}+p^e q^{j_2-1}}$  for some  $1 \leq i_1, j_1, i_2, j_2 \leq d$ . Then

$$\alpha^{k(q^{i_1-1}+p^e q^{j_1-1})} = \alpha^{k(q^{i_2-1}+p^e q^{j_2-1})}. \tag{4.2}$$

If one of  $i_1 = i_2$  and  $j_1 = j_2$  holds, then clearly the other equality holds as well. If the exponents on both sides of (4.2) are equal, then  $q^{i_1-1} - q^{i_2-1} = p^e(q^{j_2-1} - q^{j_1-1})$ . If  $j_1 \neq j_2$ , then the exponent of  $p$  in the prime factorization of the right-hand side of this last equation is equal to  $e \pmod f$ , but on the left-hand side the exponent is  $0 \pmod f$ .

$f$ , which is a contradiction. Hence  $j_i = j_2$ , and so  $i_1 = i_2$ . If  $j_1, j_2 \leq d - 2$ , then both exponents in (4.2) are at most

$$k(q^{d-1} + p^e q^{d-3}) < (q - 1)(q^{d-1} + q^{d-2}) \leq q^d - 1,$$

so the exponents must be equal. But this implies  $i_1 = i_2$  and  $j_1 = j_2$ .

If  $d \geq 5$  then, given any solution of (4.2), there exists  $m \in \{0, 1, 2, 3, 4\}$  so that  $j_k + m - 1 \leq 2d - 2$  and  $j_k + m - 1 \notin \{d - 1, d\}$  holds for  $k = 1, 2$ . Hence, raising (4.2) to the  $q^m$ th power and replacing the terms  $q^{i_k+m-1}, q^{j_k+m-1}$  by  $q^{i_k+m-1-d}, q^{j_k+m-1-d}$ , respectively, in the case that these exponents are greater than  $d$ , we obtain a solution of (4.2) with  $j_1, j_2 \leq d - 2$ . Therefore, for the original  $j_1, j_2$  we have  $j_1 + m - 1 \equiv j_2 + m - 1 \pmod{d}$ , implying  $j_1 = j_2$ . Similarly, if  $3 \leq d \leq 4$  then, given any solution of (4.2), there exists  $m \in \{0, 1, 2\}$  so that  $j_k + m - 1 \leq 2d - 2$  and  $j_k + m - 1 \neq d$  holds for  $k = 1, 2$ . Hence it is enough to consider solutions of (4.2) with  $j_1, j_2 \leq d - 1$ , and at least one of the exponents of  $\alpha$  on both sides of (4.2) is greater than  $q^d - 1$ . However, if  $j_1 \leq d - 1$  then

$$k(q^{i_1-1} + p^e q^{j_1-1}) > q^d - 1$$

is possible if and only if  $k = q - 1, i_1 = d, j_1 = d - 1$ , and again  $k(q^{i_1-1} + p^e q^{j_1-1}) < 2(q^d - 1)$ . Hence the only case to consider is

$$(q - 1)(q^{d-1} + p^e q^{d-2}) = (q - 1)(q^{i_2-1} + p^e q^{j_2-1}) + q^d - 1.$$

Here the left-hand side is divisible by  $p$ , so we must have  $i_2 = 1$  and  $p = 2$ . Thus our equation is equivalent to  $2^e(q^{d-2} - q^{j_2-1}) = (q^{d-1} + q - 2)/(q - 1)$ . Using the fact that  $2^e \geq 2$  and  $j_2 \leq d - 2$  (because  $j_2 \neq j_1$ ), the left-hand side of the last equation is greater than the right-hand side, unless  $d = 3, q = 4, j_2 = 1$ . This is the exception identified in the statement of the lemma. The last subcase is  $d = 2$ . Since  $i_1 \neq i_2, j_1 \neq j_2, \alpha^{k(1+p^e q)} = \alpha^{k(q+p^e)}$ . But this implies that  $k(1 + p^e q - q - p^e) = k(q - 1)(p^e - 1)$  is a multiple of  $q^2 - 1$ , a contradiction since it is not divisible by a  $\text{ppd}(p; 2f)$  prime.

Finally, if  $W = V^* \otimes V^\tau$ , then the eigenvalues of  $s$  in  $W \otimes \mathbf{F}_{q^d}$  are  $\omega^{-q^{i-1} + p^e q^{j-1}}$ , for  $1 \leq i, j \leq d$ . Suppose that  $\omega^{-q^{i_1-1} + p^e q^{j_1-1}} = \omega^{-q^{i_2-1} + p^e q^{j_2-1}}$  for some  $1 \leq i_1, j_1, i_2, j_2 \leq d$ . Then

$$\alpha^{k(q^{i_2-1} + p^e q^{j_1-1})} = \alpha^{k(q^{i_1-1} + p^e q^{j_2-1})}.$$

As in the previous case, the only solution is  $i_1 = i_2, j_1 = j_2$  if  $(d, q) \neq (3, 4)$ . □

As stated, Theorem 4.4 identifies a sufficient condition for an element of  $H$  to be suitable; its statement can be readily adapted to  $G$ .

**LEMMA 4.5.** *The expected sample size for step 1 is  $O(1/4d^2 \ln q)$  and the expected running time is  $O(\xi d^2 \log q + \rho_q d^8 \log^2 q)$ .*



**PROOF.** By [14], the probability that the order of a random  $s \in H$  is divisible by a  $\text{ppd}(q; d)$  number is greater than  $1/2d$ . If the order of  $s$  is divisible by a  $\text{ppd}(q; d)$ , then  $C_H(s)$  is the intersection of  $H$  with the cyclic subgroup generated by a Singer cycle; further, the order of  $s$  is largest if and only if  $C_H(s) = \langle s \rangle$ . The probability that this occurs is greater than  $1/(2 \ln q^d)$ , since  $\varphi(k) > k/(2 \ln k)$  for all  $k > 2$  (see [13, p. 227]). Hence the probability that a random  $s \in H$  satisfies the order requirement of Theorem 4.4 is greater than  $1/(4d^2 \ln q)$ . Combining with Lemma 4.3, we obtain the statement of this lemma.  $\square$

**4.3. The base change matrix** Let  $s \in H$  have order a multiple of  $(q^d - 1)/(q - 1)$ . Consider a basis  $b_1, b_2, \dots, b_d$  of the natural module  $V$  of  $H$  in which  $s$  is represented by the matrix

$$S = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & a_{d-1} \end{pmatrix}. \tag{4.3}$$

This is the rational canonical form, or companion matrix, of the *left* action of  $s$  in  $V^*$ , but the  $b_i$  are row vectors, a basis for the *right* action of  $H$  on  $V$ . The characteristic polynomial of  $s$  is  $x^d - \sum_{i=0}^{d-1} a_i x^i$ , with the entries  $a_i$  in the last column of  $S$ .

Let  $s$  have eigenvalues  $\omega, \omega^q, \dots, \omega^{q^{d-1}}$  in  $V \otimes \mathbf{F}_{q^d}$ , and corresponding eigenvectors  $e_1, e_2, \dots, e_d$  satisfying  $e_i^\sigma = e_{i+1}$ . We now determine the base change matrix between the bases  $b_1, b_2, \dots, b_d$  of  $V$  and  $e_1, e_2, \dots, e_d$  of  $V \otimes \mathbf{F}_{q^d}$ , and obtain structural information about the matrix of an element of  $H$  in the basis  $e_1, e_2, \dots, e_d$ .

**LEMMA 4.6.** *The base change matrix between  $b_1, b_2, \dots, b_d$  and  $e_1, e_2, \dots, e_d$  has the form*

$$B = \begin{pmatrix} \mu & \mu\omega & \mu\omega^2 & \cdots & \mu\omega^{d-1} \\ \mu^q & (\mu\omega)^q & (\mu\omega^2)^q & \cdots & (\mu\omega^{d-1})^q \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu^{q^{d-1}} & (\mu\omega)^{q^{d-1}} & (\mu\omega^2)^{q^{d-1}} & \cdots & (\mu\omega^{d-1})^{q^{d-1}} \end{pmatrix}$$

for some nonzero  $\mu \in \mathbf{F}_{q^d}$ .

**PROOF.** Let  $e_1 = (\alpha_1, \dots, \alpha_d)$  in the basis  $b_1, b_2, \dots, b_d$ . Then  $e_1 s = \omega e_1$  implies that

$$(\alpha_1, \dots, \alpha_d)S = (\omega\alpha_1, \dots, \omega\alpha_d).$$

Also, by (4.3),

$$(\alpha_1, \dots, \alpha_d)S = (\alpha_2, \dots, \alpha_d, \beta),$$

with  $\beta = \sum_{i=1}^d \alpha_i a_{i-1}$ . Comparing the first  $d - 1$  corresponding entries in the two vectors on the right-hand sides of these equations, we obtain  $\alpha_i = \alpha_1 \omega^{i-1}$  for  $2 \leq i \leq d$ . Hence, with the notation  $\mu := \alpha_1$ , the first row of the base change matrix  $B$  is  $(\mu, \mu\omega, \dots, \mu\omega^{d-1})$ . Since  $e_i^\sigma = e_{i+1}$  for all  $i$ , the other rows of  $B$  can be obtained by taking the  $q$ th power of entries in the previous row.  $\square$

**LEMMA 4.7.** *Let  $h \in H$  and let  $A = (a_{ij})$  be the matrix of  $h$  in the basis  $e_1, \dots, e_d$ . For  $i, j \in \{1, \dots, d\}$ ,*

$$a_{i+1, j+1} = a_{ij}^q$$

(where the index  $d + 1$  is interpreted as 1).

**PROOF.** Let  $c_1, c_2, \dots, c_d$  be the column vectors of  $B^{-1}$ , where  $B$  is the base change matrix defined in Lemma 4.6. Then  $e_1 c_1 = 1$  and, for  $2 \leq j \leq d$ ,  $e_j c_1 = e_1^{\sigma^{j-1}} c_1 = 0$ . Applying  $\sigma^{i-1}$  to these equations, we obtain  $e_1^{\sigma^{i-1}} c_1^{\sigma^{i-1}} = e_i c_1^{\sigma^{i-1}} = 1$  and  $e_j c_1^{\sigma^{i-1}} = 0$  for all  $j \neq i$ . Hence  $c_i = c_1^{\sigma^{i-1}}$  for  $2 \leq i \leq d$ .

If  $M$  is the matrix of  $h$  in the basis  $b_1, b_2, \dots, b_d$ , then its entries are in  $\mathbf{F}_q$ , and  $A = BMB^{-1}$ . Thus  $a_{ij} = e_i M c_j$  and  $a_{ij}^q = e_i^\sigma M^\sigma c_j^\sigma = e_{i+1} M c_{j+1} = a_{i+1, j+1}$ .  $\square$

**LEMMA 4.8.** *Let  $h \in H$ , and let  $A = (a_{ij})$  be the matrix of  $h$  in the basis  $e_1, \dots, e_d$ .*

- (a) *For  $i, j \in \{1, \dots, d\}$ ,  $\text{Prob}(a_{ij} = 0) < 4/q^d$ . If  $q \geq 3$  then  $\text{Prob}(a_{ij} = 0) < 2/q^d$ .*
- (b)  *$\text{Prob}(\text{all } a_{ij} \neq 0) > 5/8$ .*

**PROOF.** Recall that the entries of  $A$  lie in  $\mathbf{F}_{q^d}$ , not  $\mathbf{F}_q$ .

- (a) By Lemma 4.7, the first row of  $A$  determines uniquely the other rows of  $A$ , and  $a_{ij} = 0$  if and only if  $a_{1, j-i+1} = 0$  (if  $j - i + 1 \leq 0$  then define  $a_{1, j-i+1}$  as  $a_{1, j-i+1} := a_{1, d+j-i+1}$ ). There are  $q^{d^2-d}$  vectors of length  $d$  over  $\mathbf{F}_{q^d}$  with a 0 entry in position  $j - i + 1$ . Not all of these vectors can occur as the first row of a matrix for  $h \in \text{GL}(d, q)$ , so

$$\text{Prob}(a_{1, j-i+1} = 0) < \frac{q^{d^2-d}}{|\text{GL}(d, q)|} = \frac{1}{q^d} \prod_{k=1}^d \frac{1}{1 - 1/q^k}.$$

Observe that

$$\begin{aligned} \prod_{k=1}^d \left(1 - \frac{1}{q^k}\right) &> \left(1 - \frac{1}{q}\right) \left(1 - \sum_{k=2}^d \frac{1}{q^k}\right) > \left(1 - \frac{1}{q}\right) \left(1 - \frac{1}{q^2} \frac{1}{1 - 1/q}\right) \\ &= \frac{q^2 - q - 1}{q^2}. \end{aligned}$$

This last fraction is greater than  $1/2$  if  $q \geq 3$  and is equal to  $1/4$  if  $q = 2$ , implying both claims of (a).

- (b) By Lemma 4.7, it is enough to estimate the probability that the first row of  $A$  consists of all nonzero entries. By (a), if  $q \geq 3$ , then the probability that the first row of  $A$  contains a zero entry is less than  $2d/q^d \leq 1/4$  (not considering the solvable case  $(d, q) = (2, 3)$ ). Similarly, if  $q = 2$  and  $d \geq 6$ , then the probability that the first row of  $A$  contains a zero entry is less than  $4d/2^d \leq 3/8$ . For  $3 \leq d \leq 5$ , we counted using GAP [7] the exact number of  $h \in \text{GL}(d, 2)$  with a zero entry in its matrix  $A$ .  $\square$

In Section 6.2, we require the following conjecture.

**CONJECTURE 4.9.** *Let  $h$  be a random element of  $H$  and let  $A = (a_{ij})$  be the matrix of  $h$  in the basis  $e_1, \dots, e_d$ . The probability that a principal  $3 \times 3$  submatrix of  $A$  has determinant 0 is less than  $c/q^d$  for an absolute constant  $c$ , unless  $d = 3m$  and the principal submatrix is from rows and columns  $i, m + i, 2m + i$  for some  $i$ .*

In the exceptional case, the first row of the submatrix uniquely determines the other entries; in all other cases, there is an entry that is ‘independent’ of the others. It appears that the value of this entry is roughly uniformly distributed in  $\mathbf{F}_{q^d}^*$  and only one value makes the determinant 0. The conjecture and these observations are supported by experiment: we considered large random samples of elements from the alternating square representation of  $\text{GL}(d, q)$  for  $q = 2, 3$  and  $d \leq 9$ .

**4.4. Avoiding division by zero** Let  $g \in G$ , let  $A = (a_{ij})$  be the matrix of  $g$  in the basis  $e_1, \dots, e_d$ , and let  $K = (\kappa_{ij,kl})$  be the matrix of  $g$  in the basis  $\mathcal{B}_0$  obtained in step 3 of Decompose.

In step 4, we determine the constants associated with our choice of basis. In doing so, we perform arithmetic operations on the entries of  $K$ ; some of these operations are not possible if certain entries of  $A$  are 0. Lemma 4.8 implies that, with high probability all entries of  $A$  are nonzero and so we can perform the computations.

In step 5, we compute the action  $A$  of arbitrary  $g \in G$  in the basis  $e_1, \dots, e_d$ . The algorithms of Sections 5–7 may not work if  $A$  has zero entries; however, if we multiply  $g$  by a random  $m \in G$ , then both  $m$  and  $gm$  are uniformly distributed (but not independent) random elements of  $G$ . Hence, with probability at least  $1/4$ , all entries in the  $d \times d$  matrices of  $m$  and  $gm$  are nonzero. If so, then we compute the action of  $m$  and  $gm$  in the basis  $e_1, \dots, e_d$ , and obtain the action of  $g$  as the ratio of these two matrices.

Summarizing, in Sections 5–7 we may assume that, for  $g \in G$ , all entries of its matrix  $A$  are nonzero. In Section 6.2 we assume Conjecture 4.9 which implies that all of the  $3 \times 3$  submatrices of  $A$  not having the exceptional indices have nonzero determinant.

## 5. The symmetric square representation

In this case,  $q$  must be odd. Suppose that  $G \leq \text{GL}(W)$  where  $W$  is the symmetric square of  $V$ , and  $s \in G$  was constructed as described in step 1 of Decompose. We now discuss steps 2–5.

**5.1. Labelling the basis** Let  $l_i = \omega^{q^{i-1}}$ , for  $1 \leq i \leq d$ , be the eigenvalues of  $s$  in its action on  $V \otimes \mathbf{F}_{q^d}$ . The eigenspaces of  $s$  on  $W$  are  $\langle e_{i,j} = e_i \otimes e_j \rangle$  for  $1 \leq i \leq j \leq d$ , and  $e_{i,j}s = \omega^{q^{i-1}+q^{j-1}}e_{i,j}$ .

We know the set  $L$  of products  $l_{i,j} := l_i l_j$  for  $1 \leq i \leq j \leq d$ . We identify the indices  $(i, j)$  corresponding to every element of  $L$ , and choose a basis  $\mathcal{B}_0 = \{f_{i,j}\}$ ,  $f_{i,j} \in \langle e_{i,j} \rangle$ , using the following procedure.

- (1) We construct the orbits  $\Omega_1, \dots, \Omega_k$  of eigenvalues under the Frobenius map  $\sigma$ . If  $d$  is odd, then there are  $(d + 1)/2$  orbits of size  $d$ ; otherwise there are  $d/2$  orbits of size  $d$  and one of size  $d/2$ .
- (2) We identify the orbit  $\Omega_m$  of size  $d$  which satisfies the following test: for all distinct pairs  $\alpha, \beta \in \Omega_m$ ,  $\gamma = \alpha \cdot \beta$  is a square, and one of the square roots of  $\gamma$  is in  $\Omega_j$  for some  $j \neq m$ .
- (3) Now we have identified that  $\Omega_m$  is the orbit of  $l_{1,1}$  under  $\sigma$ . We choose an arbitrary element of  $\Omega_m$  as  $l_{1,1}$  and label  $l_{j,j} = l_{1,1}\sigma^{j-1}$ .
- (4) For  $i \neq j$ , we evaluate  $\gamma = l_{i,i}l_{j,j}$ ; record  $l_{i,j}$  as the one of  $\pm\sqrt{\gamma}$  in  $L$ .
- (5) From each orbit of eigenvalues  $\Omega$ , we pick an arbitrary  $l_{i,j} \in \Omega$  and compute its eigenspace  $\langle e_{i,j} \rangle$ . We choose the vector  $f_{i,j} \in \langle e_{i,j} \rangle$  whose first nonzero coordinate is equal to 1.
- (6) For the other elements  $l_{i,j}^{\sigma^r} \in \Omega$ , we compute  $f_{i+r,j+r} := f_{i,j}^{\sigma^r}$ .

Since we identified  $\omega^2 = l_{1,1}$ , we can compute  $\omega$  and the base change matrix  $B$  of Section 4.3. We may choose  $\mu := 1$  as the  $(1, 1)$  entry of  $B$ .

**LEMMA 5.1.** *The cost of this procedure is  $O(\rho_q d^9 \log^2 d \log q)$ .*

Observe that the largest exponent for  $d$  comes from step 5, where we compute  $d$  eigenvectors at a cost  $O(\rho_q d n^3) = O(\rho_q d^8 \log^2 d)$  for each.

**5.2. Determining the constants** The outcome of the last step is the following: for  $1 \leq i \leq j \leq d$ , we now know that

$$f_{i,j} = c_{i,j} e_{i,j}$$

for some  $c_{i,j} \in \mathbf{F}_{q^d}$ . We now describe a procedure to determine these constants.

Our choice of  $f_{i,j}$  implies that  $c_{i+1,j+1} = c_{i,j}^q$ . Since we can multiply all basis vectors by the same scalar, we may assume that  $c_{1,1} = 1$  (implying that  $c_{2,2} = c_{3,3} = \dots = c_{d,d} = 1$ ). We have no further control over the  $c_{i,j}$ . Our procedure must compute their values.

We choose random  $g \in G$  and compute the matrix  $K = (\kappa_{ij,kl})$  representing the action of  $g$  on  $W$  with respect to the basis  $\mathcal{B}_0 = \{f_{i,j}\}$ . Let  $A = (a_{ij})$  be the matrix of  $g$  in the basis  $e_1, \dots, e_d$ . *A priori*, we do not know the entries of  $A$ , but we may assume that all entries of  $A$  are nonzero.

If  $e_i g = \sum_{j=1}^d a_{ij} e_j$ , then

$$(e_i \otimes e_j)g = \left( \sum_{s=1}^d a_{is} e_s \right) \otimes \left( \sum_{t=1}^d a_{jt} e_t \right)$$

and so

$$c_{ij} e_{ij} g = f_{ij} g = \sum \kappa_{ij, st} f_{st} = \sum \kappa_{ij, st} c_{st} e_{st}.$$

The basic equation for  $\kappa_{ij, k\ell}$  is

$$\kappa_{ij, k\ell} = \frac{c_{ij}}{c_{k\ell}} (a_{ik} a_{j\ell} + a_{i\ell} a_{jk} (1 - \delta_{kl})) \quad (5.1)$$

where  $\delta_{kl} = 1$  if  $k = l$ , otherwise 0.

By choosing specific values of the indices, this equation allows us to readily deduce the following formulas.

(i)

$$\kappa_{ii, jj} = \frac{c_{ii}}{c_{jj}} a_{ij}^2 = a_{ij}^2.$$

(ii)

$$\kappa_{ii, ii} = a_{ii}^2.$$

Hence the values  $a_{ij}$  are determined up to a sign ambiguity.

(iii)

$$\kappa_{ij, jj} = \frac{c_{ij}}{c_{jj}} a_{ij} a_{jj} = c_{ij} a_{ij} a_{jj}.$$

(iv)

$$\kappa_{ii, ij} = 2 \frac{c_{ii}}{c_{ij}} a_{ii} a_{ij} = \frac{2}{c_{ij}} a_{ii} a_{ij}.$$

We square (iii) and substitute (i), (ii) to get

(v)

$$c_{ij}^2 = \frac{\kappa_{ij, jj}^2}{\kappa_{jj, jj} \kappa_{ii, jj}} c_{jj} c_{ii} = \frac{\kappa_{ij, jj}^2}{\kappa_{jj, jj} \kappa_{ii, jj}}.$$

Taking the product and ratio of (iv) and (iii), and using (i) and (v), we get

(vi)

$$a_{ii} a_{jj} = \frac{\kappa_{ij, jj} \kappa_{ii, ij}}{2 \kappa_{ii, jj}}.$$

(vii)

$$\frac{a_{ii}}{a_{jj}} = \frac{\kappa_{ii, ij} \kappa_{ij, jj}}{2 \kappa_{jj, jj} \kappa_{ii, jj}}.$$

Taking  $j = k$ , Equation (5.1) gives

$$\kappa_{ij,j\ell} = \frac{c_{ij}}{c_{j\ell}}(a_{ij}a_{j\ell} + a_{i\ell}a_{jj}).$$

Rearranging and multiplying by  $a_{ii}/a_{jj}$ , we get

(viii)

$$\frac{c_{ij}}{c_{j\ell}}a_{i\ell}a_{ii} = \frac{a_{ii}}{a_{jj}}\left(\kappa_{ij,j\ell} - c_{ij}^2 \frac{a_{ij}}{c_{ij}} \frac{a_{j\ell}}{c_{j\ell}}\right).$$

Using (iv)–(vii), the right-hand side of (viii) can be expressed as a function of the  $\kappa_{ij,k\ell}$ .

We now describe the procedure to extract the  $c_{ij}$ .

- (1) If  $d$  is even, let  $m = d/2$ , else  $m = d$ .
- (2) We evaluate (viii) for  $i = 1, j = 2, \dots, m, \ell = (2j - 1)$  to obtain

$$\frac{c_{1j}}{c_{j,2j-1}}a_{1,2j-1}a_{11}$$

as a function of the  $\kappa_{ij,k\ell}$ .

- (3) Since  $c_{1j}/c_{j,2j-1} = c_{1j}^{(1-q^{j-1})}$  is a power of  $c_{1j}$  with even exponent, by (v) we can express it as a function of the  $\kappa_{ij,k\ell}$ .
- (4) Thus we obtain  $a_{1,2j-1}a_{11}$  as a function of the  $\kappa_{ij,k\ell}$ , and (iv) now yields the value of  $c_{1,2j-1}$  without ambiguity.
- (5) If  $d$  is even, then we must compute the values  $c_{1,2j}$ . By (v), we know  $c_{12}^2$ , but  $c_{12}$  cannot be computed without ambiguity: writing  $g \in G$  in the basis  $e_1, \dots, e_d$  or in  $e_1, -e_2, e_3, \dots, -e_d$  yields the same action on  $W$ . Hence we choose an arbitrary square root of  $c_{12}^2$  from (v) to obtain  $c_{12}$ .
- (6) Evaluating (viii) with  $i = 1, j = 2, \ell = 4, 6, \dots, d$ , we obtain  $a_{1\ell}a_{11}$  as a function of  $c_{12}, c_{2\ell} = c_{1,\ell-1}^q$ , and the  $\kappa_{ij,k\ell}$ , without further ambiguity (apart from the sign of  $c_{12}$ ).
- (7) Now (iv) yields the value of  $c_{1\ell}$ . The values  $c_{ij}$ , for  $i > 1$ , can be obtained as  $c_{ij} = c_{1,j-i+1}^{q^{i-1}}$ .

**LEMMA 5.2.** *The cost of this procedure is  $O(\xi + \rho_q d^8 \log^2 d)$ .*

The most expensive part of this procedure is writing  $g$  with respect to the basis  $\mathcal{B}_0 = \{f_{i,j}\}$ .

**5.3. Evaluating images** Assume we now wish to carry out step 5 of Decompose: namely, we want to construct the projective image of an arbitrary  $g \in G$ .

We first compute the matrix  $K = (\kappa_{ij,kl})$  representing the action of  $g$  on  $W$  with respect to the basis  $\mathcal{B}_0 = \{f_{i,j}\}$ . We then compute the  $a_{ij}$  in terms of the  $\kappa_{rs,tu}$ : for given  $i, j$ , we compute  $a_{11}a_{ij} = (a_{11}/a_{ii})(a_{ii}a_{jj})$  using (vii) and (iv). Hence the  $a_{ij}$  values are recovered up to a scalar multiple  $a_{11}$ . By (ii), the value of  $a_{11}$  can be computed only up to a sign ambiguity.

**LEMMA 5.3.** *The cost of this procedure is  $O(\rho_q d^8 \log^2 d)$ .*

If necessary, we compute the images of  $m$  and  $gm$  for random  $m \in G$  as discussed in Section 4.4.

## 6. The alternating square representation

Suppose that  $G \leq \text{GL}(W)$  where  $W$  is the alternating square of  $V$ , and  $s \in G$  is constructed as described in step 1 of `Decompose`. We now discuss steps 2–5.

**6.1. Labelling the basis** Let  $l_i = \omega^{q^{i-1}}$ , for  $1 \leq i \leq d$ , be the eigenvalues of  $s$  in its action on  $V \otimes \mathbf{F}_{q^d}$ . A basis of  $W$  is the set of vectors  $e_{i,j} = e_i \wedge e_j$  for  $1 \leq i < j \leq d$ , and  $e_{i,j}s = \omega^{q^{i-1}+q^{j-1}} e_{i,j}$ .

We know the set  $L$  of products  $l_i l_j$  for  $1 \leq i \leq j \leq d$ . We identify the indices  $(i, j)$  corresponding to every element of  $L$ , and choose a basis  $\mathcal{B}_0 = \{f_{i,j}\}$ ,  $f_{i,j} \in \langle e_{i,j} \rangle$ , using the following procedure.

- (1) We construct the orbits of the eigenvalues under the Frobenius map  $\sigma$ .
- (2) We choose an orbit of length  $d$ , and label an element of this orbit as  $l_1 l_2$ . Taking  $q$ th powers determines  $l_2 l_3$  and  $l_3 l_4$ .
- (3) We identify  $l_1 l_4 = (l_1 l_2)(l_3 l_4)/(l_2 l_3)$ , and by taking  $q$ th powers we identify  $l_2 l_5$  and  $l_3 l_6$ .
- (4) We identify  $l_1 l_6 = (l_1 l_4)(l_3 l_6)/(l_3 l_4)$ , and by taking  $q$ th powers we identify  $l_2 l_7$  and  $l_3 l_8$ .
- (5) In the same manner, we identify  $l_1 l_{2i}$  for  $i \leq d/2$ . It remains to identify  $l_1 l_{2i+1}$ .
- (6) Assume  $d = 2k + 1$ . By taking  $q$ th powers, we identify  $l_2 l_{2k+1}$  and  $l_3 l_1$ . We compute  $(l_1 l_3)/(l_1 l_2)^{q-1} = \omega^2$ . From  $\omega^2$ , we compute  $\omega^{q^{i-1}+q^{j-1}}$  for all  $i, j$ . If these are not elements of  $L$ , then we select another  $L$ -orbit of length  $d$  and pick another candidate for  $l_1 l_2$ .
- (7) Assume  $d = 2k$ . We know that  $(l_1 l_2)(l_3 l_4) = (l_1 l_3)^{q+1}$ . We choose an orbit of length  $d$  in  $L$  and an element  $x$  of this orbit such that  $x^{q+1} = (l_1 l_3)^{q+1}$ . We compute  $x/(l_1 l_2)^{q-1}$ . If our choice of  $x$  is valid as  $l_1 l_3$ , then this last ratio is  $\omega^2$ . From  $\omega^2$ , we compute  $\omega^{q^{i-1}+q^{j-1}}$  for all  $i, j$ . If these are not elements of  $L$ , then we select another  $L$ -orbit of length  $d$  and pick another candidate for  $l_1 l_3$ . The case  $d = 4$  is exceptional: one orbit has size 4, the only other has size 2, and we choose  $x$  from the orbit of length 2.
- (8) From each orbit  $\Omega$  of eigenvalues, we pick an arbitrary  $l_{i,j} \in \Omega$  and compute its eigenspace  $\langle e_{i,j} \rangle$ . We choose the vector  $f_{i,j} \in \langle e_{i,j} \rangle$  whose first nonzero coordinate is equal to 1.
- (9) For the other elements  $l_{i,j}^{\sigma^r} \in \Omega$ , we compute  $f_{i+r,j+r} := f_{i,j}^{\sigma^r}$ .

During this procedure, we computed  $\omega^2$ , so we can compute the base change matrix  $B$  of Section 4.3.

**LEMMA 6.1.** *The cost of this procedure is  $O(\rho_q d^9 \log^2 d \log q)$ .*

**6.2. Determining the constants** The outcome of the last step is the following: for  $1 \leq i < j \leq d$ , we now know

$$f_{i,j} = c_{i,j} e_{i,j}$$

for some  $c_{i,j} \in \mathbb{F}_{q^d}$ . We now describe a procedure to determine these constants.

Our choice of  $f_{i,j}$  implies that  $c_{i+1,j+1} = c_{i,j}^q$ . Since we can multiply all basis vectors by the same scalar, we may assume that  $c_{1,2} = 1$  (implying  $c_{2,3} = c_{3,4} = \dots = c_{d-1,d} = 1$ ). We have no further control over the  $c_{i,j}$ . Our procedure must compute their values.

We choose random  $g \in G$  and compute the matrix  $K = (\kappa_{ij,kl})$  representing the action of  $g$  on  $W$  with respect to the basis  $\{f_{i,j}\}$ .

If  $e_i g = \sum_{j=1}^d a_{ij} e_j$ , then

$$(e_i \otimes e_j)g = \left( \sum_{s=1}^d a_{is} e_s \right) \wedge \left( \sum_{t=1}^d a_{jt} e_t \right)$$

and so

$$c_{ij} e_{ij} g = f_{ij} g = \sum \kappa_{ij,st} f_{st} = \sum \kappa_{ij,st} c_{st} e_{st}.$$

The basic equation for  $\kappa_{ij,kl}$  is

$$\kappa_{ij,kl} = \frac{c_{ij}}{c_{kl}} (a_{ik} a_{jl} - a_{il} a_{jk}). \tag{6.1}$$

Underpinning our procedure is the observation that, in dimension 3, the exterior square is the dual of the natural module. This means that for distinct triples  $i, j, k$ , the matrices

$$B_{ijk} = \begin{pmatrix} a_{ii} & \frac{c_{ik}}{c_{jk}} a_{ij} & \frac{c_{ij}}{c_{jk}} a_{ik} \\ \frac{c_{jk}}{c_{ik}} a_{ji} & a_{jj} & \frac{c_{ij}}{c_{ik}} a_{jk} \\ \frac{c_{jk}}{c_{ij}} a_{ki} & \frac{c_{ik}}{c_{ij}} a_{kj} & a_{kk} \end{pmatrix}, \quad C_{ijk} = \begin{pmatrix} \kappa_{jk,jk} & -\kappa_{ik,jk} & \kappa_{ij,jk} \\ -\kappa_{jk,ik} & \kappa_{ik,ik} & -\kappa_{ij,ik} \\ \kappa_{jk,ij} & -\kappa_{ik,ij} & \kappa_{ij,ij} \end{pmatrix} \tag{6.2}$$

satisfy  $B_{ijk} C_{ijk} = \det(B_{ijk}) \cdot I$ , because by (6.1) the entries of  $C_{ijk}$  are the appropriate  $2 \times 2$  minors of  $B_{ijk}$ . This implies  $\det(B_{ijk} C_{ijk}) = \det(B_{ijk}) \cdot \det(C_{ijk}) = \det(B_{ijk})^3$ . Moreover, if  $C_{ijk}$  is invertible, then  $B_{ijk} = \sqrt{C_{ijk} C_{ijk}^{-1}}$ ; so we can compute  $B_{ijk}$  up to a sign ambiguity. Conjecture 4.9 implies that we may assume that  $B_{ijk}$  is invertible unless  $j = i + d/3$  and  $k = i + 2d/3$ .

We now describe the procedure to determine the constants  $c_{i,j}$ . Recall that we can assume  $c_{12} = 1$  (and consequently  $c_{i,i+1} = c_{12}^{q^{i-1}} = 1$  for all  $i \leq d - 1$ ).



- (1) For  $k = 4, 6, \dots, 2\lfloor d/2 \rfloor$ , we compute  $B_{132}$  and  $B_{13k}$ . We choose the square root of determinants so that the common entries  $B_{132}[1, 1] = B_{13k}[1, 1] = a_{11}$  and  $B_{132}[2, 2] = B_{13k}[2, 2] = a_{33}$  are equal. Now  $B_{132}[2, 1] = a_{31}c_{32}/c_{12} = a_{31}$  and  $B_{13k}[2, 1] = a_{31}c_{3k}/c_{1k} = a_{31}c_{1,k-2}^{q^2}/c_{1k}$ . Taking the ratio, we obtain  $c_{1k} = c_{1,k-2}^{q^2} B_{132}[2, 1]/B_{13k}[2, 1]$  and so recursively we can compute  $c_{14}, c_{16}, \dots, c_{1,2\lfloor d/2 \rfloor}$ . Since  $c_{ij} = c_{i-1,j-1}^q$ , we obtain all  $c_{ij}$  with  $i < j$  and  $j - i$  odd.
- (2) For  $k = 5, 7, \dots, 2\lfloor d/2 \rfloor - 1$ , we compute  $B_{123}$  and  $B_{12k}$ , again choosing the square roots of determinants so that the common entries  $B_{123}[1, 1] = B_{12k}[1, 1] = a_{11}$  and  $B_{123}[2, 2] = B_{12k}[2, 2] = a_{22}$  are equal. Thus  $B_{123}[2, 1] = a_{21}c_{23}/c_{13} = a_{21}/c_{13}$  and  $B_{12k}[2, 1] = a_{21}c_{2k}/c_{1k} = a_{21}c_{1,k-1}^q/c_{1k}$ . The ratio yields

$$c_{1k} = c_{13}c_{1,k-1}^q \frac{B_{123}[2, 1]}{B_{12k}[2, 1]}. \tag{6.3}$$

Since  $c_{1,k-1}$  is already known, we obtain  $c_{1k}$  as the value of  $c_{13}$  multiplied by a known quantity.

- (3) If  $d$  is odd, then  $c_{13} = c_{1,d-1}^{q^2}$  is known, and we obtain all  $c_{1j}$  without ambiguity. The other  $c_{ij}$  are computed by the formula  $c_{ij} = c_{i-1,j-1}^q$ .
- (4) If  $d$  is even, then we compute  $B_{123}$  and  $B_{124}$ . Now (6.3), with the value  $k = 4$ , gives  $c_{14} = c_{13}^{q+1} B_{123}[2, 1]/B_{124}[2, 1]$ . Since  $c_{14}$  is already known, we obtain  $c_{13}^{q+1}$ . Using the formula  $c_{ij} = c_{i-1,j-1}^q$ , and the values of  $c_{1k}$  and  $c_{13}^{q+1}$ , we obtain  $c_{ij}$  with  $i < j$  and  $j - i$  even as  $c_{ij} = c_{13}\overline{c_{ij}}$  if  $i$  is odd, and as  $c_{ij} = c_{13}^{-1}\overline{c_{ij}}$  if  $i$  is even, where  $\overline{c_{ij}}$  is a known element of  $\mathbf{F}_{q^d}$ .

We do not compute the value for  $c_{13}$  because  $(q + 1)$ -st roots cannot be computed in polynomial time. Instead we ignore both  $c_{13}$  and  $c_{13}^{-1}$ , and take  $c_{ij} = \overline{c_{ij}}$  for all  $i < j$  with  $j - i$  even.

The justification is the following. Given  $g \in G$ , we want to compute the coefficients in the linear combination  $e_m g = \sum_{\ell} a_{m\ell} e_{\ell}$ . If we know  $c_{ij}$ , then  $a_{m\ell}$  can be computed from an appropriate entry of some  $B_{ijk}$ . If we use  $\overline{c_{ij}}$  in place of  $c_{ij}$ , then we obtain  $c_{13}a_{m\ell}$  when both  $m$  and  $m - \ell$  are odd,  $a_{m\ell}/c_{13}$  when  $m$  is even and  $m - \ell$  is odd, and  $a_{m\ell}$  when  $m - \ell$  is even. Thus, instead of  $A = (a_{m\ell})$ , we construct the conjugate of  $A$  by the diagonal matrix  $\text{Diag}(1, c_{13}, 1, c_{13}, \dots, 1, c_{13})$ . Equivalently, this is the action of  $g$  in the basis  $e_1, e_2/c_{13}, e_3, e_4/c_{13}, \dots, e_{d-1}, e_d/c_{13}$ .

**LEMMA 6.2.** *The cost of this procedure is  $O(\xi + \rho_q d^8 \log^2 d \log q)$ .*

**6.3. Evaluating images** Given  $g \in G$ , we compute the matrix  $K = (\kappa_{ij,kl})$  representing the action of  $g$  on  $W$  with respect to the basis  $\mathcal{B}_0 = \{f_{i,j}\}$ . We determine the first row of  $A$  by computing the matrices  $B_{12k}$  for  $3 \leq k \leq d$ . Following Lemma 4.7, the other entries of  $A$  can be obtained by taking  $q$ th powers.

**LEMMA 6.3.** *The cost of this procedure is  $O(\rho_q d^8 \log^2 d \log q)$ .*

If necessary, we compute the images of  $m$  and  $gm$  for random  $m \in G$  as discussed in Section 4.4.

### 7. The adjoint representation

Let  $V^* \otimes V$  have basis  $B_0 := \{e_i \otimes e_j \mid 1 \leq i, j \leq d\}$ . Recall from Section 2 our definition of the adjoint module  $W := U/W_1$ . Consider a basis for  $W$  which is the union of the set  $B_1$  of  $d^2 - d$  vectors  $e_i \otimes e_j + W_1, i \neq j$ , and a set  $B_2$  of  $d - 1$  or  $d - 2$  vectors of the form  $x + W_1$  for some  $x \in \langle e_i \otimes e_i \mid 1 \leq i \leq d \rangle$ . We can compute the subspaces  $\langle e_i \otimes e_j + W_1 \rangle$ ; by choosing the vectors with first coordinates 1 from these subspaces, we construct  $B_1$ . Choosing the remaining basis vectors from the 1-eigenspace of  $s$  on  $W$ , we construct  $B_2$ .

For  $g \in G$ , let  $K_1$  and  $K_2$  be the matrices of  $g$  on  $V^* \otimes V$  with basis  $B_0$ , and on  $W$  with basis  $B_1 \cup B_2$ , respectively. Independent of the particular choice of  $B_2$ , the  $(d^2 - d) \times (d^2 - d)$  submatrices of  $K_1$  and  $K_2$ , determined by the basis vectors  $e_i \otimes e_j$  and  $e_i \otimes e_j + W_1$ , respectively, are identical.

**7.1. Labelling the basis** Let  $l_i = \omega^{q^{i-1}}$ , for  $1 \leq i \leq d$ , be the eigenvalues of  $s$  in its action on  $V \otimes \mathbb{F}_{q^d}$ . The one-dimensional eigenspaces of  $s$  on  $W$  are  $\langle e_{i,j} \rangle$  for  $i \neq j$ , and  $e_{i,j}s = \omega^{q^{i-1} - q^{j-1}} e_{i,j}$ .

We know the set  $L$  of products  $l_{i,j} := l_i l_j^{-1}$  for  $1 \leq i, j \leq d, i \neq j$ . We identify the indices  $(i, j)$  corresponding to every element of  $L$ , and choose a basis for  $W$  using the following procedure.

- (1) We construct the  $d - 1$  orbits of elements of  $L$  under the Frobenius map  $\sigma$ .
- (2) We choose one of these orbits and declare an entry from this orbit as  $l_{1,2}$ . For  $i \in \{2, \dots, d - 1\}$ , we label  $l_{i,i+1} = l_{i-1,i}^q$ , and  $l_{d,1} = l_{d-1,d}^q$ .
- (3) For  $k \in \{2, \dots, d - 1\}$ , we perform the following:
  - we evaluate  $v := l_{1,k} l_{1,2}^{q^{(k-1)}}$ ;
  - if  $v \notin L$ , then we choose a different orbit;
  - otherwise we label  $v$  as  $l_{1,k+1}$ ;
  - for  $j \in \{2, \dots, d - k\}$ , we identify  $l_{j,j+k} := l_{j-1,j+k-1}^q$ . Also we identify  $l_{d+1-k,1} := l_{d-k,d}^q$ ;
  - for  $j \in \{d + 2 - k, \dots, d\}$ , we identify  $l_{j,j-d+k} := l_{j-1,j-d+k-1}^q$ .
- (4) For each of the  $d - 1$  orbits on  $L$ , we pick a representative  $l_{i,j}$  and compute its eigenspace  $\langle e_{i,j} \rangle$ . We choose the vector  $f_{i,j} \in \langle e_{i,j} \rangle$  whose first nonzero coordinate is equal to 1.

Since we can assume that the first coordinate of each  $e_i$  is 1, the vector  $f_{i,j}$  corresponds precisely to  $e_i \otimes e_j$ , not just to a scalar multiple of it. For other eigenvalues  $l_{i,j}^{\sigma^r}$ , we compute  $f_{i+r,j+r} := f_{i,j}^{\sigma^r}$ . Let  $B_1 := \{f_{i,j} \mid 1 \leq i \neq j \leq d\}$ .

- (5) We compute the 1-eigenspace of  $s$  and choose an arbitrary basis  $B_2$  for it. Then  $B_1 \cup B_2$  is a basis of  $W$ .

**LEMMA 7.1.** *The cost of this procedure is  $O(\rho_q d^9 \log^2 d \log q)$ .*

Since the exponent of  $\omega$  in  $l_{i,j}$  is a multiple of  $q - 1$ , we cannot compute  $\omega$  in polynomial time. Hence we cannot compute the base change matrix  $B$  of Section 4.3, but instead use the algorithm of [8] to perform the final base change.

**7.2. Evaluating images** Given  $g \in G$ , we compute the matrix representing the action of  $g$  on  $W$  with respect to the basis  $B_1 \cup B_2$ . Let  $K_1$  be the matrix of  $g$  in the basis  $B_0$ , let  $K_2 = (\kappa_{ij,kl})$  be the matrix of  $g$  in the basis  $B_1 \cup B_2$ , let  $A = (a_{ij})$  be the matrix of  $g$  in the basis  $\{e_i\}$ , and let  $A^* = (a_{ij}^*)$  be the matrix of  $g\varphi$  in the basis  $\{e_i\}$ . Here  $\varphi$  is a graph automorphism, namely an inverse transpose map, but it is taken with respect to the basis  $b_1, \dots, b_n$  defined in Section 4.3. Thus  $A$  and  $A^*$  are not inverse transposes of each other. The goal is to recover a scalar multiple of  $A$ . The basic equation in  $K_1$  is

$$\kappa_{ij,kl} = a_{ik}^* a_{jl}. \tag{7.1}$$

The  $(d^2 - d) \times (d^2 - d)$  submatrix of  $K_2$  indexed by  $B_1$  is the corresponding submatrix of  $K_1$ , so we may use (7.1) for  $i \neq j$  and  $k \neq \ell$ .

To determine  $a_{11}^* a_{ks}$  for any  $s, k$ , we use the following equations:

$$\begin{aligned} \kappa_{1k,1s} &= a_{11}^* a_{ks}, & k, s \geq 2, \\ \kappa_{21,2s} &= a_{22}^* a_{1s}, & s \neq 2, \\ \kappa_{2k,21} &= a_{22}^* a_{k1}, & k \neq 2, \\ \kappa_{31,32} &= a_{33}^* a_{12}, \\ \kappa_{32,31} &= a_{33}^* a_{21}, \\ a_{11}^*/a_{22}^* &= \kappa_{1j,1\ell}/\kappa_{2j,2\ell} & \text{for some } j, \ell \notin \{1, 2\}, \\ a_{11}^*/a_{33}^* &= \kappa_{1j,1\ell}/\kappa_{3j,3\ell} & \text{for some } j, \ell \notin \{1, 3\}. \end{aligned}$$

**LEMMA 7.2.** *The cost of this procedure is  $O(\rho_q d^8 \log^2 d)$ .*

If necessary, we compute the images of  $m$  and  $gm$  for random  $m \in G$  as discussed in Section 4.4.

### 8. The other representations

We now discuss the outstanding cases: namely,  $W = V \otimes V^\tau$  and  $W = V^* \otimes V^\tau$ . Recall that  $H \leq \text{GL}(d, q)$  with  $q = p^f$ . The operation  $\tau$  is to take  $p^e$ th powers of the entries in the matrices representing the group elements, for some fixed positive  $e < f$ , and so  $W$  is now irreducible.

As before, we can assume that the first coordinate of the basis vector  $e_1$  is 1; since all other  $e_i$  are obtained by the Frobenius map  $\sigma$ , their first coordinate is also 1. Consequently, the first coordinates of the  $d^2$  vectors  $e_i \otimes e_j$  are 1. These  $d^2$  vectors form a basis of  $W$ . Hence, we can compute the one-dimensional subspaces  $\langle e_i \otimes e_j \rangle$  and, by picking the vectors with first coordinates 1 from these subspaces, we choose the vectors  $e_i \otimes e_j$ .

**8.1. Labelling the basis when  $W = V \otimes V^\tau$**  Let  $l_i = \omega^{q^{i-1}}$ , for  $1 \leq i \leq d$ , be the eigenvalues of  $s$  in its action on  $V \otimes \mathbf{F}_{q^d}$ . The eigenspaces of  $s$  on  $W$  are  $\langle e_{i,j} = e_i \otimes e_j \rangle$  for  $1 \leq i, j \leq d$ , and  $e_{i,j}s = \omega^{q^i+q^j p^e} e_{i,j}$ .

We know the set  $L$  of products  $l_{i,j} := l_i(l_j)^{p^e}$  for  $1 \leq i, j \leq d$ . We identify the indices  $(i, j)$  corresponding to every element of  $L$ , and choose a basis  $\mathcal{B}_0 = \{f_{i,j}\}$ ,  $f_{i,j} \in \langle e_{i,j} \rangle$ , using the following procedure.

- (1) We construct the orbits of eigenvalues under the Frobenius map  $\sigma$ .
- (2) We choose one of these orbits and declare an entry  $\lambda$  from this orbit as  $l_{1,1}$ .
- (3) If there exists an eigenvalue  $\nu$  where  $\nu^{1+p^e} = \lambda^{1+q p^e}$  and  $\lambda^{q+1} \nu^{-1}$  is also an eigenvalue, then we identify  $l_{1,2}$  as  $\nu$  and  $\lambda^{q+1} \nu^{-1}$  as  $l_{2,1}$ . Otherwise we choose a new orbit.
- (4) For  $i \in \{2, \dots, d\}$ , we label  $l_{i,i} = l_{i-1,i-1}^q$ .
- (5) For  $i \in \{2, \dots, d-1\}$ , we label  $l_{i,i+1} = l_{i-1,i}^q$ , and  $l_{d,1} = l_{d-1,d}^q$ .
- (6) For  $i \in \{2, \dots, d-1\}$ , we label  $l_{i+1,1} := l_{1,1} l_{i,i} l_{i+1,i+1} / l_{i,i+1} l_{1,i}$  and  $l_{1,i+1} := l_{1,1} l_{i+1,i+1} / l_{i+1,1}$ .
- (7) The remaining values  $l_{i,j}$  can now be identified by taking  $q$ th powers of the already labelled elements of their orbits.
- (8) For each orbit on  $L$ , we pick a representative  $l_{i,j}$  and compute its eigenspace  $\langle e_{i,j} \rangle$ . We choose the vector  $f_{i,j} \in \langle e_{i,j} \rangle$  whose first nonzero coordinate is equal to 1.

Since we can assume that the first coordinate of each  $e_i$  is 1, the vector  $f_{i,j}$  corresponds *precisely* to  $e_i \otimes e_j$ , not just to a scalar multiple of it. For other eigenvalues  $l_{i,j}^{\sigma^r}$ , we compute  $f_{i+r,j+r} := f_{i,j} \sigma^r$ .

**LEMMA 8.1.** *The cost of this procedure is  $O(\rho_q d^9 \log^2 d \log q)$ .*

We use the algorithm of [8] to perform the final base change.

**8.2. Evaluating images when  $W = V \otimes V^\tau$**  Given  $g \in G$ , we compute its  $d^2 \times d^2$  matrix  $K = (\kappa_{ij,kl})$  in the basis  $\{e_i \otimes e_j\}$ . Let  $A = (a_{ij})$  be the matrix of  $g$  in the basis  $\{e_i\}$ . The goal is to recover a scalar multiple of  $A$ . The basic equation for  $\kappa_{ij,kl}$  is

$$\kappa_{ij,kl} = a_{ik} a_{jl}^{p^e}.$$

We choose an arbitrary nonzero entry  $\kappa_{i_0 j_0, k_0 \ell_0}$  in  $K$ . For this fixed  $j_0, \ell_0$ , the matrix with  $(i, k)$  entry  $\kappa_{i j_0, k \ell_0} = a_{ik} a_{j_0 \ell_0}^{p^e}$  is a projective image of  $g$ .

**LEMMA 8.2.** *The cost of this procedure is  $O(\rho_q d^8 \log^2 d)$ .*

**8.3. Labelling the basis when  $W = V^* \otimes V^\tau$**  Labelling the basis for this case is essentially identical to that described in Section 8.1.

**8.4. Evaluating images when  $W = V^* \otimes V^\tau$**  Given  $g \in G$ , we compute its  $d^2 \times d^2$  matrix  $K = (\kappa_{ij,kl})$  in the basis  $\{e_i \otimes e_j\}$ . Let  $A = (a_{ij})$  be the matrix of  $g$  in

TABLE 1. Performance of implementation for some groups.

$d$	$q$	$G$	Setup	Image
5	$7^{10}$	Symmetric square	1.0	0.02
		Alternating square	0.3	0.01
		Adjoint	1.6	0.05
		$V \otimes V^\tau$	1.5	0.05
10	$5^6$	Symmetric square	15.7	0.4
		Alternating square	6.0	0.2
		Adjoint	34.5	1.1
		$V \otimes V^\tau$	73.6	1.0
15	$3^2$	Symmetric square	50.1	1.7
		Alternating square	23.4	1.3
		Adjoint	140.2	5.2
		$V \otimes V^\tau$	150.7	1.2

the basis  $\{e_i\}$  and let  $A^* = (a_{ij}^*)$  be the matrix of  $g\varphi$  in the basis  $\{e_i\}$  for a graph automorphism  $\varphi$ . The goal is to recover a scalar multiple of  $A^*$ . The basic equation for  $\kappa_{ij,kl}$  is

$$\kappa_{ij,kl} = a_{ik}^* a_{jl}^e.$$

We choose an arbitrary nonzero entry  $\kappa_{i_0 j_0, k_0 \ell_0}$  in  $K$ . For this fixed  $j_0, \ell_0$ , the matrix with  $(i, k)$  entry  $\kappa_{ij_0, k\ell_0} = a_{ik}^* a_{j_0 \ell_0}^e$  is a projective image of  $g$ .

**LEMMA 8.3.** *The cost of this procedure is  $O(\rho_q d^8 \log^2 d)$ .*

## 9. Implementation and performance

We have implemented our algorithms in MAGMA. We use the algorithm of [6] to generate random elements and in step 1 choose a sample of size  $4d^2$ .

The computations reported in Table 1 were carried out using MAGMA V2.13 on a Pentium IV 2.8 GHz processor. Let  $G$  act as the named representation of  $H := \text{GL}(d, q)$ . The first four steps of Algorithm `Decompose` provide the data structure of Theorem 2.1. Having computed this data, we can now compute the projective image of  $g \in G$ . In the final two columns, we list the CPU times in seconds to set up the data structure, and to evaluate the image of a randomly chosen element of  $G$ .

## Acknowledgement

We thank the referee for helpful comments and suggestions.

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KAY MAGAARD, School of Mathematics, University of Birmingham,  
Birmingham B15 2TT, UK

e-mail: [k.magaard@bham.ac.uk](mailto:k.magaard@bham.ac.uk)

E. A. O’BRIEN, Department of Mathematics, University of Auckland, Auckland,  
New Zealand

e-mail: [obrien@math.auckland.ac.nz](mailto:obrien@math.auckland.ac.nz)

ÁKOS SERESS, Department of Mathematics, The Ohio State University,  
Columbus, OH 43210, USA

e-mail: [akos@math.ohio-state.edu](mailto:akos@math.ohio-state.edu)