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June 19, 2017

Abstract

As is known, Darcy's model for fluid flows in isotropic homogeneous porous media gives rise to singularities in the velocity field for essentially two-dimensional flow configuration, like flows over corners. Considering this problem from the modelling viewpoint, the present study aims at removing this singularity, which cannot be regularized via conventional generalizations of the Darcy model, like Brinkman's equation, without sacrificing Darcy's law itself for unidirectional flows where its validity is well established experimentally. The key idea is that, as confirmed by a simple analogy, the permeability of a porous matrix with respect to flow is not a constant independent of the flow but a function of the flow field (its scalar invariants), decreasing as the curvature of the streamlines increases. This introduces a completely new class of models where the flow field and the permeability field are linked and, in particular problems, have to be found simultaneously.

Keywords mathematical modelling, porous media, Darcy's law, velocity singularity, curvature of streamlines

Introduction

As discovered experimentally in 1855-56 by Henry Darcy and his collaborator Charles Ritter, for a unidirectional flow of water down a column of sand, "for identical sands, one can assume that the discharge is directly proportional to the head and inversely proportional to the thickness of the layer traversed".[1] In other words, $Q = -K_S(p_{outlet} - p_{inlet})/H$, where Q is the volumetric discharge across a porous layer of thickness H driven by the pressure difference $p_{inlet} - p_{outlet} > 0$, and K_S is a scalar coefficient of proportionality. Importantly, experiments also show that Q is proportional to the area S of the cross-section through which the flow is passing, provided this cross-section is large compared with the pore sizes. This indicates that, if viewed macroscopically, the flow inside the porous sample is uniform ('plug flow'), so that, introducing the averaged flow velocity as the volumetric flux through a unit area of the cross-section, $u = Q/S$, one has

$$u = -K \frac{p_{outlet} - p_{inlet}}{H}, \tag{1}$$

where $K = K_S/S$ is known as hydraulic conductivity. In applications, it is convenient to factor out the fluid's viscosity μ and represent the hydraulic conductivity as $K = \chi/\mu$, where χ is known as the permeability of the medium. In many experiments performed since Darcy's time, in particular in the last three decades,[2–7] it has been shown that, once

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the pressure difference across a sample of porous medium becomes too high, and the flow in the pores is no longer the creeping one, the velocity-pressure relationship deviates from the simple linear dependence (1). At the same time, for most porous media and the flow parameters of practical interest, in particular in engineering applications, the relationship (1) holds, and in what follows we will take that, for the media we are dealing with and the unidirectional flows with the rates we are interested in, (1) is an experimental fact.

In the limit $H \to 0$, the ratio $(p_{\text{outlet}} - p_{\text{inlet}})/H$ on the right-hand side of (1) turns into a directional derivative, $\partial p/\partial \hat{\mathbf{u}} \equiv \hat{\mathbf{u}} \cdot \nabla p$, where $\hat{\mathbf{u}} = \mathbf{u}/|\mathbf{u}|$ is a unit vector in the direction of the flow velocity $\mathbf{u}$. The left-hand side of (1) can be obviously written down as $u = \mathbf{u} \cdot \hat{\mathbf{u}}$, so that, in the limit $H \to 0$, equation (1) takes the form $\mathbf{u} \cdot \hat{\mathbf{u}} = -(\chi/\mu)\hat{\mathbf{u}} \cdot \nabla p$, or simply

$$\mathbf{u} = -(\chi/\mu)\nabla p, \tag{2}$$

which came to be known as Darcy's law. It should be noted that the above derivation implies the continuum limit as it assumes the fluid's pressure to be a differentiable function of spatial coordinates. It also presumes that the porous medium is, like Darcy's sand, isotropic, so that the flow is parallel to the pressure gradient and χ is independent of the flow direction. Assuming further that the fluid is incompressible and its velocity $\mathbf{u}$ is a continuously differentiable function of the spatial coordinates, in addition to (2) we will also have a continuity equation in the form

$$\nabla \cdot \mathbf{u} = 0. \tag{3}$$

Then, if the porous medium is homogeneous, we have that $\chi = \chi_0$ is a constant, and, in

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this case, the fluid's pressure satisfies Laplace's equation $\nabla^2 p = 0$. The model (2)–(3) for the bulk flow with $\chi = \chi_0$ allows one to formulate and solve a variety of problems, and the resulting predictions have been shown to be qualitatively correct. As for the model's quantitative validation and accuracy, strictly speaking, it is only for the unidirectional flows, as in Darcy's original experiment, that the model has been experimentally tested and its limits of applicability with regard to the flow rate for particular media were determined.[7]

In the simple derivation given above, which sums up the essence of how Darcy's law was originally derived, the formal assumption that p and then $\mathbf{u}$, χ and, if variable, μ are continuously differentiable functions of the spatial coordinates tells us nothing about the limits of applicability of the notions involved and hence the model itself. To elucidate the conditions in which the model can be expected to work, the system (2)–(3) has been re-derived theoretically several times, with each derivation using its own broadly equivalent plausible assumptions.[8–12] In particular, the derivation via the technique of volume averaging[13] highlights that, as always in mechanics of multiphase media, the macroscopic scale ('Darcy scale') L on which the flow is described in terms of averaged quantities must be well separated from the microscopic scale a ('pore-scale') where the actual fluid motion takes place. Then an intermediate scale of averaging and hence averaged quantities can be introduced, so that, formally, the continuum mechanics description of flow in porous media follows from an asymptotic limit $a/L \to 0$. This limit can be called a 'secondary' continuum limit with the 'primary' one being the limit $l_{\mathrm{mol}}/a \to 0$ (l_{mol} is the molecular length scale) allowing the continuum mechanics description of the pore-scale flow.

It should be emphasized here that various averaging schemes used to re-derive Darcy's

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law are nothing but different ways of organizing our thinking about the process in terms of the assumptions all of them invariably involve. Darcy's law was obtained phenomenologically and, for unidirectional flows, validated experimentally, so that the fact that it can also be obtained, with a number of assumptions, from some averaging scheme is an argument giving credibility to this scheme, not the other way round. We will introduce a new class of models phenomenologically generalizing Darcy's model such that the flow field they predict on the macroscpic/Darcy scale is singularity-free.

Singularity

A number of recent applications required the use of the system (2)–(3) in the situations where the flow is *essentially* two-dimensional, i.e. very far from the Darcy's original configuration. For example, when a fluid imbibes into a porous medium from a confinement, e.g. a container or a drop sitting on a porous substrate (Fig. 1a), the boundary confining the flow on the Darcy scale appears to have a corner with an angle exceeding π if measured through the liquid.[14] Although the porous medium here occupies only a half-space, the flow entering it normally to its boundary has to turn around as this boundary is a streamline, so that in the porous medium one effectively has a flow round a corner. One has a similar situation once the wetting front propagating through a porous medium fragments as one section of it is brought to a halt, for example as a result of it being stuck in the threshold mode of motion,[15] whilst the neighbouring section continues to move (Fig. 1b). Then, on the Darcy scale, the advancing fluid appears to have a flow domain with a corner with the angle of $3\pi/2$. In all physically realistic situations, in these flow configurations

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one should expect the fluid's velocity to remain finite everywhere in the flow domain, and this feature is to be reproduced by the quantitative model of the flow.

In a distilled form, the situation sketched above occurs when we simply have a solid wedge inserted into a porous medium (Fig. 1c). When we consider the flow near a corner, we can immediately see a problem. Indeed, for a domain shown in Fig. 1c equations (2), (3) with constant $\chi = \chi_0$ reduce to Laplace's equation for the pressure, $\nabla^2 p = 0$, and, with the impermeability conditions on the boundary, for a flow near a corner, we have

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} = 0, \quad (r > 0, \ -\alpha < \theta < \alpha); \tag{4}$$

$$\frac{\partial p}{\partial \theta} = 0, \quad (r > 0, \ \theta = \pm\alpha), \tag{5}$$

where (r, θ) are the polar coordinates with the origin at the corner. Looking for a separable solution of the form $p = r^q f(\theta)$, where, for p to be finite, $q \geq 0$, one arrives at a series of solutions of which we will be interested only in the leading one as $r \to 0$. Then, for the pressure we have

$$p = A r^{\pi/(2\alpha)} \sin\left(\frac{\pi\theta}{2\alpha}\right), \tag{6}$$

so that the components of the corresponding velocity field are given by

$$u_r = -\frac{\chi_0 \pi}{2\alpha\mu} A r^{\pi/(2\alpha)-1} \sin\left(\frac{\pi\theta}{2\alpha}\right), \tag{7}$$

$$u_\theta = -\frac{\chi_0 \pi}{2\alpha\mu} A r^{\pi/(2\alpha)-1} \cos\left(\frac{\pi\theta}{2\alpha}\right). \tag{8}$$

This is the exact global solution for the flow domain bounded by a wedge (Fig. 1c) and

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the leading term of the local asymptotics as $r \to 0$ if one has the wedge as part of the boundary in some more complex global problem. In the latter case, a numerical analysis of a particular flow similar to that sketched in Fig. 1a[16] confirmed that, in general, as one would expect, $A \neq 0$. For $2\alpha = \pi$, we obviously have, if $A \neq 0$, a uniform flow past a plane boundary, i.e. the situation for which Darcy's law was originally discovered, and a trivial case of no flow if $A = 0$. However — and this is what is of interest to us in this solution — for $2\alpha > \pi$ the velocity field becomes singular with both components of the velocity going to infinity as $r \to 0$. In other words, we have a clearly unphysical feature in an otherwise well tested model.

[Figure 1 about here.]

It is noteworthy that for single-phase flows the above singularity, albeit unacceptable from the physical point of view, is a relatively minor issue from the practical standpoint as it is integrable and hence as a result the mass flux described by (7)–(8) remains finite, although clearly overpredicted. The situation becomes serious when what matters is not an integral but the values of the velocity. This happens, such as when one deals with two-phase flows where in the boundary conditions at the liquid-fluid interface[15, 17] involve the velocity itself and not its integral. Then, the singularity in the velocity distribution becomes a major obstacle for the realistic modelling of the propagation of liquid-fluid interfaces in the situations where the flow becomes essentially two-dimensional. For example, the situation shown in Fig. 1b becomes impossible to describe since, once one section of the interface is brought to a halt, cannot move to create a corner as otherwise its velocity close to the section at rest would be unbounded.

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The above singularity is, of course, well known, but so far the research effort, both in the description of flows in porous media[18,19] as well as in numerous other problems involving elliptic equations in domains with corners,[20–23] has been focussed on designing numerical algorithms which, instead of ignoring the singularity in an expectation that its overall effect is negligible, incorporate it into numerical codes. The resulting numerical solution would then accurately approximate the singular analytical one, whose leading-order terms are given by (6), (7)–(8), without making the latter any more acceptable from the physical point of view. Here we are interested in the modelling side where, in order to be able to use Darcy's model for essentially two-dimensional flows, one has to *remove* the singularity while preserving the classical Darcy law for unidirectional flows for which it was well tested experimentally.

Modelling

The singularity appearing in the essentially two-dimensional flows makes it necessary to take a more general view of the modelling of the flow on the Darcy scale. For the flow regimes we are interested in, on the pore scale we have a linear problem described by the Stokes equations, so that the averaged quantities will also be related linearly. Then, following,[24] in a general case, one can state[25] that for $n = 1, 2, 3$ the pressure gradient and velocity are related by

$$\nabla_n p = \mu \left(-b_n^i u_i + b_n^{ij} \nabla_i u_j + b_n^{ijk} \nabla_i \nabla_j u_k + \ldots \right),$$ (9)

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where the tensor coefficients b_n^i, b_n^{ij}, b_n^{ijk}, etc, reflect (possible) anisotropy of the porous matrix. The form (9) has Darcy's law for an isotropic matrix, as its simplest case with $b_n^i = \delta_n^i/\chi$, where δ_n^i is Kronecker's delta-symbol, and all other coefficients are zero. The expansion (9) also includes Brinkman's model,[26] which corresponds to all b's being zero except $b_n^i = \delta_n^i/\chi$ and $b_n^{ijk} = \delta_n^k g^{ij}/\chi$, where g^{ij} are contravariant components of the metric tensor — then, for an isotropic matrix, a term $\chi\nabla^2\mathbf{u}$ is added to the right-hand side of (2), often with an additional empirical factor known as 'Brinkman viscosity'. Mathematically, the use of Brinkman's generalization would be equivalent to an 'upgrade' of the Euler to the Navier-Stokes equations in fluid mechanics. However, with this and other generalizations based on (9) and introducing higher derivatives of $\mathbf{u}$, we no longer have Darcy's law for the unidirectional flow: higher-order derivatives will require additional boundary conditions on the solid walls and the velocity profile in a pipe filled by a porous material will no longer be that of a plug flow. However, for a unidirectional flow Darcy's law with its plug-flow velocity profile is a well-tested experimental fact, and we would like our model to be able to describe it. Thus, we are back with the same problem of having to preserve (2) for unidirectional flows and to remove the singularity mentioned earlier. The conventional generalizations of Darcy's model summarized in (9) do not address the problem.

Also, unlike fluid mechanics of viscous fluids, where singularities at corners, such as contact lines or free-surface cusps, can be regularized by incorporating 'extra' physics into the boundary conditions,[27] the boundary condition required on an impermeable wall by the Darcy model is only the condition of impermeability, and it cannot accommodate any 'extra' physics to remove the singularity.

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Outline of an alternative approach

An avenue that at this stage is open for exploration is to consider the dependence of permeability χ of the porous matrix on the flow that passes through this matrix. Actually, to a fresh eye, this idea should look much more natural than the opposite statement. Indeed, why should a porous matrix resist the flow in the same way regardless of the character of this flow? We know that this is not the case where the porous matrix is, for example, a regular lattice as its properties are then anisotropic but the notion of anisotropy in this context deals just with the flow direction which is only the most basic of the flow characteristics. If the matrix is isotropic, it is other characteristics that have to come into play.

As a local scalar characteristic χ can depend on the local scalar invariants of the velocity field. This idea immediately gives that, in the general case, at every point the permeability can depend only on the curvature κ and torsion τ of the streamline at this point, $\chi = \chi(\kappa, \tau)$. The dependence of χ on the modulus of velocity $|\mathbf{u}|$ is ruled out as the underlying Stokes flow on the pore scale implies a linear dependence on velocity (9). The general form $\chi = \chi(\kappa, \tau)$ can be further simplified given that the velocity singularity appears in a plane-parallel flow where $\tau = 0$ for all streamlines, i.e. τ is irrelevant. As a result, we have $\chi = \chi(\kappa)$, or, in a more convenient form

$$\chi = \chi_0 \, \overline{\chi}(\overline{\kappa}), \qquad \overline{\kappa} = \ell \kappa, \tag{10}$$

where ℓ is a coefficient with the dimension of length characterizing the dependence of permeability on the curvature of streamlines for a given porous medium and κ is the

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streamline's curvature which can be expressed in terms of the flow velocity $\mathbf{u}$ as

$$\kappa = |\mathbf{u}|^{-3}|\mathbf{u} \times (\mathbf{u} \cdot \nabla \mathbf{u})|. \tag{11}$$

To close the system of equations in the bulk (2), (3), (10), (11), we need only to specify the function $\overline{\chi}(\overline{\kappa})$. Combining equations (2), (3), (10), (11), we can also write them down in an equivalent and more convenient form as

$$\nabla^2 p + \nabla \ln \overline{\chi} \cdot \nabla p = 0, \tag{12}$$

$$\kappa = |\nabla p|^{-3}|\nabla p \times (\nabla p \cdot \nabla \nabla p)|, \tag{13}$$

leaving us still with the need to specify $\overline{\chi}$ as a function of κ. What is required now is an indication pointing the direction in which $\overline{\chi}$ will vary from $\overline{\chi} = 1$ for unidirectional flow ($\kappa = 0$) as κ increases.

Thus, to describe 2D/3D flows in porous media without unphysical singularities in the velocity field, we need a new class of models which contains Darcy's classical model (2)–(3) as the simplest particular case. The porous matrix has to be characterized not just by a constant χ_0 as its permeability; one has to specify a permeability *function* $\chi = \chi(\kappa)$, and below we will examine the properties this function must possess and whether or not the problems that arise from this class of models are solvable.

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Physical mechanism: A suggestive analogy

In order to qualitatively test our conjecture that it is the dependence of the porous medium's permeability on the flow geometry that regularizes the singularity and obtain some information about the function $\overline{\chi}(\overline{\kappa})$, we will consider a convenient simple model. Darcy's law (2) can be thought of as if the porous matrix acts effectively as a set of parallel capillaries or channels directed along ∇p. Then, taking a Poiseuille flow in a channel of width a and averaging the flow's velocity across it, we obtain

$$\langle \mathbf{u} \rangle = -K_{\parallel} \langle \nabla p \rangle, \qquad K_{\parallel} = \frac{a^2}{12\mu}, \tag{14}$$

where $\langle f \rangle = (1/a) \int_{-a/2}^{a/2} f \, dy$ and $\langle \nabla p \rangle = \nabla p$ as for the Poiseuille flow in a straight channel the pressure is uniform across the channel. The hydraulic conductivity K of a porous matrix can then be represented as the product of $K_{\parallel}$ and the number of effective channels per unit length. (Alternatively, one can calculate $K_{\parallel}$ for a representative 'effective capillary' and then multiply it by the number of effective capillaries per unit area.)

[Figure 2 about here.]

Now, considering a *bent* channel of width a with the radius of curvature R of its centerline (Fig. 2). Then, after simple calculations we arrive at

$$\langle \mathbf{u} \rangle = -K_{((} \langle \nabla p \rangle, \tag{15}$$

$$K_{((} = \frac{a^2}{4\mu\delta} \left[\frac{1}{G} - \frac{G}{\delta^2} \left(1 - \frac{\delta^2}{4} \right)^2 \right], \quad G = \ln \left(\frac{1 + \delta/2}{1 - \delta/2} \right), \tag{16}$$

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where $\delta = a/R$ and $\langle f \rangle = (1/a) \int_{R-a/2}^{R+a/2} f\, dr$. The subscripts $\|$ and $((\ $ in (14) and (15)–(16) refer to the straight and curved channel, respectively. One can easily see that $K_{((} < K_{\|}$ and $K_{((} \to K_{\|}$ as $\delta \to 0$, whilst in the opposite limit $\delta \to 2$ corresponding to the maximum possible curvature of the centerline (the radius of curvature of the inner wall tends to zero), one has $K_{((} \to 0$. For a porous medium, this indicates that, as one would expect, $\overline{\chi} \to 1$ as $\overline{\kappa} \to 0$ and $\overline{\chi} \to 0$ as $\overline{\kappa} \to \infty$, as on the Darcy scale the radius of curvature comparable with the pore scale (or even the scale of averaging) is zero in the (secondary) continuum limit, i.e. the limit that produces the continuum model on the Darcy scale. Thus, the simple channel-based analogy, where the variation of δ from 0 to 2 mimics the variation of the streamline curvature in the Darcy-scale description from 0 to ∞, qualitatively confirms our conjecture that the vanishing of permeability as the curvature of the streamlines increases could indeed be the mechanism regularizing the flow field which otherwise is unphysically singular.

It should be noted that the above analogy should not be taken literally as even a unidirectional flow through a porous medium involves, on the pore scale, flows through a multitude of curved pores (but the Darcy-scale streamlines are, of course, straight), so that the curvature effect on the pore-scale hydraulic conductivity is already accounted for in χ_0. The essence of the argument is that, once on the Darcy scale we have curved streamlines, the 'weight' of the contributions from the pores with lower hydraulic conductivity increases, resulting in the reduction of χ, which is the average measure of it.

The leading-order deviation of $\overline{\chi}(\overline{\kappa})$ from 1 for small $\overline{\kappa}$ can be inferred from (15)–(16). Indeed, to leading order as $\delta \to 0$ the number of 'effective channels' remains the same as for $\delta = 0$ and, expanding $K_{((}$ in a Taylor series as $\delta \to 0$, we arrive at a correction to $K_{\|}$

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quadratic in δ. This gives that for $\overline{\chi}(\overline{\kappa})$ we should have

$$\overline{\chi} = 1 - C\overline{\kappa}^2 + \ldots, \qquad \text{as } \overline{\kappa} \to 0, \tag{17}$$

where $C > 0$ is a dimensionless coefficient specific to a particular porous matrix. The same approach for the limit $\overline{\kappa} \to \infty$ does not work as the number of 'effective channels' will obviously vary with curvature in a way that we do not know, and nor do we know their number at $\overline{\kappa} = \infty$. Our consideration of the porous medium in terms of effective channels indicates only that $\overline{\chi}(\overline{\kappa}) \to 0$ as $\overline{\kappa} \to \infty$.

Regularization of the flow field and the problem's solvability

Consider the vicinity of a corner where, to be physically meaningful, the solution must be regular. We will use the same polar coordinate system as before and, to non-dimensionalize the problem, take ℓ as the characteristic length and assume that the characteristic scales U and P for velocity and pressure satisfy $\chi_0 P/(\mu U \ell) = 1$. In the dimensionless form, equations (12), (13) will stay exactly the same and in what follows we will, as before, use overbars to mark dimensionless quantities. As before, we will be considering a local solution as $\overline{r} \to 0$ in the separable form $\overline{p} = \overline{r}^q f(\theta)$, $q \geq 0$. Then, expression (13) for the curvature takes the form

$$\overline{\kappa} = \frac{\left| f'\left(qff'' + (1-q)f'^2 + q^2 f^2\right) \right|}{\overline{r}\left(q^2 f^2 + f'^2\right)^{3/2}}, \tag{18}$$

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where primes denote differentiation with respect to θ. From this expression and the separable form of the solution for the pressure it is clear that (a) regularization of the velocity field, which in the dimensionless form is given by $\overline{\mathbf{u}} = -\overline{\chi}(\overline{\kappa})\overline{\nabla}\overline{p}$, and (b) the relative magnitude of the two terms in (12) as $\overline{r} \to 0$ are both determined by the asymptotic behaviour of $\overline{\chi}(\overline{\kappa})$ as $\overline{\kappa} \to \infty$. We can easily see that, if $\overline{\chi} \propto \ln \overline{\kappa}$ as $\overline{\kappa} \to \infty$, the second term in (12) is negligible compared to the first one as $\overline{r} \to 0$ so that, to leading order, the pressure will be again given by (6) and the factor $\overline{\chi}(\overline{\kappa})$ is insufficient to suppress the singularity (7)–(8). An algebraic decay of $\overline{\chi}(\overline{\kappa})$ as $\overline{\kappa} \to \infty$, such that $\overline{\chi} \propto \overline{\kappa}^{-s}$ can regularize the velocity filed, with the weakest estimate $s \geq 1$, and, since in this case both terms in (12) are of the same order, one has to use/know the full function $\overline{\chi}(\overline{\kappa})$ for $0 \leq \overline{\kappa} < \infty$. Finally, if $\overline{\chi}$ decays faster than algebraically as $\overline{\kappa} \to \infty$, e.g.

$$\overline{\chi} = \exp(-C\overline{\kappa}^2), \tag{19}$$

which satisfies both (17) and $\overline{\chi} \to 0$ as $\overline{\kappa} \to \infty$, then we need to know only this fact as in this case the two terms in (12) are not comparable as $\overline{r} \to 0$, and we neglect the first compared to the second one.

The form of the function $\overline{\chi}(\overline{\kappa})$ is to be determined either experimentally or conjectured phenomenologically with the subsequent experimental verification of the model's predictions. One would expect that an algebraic dependence of $\overline{\chi}$ on $\overline{\kappa}$, where

$$\overline{\chi} = \frac{1}{1 + C\overline{\kappa}^2}$$

is the simplest suitable choice, is most likely, but at this stage the form of the function

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$\overline{\chi}(\overline{\kappa})$ is an open question.

An issue that one needs to address before embarking on a search for $\overline{\chi}(\overline{\kappa})$ is whether the very non-linear problems involving such a function are solvable. In an attempt to shed some light on this issue, we will show that even in the most extreme limiting case, namely where $\overline{\chi}$ vanishes exponentially as $\overline{\kappa} \to \infty$, the solution exists and hence that the flow field can be regularized, leaving the less extreme, more realistic but also more cumbersome and computationally demanding case of an algebraically decaying $\overline{\chi}$, which requires the knowledge of/assumptions about the full function $\overline{\chi}(\overline{\kappa})$, for a future study.

With the exponentially decaying $\overline{\chi}(\overline{\kappa})$, to leading order as $\overline{r} \to 0$, we have

$$\overline{\nabla}\overline{\kappa} \cdot \overline{\nabla}\overline{p} = 0, \qquad (\overline{r} > 0, \ -\alpha < \theta < \alpha), \tag{20}$$

where we need to use (18), $\overline{p} = \overline{r}^q f(\theta)$ and the boundary conditions of impermeability on the sides of the corner, $\partial\overline{p}/\partial\theta = 0$ on $\theta = \pm\alpha$. Explicitly, this gives

$$\left(q^2 f^2 + f'^2\right) f f' f''' - 2 f f'^2 f''^2 + q^2 f^3 f''^2 - 5 q^2 f^2 f'^2 f''$$

$$+ f'^4 f'' + (3q^2 - 1) f f'^4 - 2q^2 f^3 f'^2 - q^4 f^5 = 0 \tag{21}$$

subject to conditions $f'(\pm\alpha) = 0$. Equation (21) is not as unassailable as it looks, first, as it is homogeneous so that the substitution $f'(\theta) = q f(\theta) y(\theta)$ reduces it to

$$y(y^2 + 1)y'' - (2y^2 - 1)y'^2 - (y^2 + 1)^2 = 0, \tag{22}$$

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subject to boundary conditions $y(\pm\alpha) = 0$. Note that, unlike the problem (4)–(5), q is not an eigenvalue and its value is provided by the global solution. Now, the substitution $y'^2 = F(y)$ reduces (22) to a simple linear equation

$$F' - \frac{2(2y^2 - 1)}{y(y^2 + 1)} F = \frac{2(y^2 + 1)}{y},$$

where the prime denotes differentiation with respect to y. After integrating this equation, going back through the substitutions and integrating the resulting equations subject to the corresponding boundary conditions, we arrive at

$$\bar{p} = \begin{cases} \pm A\bar{r}^q \exp\left(q \int \dfrac{\sqrt{\sin^2\alpha - \sin^2\theta}}{\sin\theta} \, d\theta \right), & \text{for } |\sin\theta| \leq |\sin\alpha| \\[2ex] \pm A\bar{r}^q \exp\left(-q \int \dfrac{\sqrt{\cos^2\alpha - \cos^2\theta}}{\cos\theta} \, d\theta \right), & \text{for } |\sin\theta| > |\sin\alpha| \end{cases} \tag{23}$$

where we have $+$ for θ in the first and third quadrant and $-$ otherwise or vice versa. In particular, for $q = 1$ as $\alpha \to \pi/2$ this solution coincides with (6) corresponding to a uniform flow past a planar wall. Notably, the deviation of (6) and (23) from this plug-flow solution as α deviates from $\pi/2$ are very different. For all corners, the flow velocity generated by (23) is regularized as the pressure gradient has to be multiplied by the permeability which, like, for example, (19), ensures that, for any q, the velocity remains finite. As already stated, it was clear from the start that, should a solution in the separable form exist, it would have no singularity in the velocity field, so that here we have just proven its existence by simply finding it.

It should be emphasized that the solution (23) is just an example corresponding to the

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extreme (i.e. faster than algebraic) decay of permeability as curvature of the streamlines tends to infinity. This example is considered here as it (a) shows that the new class of problems we introduce not only yields solutions but even allows, albeit in extreme cases, finding them analytically, and (b) does not require knowing the function $\overline{\chi}(\overline{\kappa})$ which is yet to be obtained experimentally or via modelling. It is only after this function is determined (or conjectured) that the full flow near the corner, as opposed to just its asymptotics as $r \to 0$ in an extreme case of faster-than-algebraic decay of χ, could be considered/computed.

Conclusions

The analysis triggered by the necessity to remove an unphysical singularity arising in the classical Darcy model with a constant permeability leads to the following conclusions:

(i) The lower permeability of a porous matrix with regard to two-dimensional flows compared to unidirectional ones suggests that the standard Darcy model with a constant permeability *overpredicts* the flow rate when applied to two-dimensional flows. The case of a flow past a corner only highlights this general feature.

(ii) In order to describe essentially two-dimensional flows in porous media, we have to introduce a fundamentally *new class of mathematical problems* where the permeability of the porous matrix is not a prescribed scalar or tensor field; it depends on the flow, more specifically, on the curvature of the flow's streamlines, and hence has to be found simultaneously with it. In more general terms, this is a particular

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case in a class of problems where the transport coefficients are dependent on local invariants of the solution. Although, as shown above, in some special cases one can even find fully analytical solutions to such problems, an analysis of a general case will require the development of dedicated numerical algorithms which, given the unusual nature of such problems, is a challenging and interesting task.

(iii) A quantitative experimental study into the permeability of porous media in essentially two-dimensional flows is called for. Such a study would make it possible to quantify the conditions where the flows can be described in terms of the standard Darcy model with an acceptable accuracy and where a more complex model from the class identified here is required.

(iv) On the theoretical side, the remaining question is that of finding the weakest condition on the decay of $\overline{\chi}(\overline{\kappa})$ as $\overline{\kappa} \to \infty$ that ensures the regularity of the velocity field for essentially two-dimensional flows, like flows round a corner, and the determination of $\chi(\kappa)$ theoretically.

Discussion

With regard to the derivation of the model, it is important to note the following two moments. The first one concerns the reasons behind the direction in which the classical Darcy model has to be generalized. The well-verified applicability of the classical Darcy law and hence the model (2)–(3) with $\chi = \chi_0$ to the description of flows through isotropic homogeneous porous media filling a straight tube or channel means that, for equations

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operating on the Darcy scale, one has the impermeability boundary condition on the solid boundaries confining the flow domain. Then, if we change the shape of these boundaries, e.g. replace one plane wall with two plane walls forming an angle, we will still have the same impermeability boundary condition on the new boundary and not, say, no-slip, as the type of boundary condition should not change with the shape of the domain where the bulk equations operate. Thus, from the basics of mathematical physics and the requirement to have (2)–(3) with $\chi = \chi_0$ intact in the situations where this model works, it follows that one has to look for the generalization of (2) that has χ dependent on the solution, i.e. on the flow, and, to remove the clearly unphysical singularities arising in essentially two-dimensional flows, χ must decrease and sufficiently rapidly as the curvature of the streamlines in the Darcy-scale description of the flow goes up. In other words, the validity of Darcy's classical law in domains confined by parallel walls together with the fact that the physics incorporated into the boundary conditions (in our case, impermeability) doesn't change with the shape of the boundary dictate the direction of generalization of (2)–(3) with $\chi = \chi_0$.

The second aspect in developing the model is to consider the physical mechanism(s) behind the decrease of permeability in response to the more and more 'curved' flow. This mechanism then has to be quantified and incorporated into the appropriate dependence of χ on the curvature κ of the Darcy-scale streamlines. The resulting function $\chi(\kappa)$ will, of course, depend on the porous matrix, the topology and geometry of pores and other characteristics. For the time being, this second aspect in developing the above class of models remains an open question. All what the present study has done in this regard is that it showed that the flow in an 'effective' curved channel, as a representative semi-

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quantitative model of the corresponding flow in a porous medium, experiences greater resistance compared with the flow in a straight channel of the same width, the analogy often used in representing Darcy's classical law. This suggests that, qualitatively, the decrease of χ with the streamline curvature increase is a trend which at least follows from the effective representation of the flow. Further exploration of the curvature-dependence of χ could go in parallel via both experimentation and theoretical consideration of some model porous matrices.

Darcy's law versus other linear laws

Mathematically, the singularity addressed in the present work also appears in other models with similar to Darcy's law linear constitutive equations, like Fourier's law for heat conduction[28] and Fick's law for diffusion.[29] In all these cases, the singularity is, obviously, also unphysical but there are differences of principle in how the issue is to be interpreted and resolved which come from analyzing the scales on which these models can be applied. In Fourier's and Fick's laws, the averaged quantities follow from the primary continuum limit, i.e. the separation of scales between the molecular scale l_{mol} and the macroscopic scale L_{macro} on which the continuum description in terms of averaged quantities is applied, making possible the averaging. Therefore, as one zooms in to the corner, the sharp edge turns into a rounded surface, with the radius of curvature R_{tip} still much larger than l_{mol}, while both Fourier's and Fick's law still remain applicable as one still has $L_{\mathrm{macro}} \ll R_{\mathrm{tip}}$. Then, the singularity in the heat or diffusion flux near the corner appears simply as a price for the simplification of the geometry of the domain in which the process is considered: this simplification is convenient to describe the far-field (compared with R_{tip}) distribution

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of temperature or concentration whilst, if the near-field picture is required, the corner and hence the singularity disappear as one zooms in whilst both the Fourier law and Fick's law remain applicable ($L_{\mathrm{macro}} \ll R_{\mathrm{tip}}$).

A genuine corner in the single-phase systems, i.e. a corner staying as long as the continuum description in the primary continuum limit remains valid, takes place where we have either a 'contact line',[27,30] where the 'contact angle' is invariably less than π, or a singularity of the free-surface curvature caused by a convergent flow, often referred to as a 'free-surface cusp'[31,32] whereas in reality it is a corner.[33] In the latter case, one has in essence a 'contact line' formed by the free surface and a plane of symmetry,[27] so that there is no flow over an angle greater than π and hence no unphysical singularities to be removed.

Once we turn to Darcy's law, it becomes clear that the zooming-in argument that works in the case of the Fourier law and Fick's law wouldn't deliver: corners rounded on a scale comparable with the scale of averaging or even smaller than the pore size routinely occur in applications. For example, a knife put into a loaf of bread or in the soil will have the curvature of its edge far smaller even than the pore size. Then, given that the averaged quantities featuring in (2)–(3) make sense for the flow considered on a length scale much larger than the pore scale, i.e. in the asymptotic limit $a/L \to 0$, the boundary rounded on the pore scale will appear as having a genuine corner for the Darcy-scale description. If one zooms in on the corner, then, as the corner is approached, the averaged quantities will become meaningless first or simultaneously with the corner becoming rounded. In other words, if we stay on the (Darcy) length scale where the continuum description in terms or averaged quantities applies, we will have a corner in the boundary of the flow

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domain, and hence the model we use must give realistic predictions for the flow field for the whole flow domain described on this scale.

On the mathematical side, there is a curious parallel between the velocity singularity (7)–(8) arising from the harmonic equation for the pressure (4) and a pressure singularity arising as a numerical artefact but also as an eigensolution in the Stokes flow where one has a *bi*-harmonic equation for the streamfunction.[34,35] The artefact resists all conventional ways of regularization (and hence has been informally termed as a persistent 'eigen-mare') and appears only after the angle of the corner exceeds a certain value, similarly to how for the standard Darcy model the singularity in the velocity field emerges when the corner angle exceeds π.

Experimental verification

The presented study opens up a promising and highly challenging line of inquiry. The velocity singularity in flows past corners indicates that in such situations the standard Darcy law *overpredicts* the mass flux, and this suggests measuring this flux in channels confined, on one side, by a corner, like in Fig. 1c, and, on the other, by a wall having, say, the shape of a streamline of the solution (7)–(8). Then, for such a channel one would have an exact theoretical solution in the framework of the standard Darcy model (6)–(8), whilst experiments would give one the flux through this channel Q as a function of two parameters: the angle 2α of the corner and the channel's width W defined, for example, as the distance from the corner edge to the opposite wall. These measurements could then be compared with the integral flux corresponding to (7)–(8) to find out where in the parameter space, i.e. for what α and W, the deviation of the flux predicted by the

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standard Darcy model from the measured one becomes measurable. Such an experiment alone would determine the limits of applicability of the standard Darcy model from the viewpoint that has not been considered before, namely not by imposing high pressure gradients until the flow on the pore scale is no longer the Stokes flow but solely by varying the flow geometry.

A special issue in the suggested experimental programme is the reproducibility of the porous matrix. Here, the most promising route seems to be to use densely packed spherical beads, as, for example, in.[36] Besides reproducibility, an additional advantage of this class of porous matrices is that each of them is characterized by a single length scale, and this will allow one to determine whether, to use the analogy with the flow in 'effective' channels or tubes once again, it is the increased resistance of these channels/tubes or the reduction in their number which is responsible for the regularization of the flow field.

After the region of the parameter space where the standard Darcy model is measurably inaccurate is established, one will face the problem of interpreting the results in terms of the flow-dependence of χ. An inverse problem of determining $\chi(\kappa)$ from the measurements of $Q(\alpha, W)$ seems to be extremely complex, even in term of how to formulate it mathematically. Therefore, a direct way of, first, conjecturing $\chi(\kappa)$ (or deducing it via some modelling) and then solving the problem of the flow in a curved channel, with the results to be compared with the experimentally measured $Q(\alpha, W)$, looks more promising. although the theoretical/computational problem of a simultaneous finding of the flow *and* the permeability of the matrix, with the dependence $\chi(\kappa)$ prescribed, is more straightforward than the inverse problem of finding $\chi(\kappa)$ from the known $Q(\alpha, W)$, it is still a very challenging task. This class of problems, where in a continuum model the transport

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coefficients depend on the vector characteristics of the solution, has never been studied before and will require the development of essentially new numerical methods.

It should be emphasized that the analysis presented in this work is phenomenological and hence, strictly speaking, it is impossible to come up with an a-priori estimate for the values of the material constants involve and their dependence on the pore-scale and even structural properties of the system, in the same way as one would not be able to produce an estimate of any validity, say, for the viscosity coefficient in a phenomenologically derived model of a multicomponent fluid, even with the full knowledge of the molecular properties of its ingredients. However, with this caveat in place, it looks plausible to suggest that in (17), which in a dimensional form is given by

$$\chi = \chi_0(1 - C\ell^2\kappa^2 + \ldots),$$

for a porous matrix characterized by a single length scale a, like the matrix made up of identical spheres, one would have, besides $\chi_0 \propto a^2$, also $\ell = a$ and the dimensionless constant C depending only on the type/structure of the matrix. This conjecture, however, requires experimental testing.

Acknowledgment

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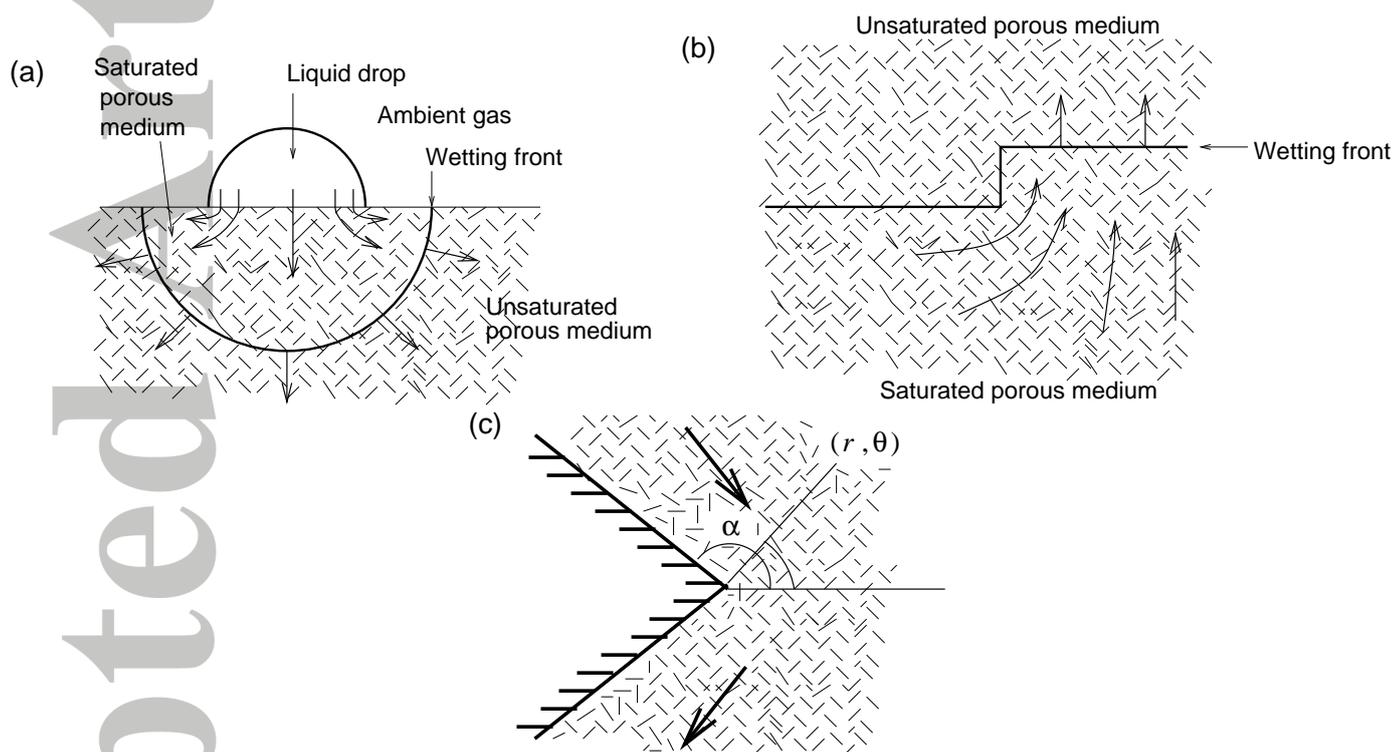
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List of Figures

Figure 1: Schematic illustration of some examples of flows over corners in porous media where, in the theoretical description in the framework of the standard Darcy model, a singularity in the velocity field takes place. (a) Imbibition of a liquid drop into an unsaturated porous substrate; (b) Fragmentation of a wetting front as a section of it is brought to a halt; (c) Flow over a solid wedge — the definition sketch for the problem (4)–(5).
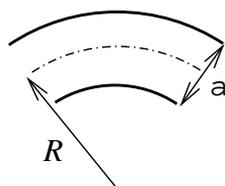
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Figure 2: An illustrative sketch for the bent channel.

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