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HAMILTON DECOMPOSITIONS OF REGULAR EXPANDERS: APPLICATIONS

DANIELA KÜHN AND DERYK OSTHUS

Abstract. In a recent paper, we showed that every sufficiently large regular digraph G on n vertices whose degree is linear in n and which is a robust outexpander has a decomposition into edge-disjoint Hamilton cycles. The main consequence of this theorem is that every regular tournament on n vertices can be decomposed into $(n-1)/2$ edge-disjoint Hamilton cycles, whenever n is sufficiently large. This verified a conjecture of Kelly from 1968. In this paper, we derive a number of further consequences of our result on robust outexpanders, the main ones are the following:

(i) an undirected analogue of our result on robust outexpanders;
(ii) best possible bounds on the size of an optimal packing of edge-disjoint Hamilton cycles in a graph of minimum degree δ for a large range of values for δ.
(iii) a similar result for digraphs of given minimum semidegree;
(iv) an approximate version of a conjecture of Nash-Williams on Hamilton decompositions of dense regular graphs;
(v) a verification of the 'very dense' case of a conjecture of Frieze and Krivelevich on packing edge-disjoint Hamilton cycles in random graphs;
(vi) a proof of a conjecture of Erdős on the size of an optimal packing of edge-disjoint Hamilton cycles in a random tournament.

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1. Introduction

1.1. **Robust expanders.** A graph or digraph G has a Hamilton decomposition if it contains a set of edge-disjoint Hamilton cycles which together cover all the edges of G. Despite the fact that the study of Hamilton decompositions has a long history, general results on Hamilton decompositions have been rare so far. The first result in this direction is Walecki's construction of a Hamilton decomposition for the complete graph on an odd number of vertices. Tillson [50] proved an analogue of this for complete digraphs. In 1968, Kelly (see e.g. [10, 38, 42]) conjectured an analogue of this for tournaments, namely that every regular tournament has a Hamilton decomposition. Note that in a digraph we allow up to two edges between any pair of vertices – at most one in each direction. So a digraph might contain cycles of length two, whereas in an oriented graph, we only allow at most one edge between any pair of vertices. A tournament T is an orientation of a

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complete (undirected) graph. T is regular if the outdegree of every vertex equals its indegree.

Very recently, we were able to prove Kelly's conjecture for all large tournaments [39] (that paper also includes a more thorough discussion of partial results towards this conjecture). Our proof led us to a far more general decomposition theorem, which involves the concept of robust outexpanders. This concept was introduced by Kühn, Osthus and Treglown [40], and was already used implicitly in [26, 27]. Roughly speaking, a digraph is a robust outexpander if for every set S which is not too small and not too large, its 'robust' outneighbourhood is at least a little larger than S. In [39] we proved that every dense regular robust outexpander has a Hamilton decomposition. Since every regular tournament is easily seen to be a robust outexpander, our result on Kelly's conjecture is a very special case of this.

More precisely, let $0 < \nu \leq \tau < 1$. Given any digraph G on n vertices and $S \subseteq V(G)$, the *ν-robust outneighbourhood* $RN^+_{\nu,G}(S)$ *of* S is the set of all those vertices x of G which have at least νn inneighbours in S. G is called a *robust (ν,τ)-outexpander* if

$$|RN^+_{\nu,G}(S)| \geq |S| + \nu n \quad \text{for all} \quad S \subseteq V(G) \quad \text{with} \quad \tau n \leq |S| \leq (1-\tau)n.$$

We can now state the main result of [39], which guarantees a Hamilton decomposition in every regular robust outexpander of linear degree.

Theorem 1.1. *For every $\alpha > 0$ there exists $\tau > 0$ such that for every $\nu > 0$ there exists $n_0 = n_0(\alpha,\nu,\tau)$ for which the following holds. Suppose that*

(i) *G is an r-regular digraph on $n \geq n_0$ vertices, where $r \geq \alpha n$;*
(ii) *G is a robust (ν,τ)-outexpander.*

Then G has a Hamilton decomposition. Moreover, this decomposition can be found in time polynomial in n.

The proof of Theorem 1.1 introduces a new method for decomposing a graph into (Hamilton) cycles, which we believe will have further applications. As a tool, it uses (amongst others) a result of Osthus and Staden [46] which states that any G as in Theorem 1.1 has an approximate Hamilton decomposition – i.e. a set of edge-disjoint Hamilton cycles covering almost all edges of G ([46] generalizes a corresponding result in [41] on dense regular oriented graphs). Given such an approximate decomposition of G, it is obviously not always possible to extend this into a Hamilton decomposition. The key idea in the proof of Theorem 1.1 is to overcome this obstacle by finding a sparse 'robustly decomposable' spanning subdigraph H^* of G, i.e. H^* has a Hamilton decomposition even if we add the edges of any very sparse spanning regular subdigraph H of $G \setminus E(H^*)$ to H^*. Now one can apply the result of [46] to obtain an approximate decomposition of $G \setminus E(H^*)$, which leaves a very sparse uncovered leftover H. The choice of H^* implies that $H^* \cup H$ has a Hamilton decomposition. Altogether, this gives the required Hamilton decomposition of G.

Some applications of Theorem 1.1 were already derived and described in [39] (namely Theorems 1.7 and 1.8 below). As also discussed in [39], Theorem 1.7 in

turn has a surprising application to finding tours for the Asymmetric Travelling Salesman Problem with large 'domination ratio', which (together with a result by Gutin and Yeo [20]) solved a problem raised e.g. by Glover and Punnen [19] as well as Alon, Gutin and Krivelevich [2]. Since robust expansion is a property shared by several important classes of graphs, it turns out that Theorem 1.1 has numerous further applications, which we discuss and derive in this paper.

For applications to undirected graphs, it is obviously helpful to have an undirected version of Theorem 1.1. For this, we introduce the analogue of robust outexpanders for undirected graphs. Let $0 < \nu \leq \tau < 1$. Given any (undirected) graph G on n vertices and $S \subseteq V(G)$, the *ν-robust neighbourhood $RN_{\nu,G}(S)$ of S* is the set of all those vertices x of G which have at least νn neighbours in S. G is called a *robust (ν,τ)-expander* if

$$|RN_{\nu,G}(S)| \geq |S| + \nu n \quad \text{for all} \quad S \subseteq V(G) \quad \text{with} \quad \tau n \leq |S| \leq (1-\tau)n.$$

With this notion, we can derive the following analogue for (undirected) robust expanders from Theorem 1.1.

Theorem 1.2. *For every $\alpha > 0$ there exists $\tau > 0$ such that for every $\nu > 0$ there exists $n_0 = n_0(\alpha,\nu,\tau)$ for which the following holds. Suppose that*

(i) *G is an r-regular graph on $n \geq n_0$ vertices, where $r \geq \alpha n$ is even;*
(ii) *G is a robust (ν,τ)-expander.*

Then G has a Hamilton decomposition. Moreover, this decomposition can be found in time polynomial in n.

One can easily derive a version of the above result for odd values of r: suppose that G satisfies all conditions of Theorem 1.2 except that r is odd. Then we can obtain a decomposition of G into one perfect matching and a set of Hamilton cycles. Indeed, the conditions ensure that G contains a perfect matching M (e.g. via Tutte's theorem). Removing M from G leaves a graph $G - M$ which satisfies the conditions of Theorem 1.2 (with slightly worse parameters), so $G - M$ has a Hamilton decomposition.

1.2. **Edge-disjoint Hamilton cycles in dense graphs.** Nash-Williams [44] conjectured that every sufficiently dense even-regular graph has a Hamilton decomposition. More precisely, he conjectured that such a decomposition exists in every $2d$-regular graph with at most $4d+1$ vertices. This is sometimes referred to as the 'Hamilton decomposition conjecture'. The following result implies an approximate version of this conjecture.

Theorem 1.3. *For every $\varepsilon > 0$ there exists n_0 such that every r-regular graph G on $n \geq n_0$ vertices, where $r \geq (1/2+\varepsilon)n$ is even, has a Hamilton decomposition.*

Theorem 1.3 is an (almost) immediate consequence of Theorem 1.2. After the current paper was completed, the conjecture of Nash-Williams was verified for all large n in a sequence of papers by Csaba, Kühn, Lo, Osthus and Treglown [13, 14, 35, 36] (these papers also contain a proof of the related '1-factorization conjecture'). The proof is far more difficult than that of Theorem 1.3, but once again

Theorem 1.2 plays a crucial role. Previous partial results related to this problem were proved e.g. Jackson [23], Perkovic and Reed [47], as well as Christofides, Kühn and Osthus [12].

The paper [12] also asymptotically determined the number of edge-disjoint Hamilton cycles which one can guarantee in a graph with given minimum degree δ. (The case $\delta = n/2$ of this question was originally raised by Nash-Williams [43, 44].) More precisely, given a graph G, let $\mathrm{ham}(G)$ denote the maximum number of edge-disjoint Hamilton cycles in G and let $\mathrm{reg}_{\mathrm{even}}(G)$ denote the largest degree of an even-regular spanning subgraph of G. So $\mathrm{ham}(G) \leq \mathrm{reg}_{\mathrm{even}}(G)/2$. Moreover, given $\delta, n \in \mathbb{N}$ with $\delta < n$, let $\mathrm{ham}(n,\delta)$ denote the minimum of $\mathrm{ham}(G)$ over all graphs on n vertices with minimum degree δ. Similarly, let $\mathrm{reg}_{\mathrm{even}}(n,\delta)$ denote the minimum of $\mathrm{reg}_{\mathrm{even}}(G)$ over all graphs on n vertices with minimum degree δ. Thus

$$\mathrm{ham}(n,\delta) \leq \mathrm{reg}_{\mathrm{even}}(n,\delta)/2 \tag{1}$$

and $\mathrm{ham}(n,\delta) = 0$ whenever $\delta < n/2$. The following result was proved in [12]. It determines $\mathrm{reg}_{\mathrm{even}}(n,\delta)$ almost exactly and gives a lower bound on $\mathrm{ham}(n,\delta)$ which approximately matches the trivial upper bound in (1). The lower bound in (i) is Theorem 12(i) in [12] and the upper bound in (i) was proved in Section 2 of [12] (see also [21]).

Theorem 1.4. *Suppose that $n, \delta \in \mathbb{N}$ and $n/2 < \delta < n$. Let*

$$g(n,\delta) := \frac{\delta + \sqrt{n(2\delta - n)}}{2}.$$

Let $g_{\mathrm{even}}(n,\delta)$ be the largest even number which is at most $g(n,\delta)$ and let $g'_{\mathrm{even}}(n,\delta)$ be the largest even number which is at most $g(n,\delta)+1$. Then

(i) $g_{\mathrm{even}}(n,\delta) \leq \mathrm{reg}_{\mathrm{even}}(n,\delta) \leq g'_{\mathrm{even}}(n,\delta)$.

(ii) *For every $\varepsilon > 0$ there exists n_0 such that whenever $n \geq n_0$ and $(1/2+\varepsilon)n < \delta < n$ then*

$$g_{\mathrm{even}}(n,\delta) - \varepsilon n \leq 2 \cdot \mathrm{ham}(n,\delta) \leq \mathrm{reg}_{\mathrm{even}}(n,\delta) \leq g'_{\mathrm{even}}(n,\delta).$$

The error bound in (ii) was subsequently improved by Hartke and Seacrest [18]. Note that if δ is close to $n/2$, then $g(n,\delta)$ is close to $n/4$. In particular, Theorem 1.4 implies an approximate solution to the problem raised by Nash-Williams. The upper bounds in Theorem 1.4 (as well as those in Theorem 1.6 below) are based on a generalization of a construction of Babai.

Theorem 1.3 together with Theorem 1.4(i) imply the following precise version of Theorem 1.4(ii) for graphs of sufficiently large minimum degree. Surprisingly, it turns out that the trivial bound (1) holds with equality in this case. Note that there are many pairs n, δ for which we have $g_{\mathrm{even}}(n,\delta) = g'_{\mathrm{even}}(n,\delta)$, and thus Theorem 1.5 even gives the exact numerical value for $\mathrm{ham}(n,\delta)$ in these cases.

Theorem 1.5. *For every $\varepsilon > 0$ there exists n_0 such that the following holds for all $n \geq n_0$.*

(i) *Every graph G on n vertices with $\delta(G) \geq (2-\sqrt{2}+\varepsilon)n$ satisfies* $\mathrm{ham}(G) = \mathrm{reg}_{\mathrm{even}}(G)/2$.

(ii) *If* $(2-\sqrt{2}+\varepsilon)n < \delta < n$, *then*

$$g_{\mathrm{even}}(n,\delta) \leq 2\cdot \mathrm{ham}(n,\delta) = \mathrm{reg}_{\mathrm{even}}(n,\delta) \leq g'_{\mathrm{even}}(n,\delta),$$

where $g_{\mathrm{even}}(n,\delta)$ *and* $g'_{\mathrm{even}}(n,\delta)$ *are as defined in Theorem 1.4.*

Note that (i) immediately implies (ii). In [13, 14, 34, 35, 36], Theorem 1.5(ii) is extended to cover the entire range when $\delta \geq n/2$. The proof relies on Theorem 1.5 for the case when δ is much larger than $n/2$ and again relies on Theorem 1.2. The paper by Kühn, Lapinskas and Osthus [34] covers the case when G is far from extremal and the sequence [13, 14, 35, 36] mentioned above covers the 'extremal cases'. Together, these results imply an exact solution to the problem of Nash-Williams mentioned earlier.

A challenging open problem would be to extend the stronger assertion (i) to the entire range $\delta(G) \geq n/2$. An approximate result towards this was recently proved by Ferber, Krivelevich and Sudakov [15].

1.3. **Edge-disjoint Hamilton cycles in dense digraphs.** We can use Theorem 1.1 to prove an analogue of Theorems 1.4 and 1.5 for digraphs. Before we can state it, we need to introduce the following notation. Given a digraph G, we write $\delta^+(G)$ for its minimum outdegree and $\delta^-(G)$ for its minimum indegree. The *minimum semidegree* $\delta^0(G)$ of G is the minimum of $\delta^+(G)$ and $\delta^-(G)$.

Similarly as for undirected graphs, given a digraph G, let $\mathrm{ham}(G)$ denote the maximum number of edge-disjoint Hamilton cycles in G and let $\mathrm{reg}(G)$ denote the largest degree of a regular spanning subdigraph of G. So $\mathrm{ham}(G) \leq \mathrm{reg}(G)$. Moreover, given $\delta, n \in \mathbb{N}$ with $\delta < n$, let $\mathrm{ham}^{\mathrm{dir}}(n,\delta)$ denote the minimum of $\mathrm{ham}(G)$ over all digraphs on n vertices with minimum semidegree δ. Similarly, let $\mathrm{reg}^{\mathrm{dir}}(n,\delta)$ denote the minimum of $\mathrm{reg}(G)$ over all digraphs on n vertices with minimum semidegree δ. Thus $\mathrm{ham}^{\mathrm{dir}}(n,\delta) \leq \mathrm{reg}^{\mathrm{dir}}(n,\delta)$ and $\mathrm{ham}^{\mathrm{dir}}(n,\delta) = 0$ whenever $\delta < n/2$.

Theorem 1.6. *Given* $\delta, n \in \mathbb{N}$ *with* $n/2 \leq \delta < n$, *let*

$$f(n,\delta) := \left\lfloor \frac{\delta + \sqrt{n(2\delta - n) + \mathbf{1}_{n\not\equiv\delta}}}{2} \right\rfloor, \quad \textit{where} \quad \mathbf{1}_{n\not\equiv\delta} := \begin{cases} 0 & \textit{if } n \equiv \delta \mod 2 \\ 1 & \textit{if } n \not\equiv \delta \mod 2. \end{cases}$$

For every $\varepsilon > 0$ *there exists* n_0 *such that the following statements hold for all* $n \geq n_0$.

(i) *If* $(1/2+\varepsilon)n < \delta < n$ *then*

$$f(n,\delta) - \varepsilon n \leq \mathrm{ham}^{\mathrm{dir}}(n,\delta) \leq \mathrm{reg}^{\mathrm{dir}}(n,\delta) = f(n,\delta).$$

(ii) *Every digraph* G *on* n *vertices with* $\delta^0(G) \geq (2-\sqrt{2}+\varepsilon)n$ *satisfies* $\mathrm{ham}(G) = \mathrm{reg}(G)$. *In particular, if* $(2-\sqrt{2}+\varepsilon)n < \delta < n$ *then*

$$\mathrm{ham}^{\mathrm{dir}}(n,\delta) = \mathrm{reg}^{\mathrm{dir}}(n,\delta) = f(n,\delta).$$

We conjecture that one can extend Theorem 1.6 to show that $\mathrm{ham}^{\mathrm{dir}}(n,\delta) = \mathrm{reg}^{\mathrm{dir}}(n,\delta)$ for any $\delta \geq n/2$. This would completely determine the values of $\mathrm{ham}^{\mathrm{dir}}(n,\delta)$, as in Lemma 4.1 we show that $\mathrm{reg}^{\mathrm{dir}}(n,\delta) = f(n,\delta)$ for any $\delta \geq n/2$.

We now state two consequences of Theorem 1.1 which we already observed in [39] and which we will use in this paper. The first is a version of Theorem 1.3 for digraphs. We will use it in our proof of Theorem 1.6.

Theorem 1.7. *For every $\varepsilon > 0$ there exists n_0 such that every r-regular digraph G on $n \geq n_0$ vertices, where $r \geq (1/2 + \varepsilon)n$, has a Hamilton decomposition.*

Theorem 1.7 follows from Theorem 1.1 since one can easily verify that regular digraphs as above are robust outexpanders (see Lemma 13.2 in [39]).

The second result states that every sufficiently dense regular oriented graph has a Hamilton decomposition. We will use it in our proof of Theorem 1.9 below.

Theorem 1.8. *For every $\varepsilon > 0$ there exists n_0 such that every r-regular oriented graph G on $n \geq n_0$ vertices with $r \geq 3n/8 + \varepsilon n$ has a Hamilton decomposition.*

Note that this implies Kelly's conjecture for large regular tournaments. As before, one can easily verify that such oriented graphs are robust outexpanders (see Lemma 13.1 in [39]).

1.4. **Random graphs and random tournaments.** Erdős conjectured a probabilistic version of Kelly's conjecture (see [49]), namely that asymptotically almost surely a random tournament T should contain $\delta^0(T)$ edge-disjoint Hamilton cycles. Here we say that a property holds asymptotically almost surely (a.a.s.) if it holds with probability tending to 1 as n tends to infinity. Note that trivially, any digraph G has at most $\delta^0(G)$ edge-disjoint Hamilton cycles. The following result confirms the conjecture of Erdős.

Theorem 1.9. *Let T be a tournament on n vertices which is chosen uniformly at random. Then a.a.s. T contains $\delta^0(T)$ edge-disjoint Hamilton cycles.*

Note that Theorem 1.9 is equivalent to the following statement: Consider the complete graph on n vertices and orient each edge randomly (where the probability for each of the two possible directions is 1/2), independently of all other edges. Let T denote the resulting tournament. Then a.a.s. T contains $\delta^0(T)$ edge-disjoint Hamilton cycles. To prove Theorem 1.9, we find a $\delta^0(T)$-regular oriented spanning subgraph T' of T and apply Theorem 1.8 to T' to obtain a Hamilton decomposition of T'.

A similar phenomenon occurs also for undirected random graphs. Let $G_{n,p}$ denote the binomial random graph with edge probability p. Bollobás and Frieze [9], showed that a.a.s. $G_{n,p}$ contains $\lfloor \delta(G_{n,p})/2 \rfloor$ edge-disjoint Hamilton cycles in the range of p where the minimum degree $\delta(G_{n,p})$ is a.a.s. bounded. Frieze and Krivelevich [17] conjectured that this result extends to the entire range of edge probabilities p. Partial results were proved in several papers, e.g. [17, 7, 29, 30, 32]. In particular, Knox, Kühn and Osthus [30] confirmed the conjecture in the range when $(\log n)^{50}/n \leq p \leq 1 - n^{-1/4}(\log n)^9$ and shortly afterwards Krivelevich and Samotij [32] covered the range when $\log n/n \leq p \leq n^{-1+\varepsilon}$. So in combination with [9], this implies that the conjecture remains open only in the case when p tends to 1 fairly quickly, i.e. when $p \geq 1 - n^{-1/4}(\log n)^9$. This special case follows

without too much work from Theorem 1.3 (in a similar way as Theorem 1.9). Altogether, this gives the following result, which confirms the conjecture of Frieze and Krivelevich. We emphasize that the main contribution to Theorem 1.10 comes from the results in [30, 32].

Theorem 1.10. *For any $p = p(n)$, a.a.s. $G_{n,p}$ contains $\lfloor \delta(G_{n,p})/2 \rfloor$ edge-disjoint Hamilton cycles.*

1.5. **Quasi-random graphs.** Robust expansion is a significant weakening of the extremely well-studied notion of quasi-randomness. The latter is usually defined in terms of the eigenvalues of a graph. More precisely, given a graph G, let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denote the eigenvalues of its adjacency matrix. The *second eigenvalue* of G is $\lambda(G) := \max_{i \geq 2} |\lambda_i|$. We say that a graph G is an (n, d, λ)*-graph* if it has n vertices, is d-regular and $\lambda(G) \leq \lambda$. It is well known that if λ is much smaller than d, then such a graph G has strong expansion properties. This means that we can apply Theorem 1.1 to quasi-random graphs in order to obtain the following result.

Theorem 1.11. *For every $\alpha > 0$ there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Let G be any (n, d, λ)-graph on $n \geq n_0$ vertices with $\lambda \leq \varepsilon n$ and such that $d \geq \alpha n$ is even. Then G has a Hamilton decomposition.*

Since it is well known that Paley graphs satisfy strong quasi-randomness conditions (see e.g. [8]), Theorem 1.11 generalizes (for large n) a recent result of Alspach, Bryant and Dyer [5] that every Paley graph has a Hamilton decomposition. Unsurprisingly, dense random regular graphs also satisfy strong quasi-random properties (see [33] for precise results). So Theorem 1.11 implies that a.a.s. random r-regular graphs have a Hamilton decomposition if r is even and linear in n. The case when r is bounded has received much attention and was settled by Kim and Wormald [28]. Frieze and Krivelevich [16] proved that ε-regular graphs which are almost regular have an approximate Hamilton decomposition. As graphs which are regular and ε-regular satisfy the conditions of Theorem 1.11, their result can be viewed as an approximate version of Theorem 1.11.

1.6. **Partite tournaments.** Jackson [23] posed the following bipartite version of Kelly's conjecture (both versions are also discussed e.g. by Bondy [10]). Here a *bipartite tournament* is an orientation of a complete bipartite graph.

Conjecture 1.12 (Jackson)**.** *Every regular bipartite tournament has a Hamilton decomposition.*

This does not seem to follow from our results. However, a k-partite version for $k \geq 4$ follows from Theorems 1.8 and 1.1. For this, we define a *regular k-partite tournament* to be an orientation of a complete k-partite graph with equal size vertex classes in which the indegree of every vertex equals its outdegree.

Corollary 1.13. *For every $k \in \mathbb{N}$ with $k \geq 4$, there is an $n_0 \in \mathbb{N}$ so that every regular k-partite tournament G on $n \geq n_0$ vertices has a Hamilton decomposition.*

Note that a regular k-partite tournament has degree $(k-1)n/2k$, so for $k \geq 5$ it satisfies the conditions of Theorem 1.8, and so Corollary 1.13 follows immediately. For $k = 4$, one cannot apply Theorem 1.8, but one can show that a regular 4-partite tournament is a robust outexpander (this follows by analyzing e.g. the proof of Lemma 13.1. in [39]). Thus one can apply Theorem 1.1 directly in this case to obtain a Hamilton decomposition. A regular 3-partite tournament is not necessarily a robust outexpander (e.g. it could be a blow-up of an oriented triangle). But we conjecture that Corollary 1.13 can be extended to all $k \geq 2$.

Conjecture 1.14. *Every regular tripartite tournament has a Hamilton decomposition.*

The corresponding question for undirected complete k-partite graphs was settled in the affirmative by Auerbach and Laskar [6], as well as Hetyei [22]. A digraph version was proved by Ng [45].

1.7. **Algorithmic aspects.** Robust expansion has turned out to be a useful and natural concept – not just for Hamilton decompositions (see e.g. [31]). This raises the question of whether this property can be recognized efficiently. The following result answers this question in the affirmative, as long as one does not need to know the exact expansion parameters. This is a similar situation as for the concept of ε-regularity. Note that for both concepts, the exact parameters are not relevant for any of the applications.

Theorem 1.15. *Given $0 < \nu \leq \tau < 1$, there is an algorithm which in time polynomial in n either decides that a digraph G on n vertices is not a robust (ν, τ)-outexpander or decides that G is a robust $(\nu^3/2560, 6\tau)$-outexpander. A similar statement holds for (undirected) robust expanders.*

The proof of Theorem 1.15 proceeds by showing that a (di-)graph G is a robust (out-)expander if and only if the reduced (di-)graph R obtained from an application of Szemerédi's regularity lemma is one. Similar ideas are also used to make the proof of Theorem 1.2 algorithmic in Section 6.

It would be interesting to know if one could obtain a characterization of robust (out-)expansion without considering R, but perhaps in terms of the eigenvalues of G and/or the codegrees (i.e. common neighbourhoods) of pairs of vertices. Such a connection would appear to be quite natural because of the well-known connection between expansion, eigenvalue separation and codegree bounds.

1.8. **Organization of the paper.** In Section 3, we derive Theorem 1.2 from its directed version (Theorem 1.1). We then deduce Theorems 1.3 and 1.5 from Theorem 1.2. In Section 4, we use Theorem 1.1 to prove Theorem 1.6. We then consider (quasi-)random structures in Section 5 to prove Theorems 1.9, 1.10 and 1.11. We give an algorithm for checking robust expansion in Section 6 as well as an algorithmic proof of Theorem 1.2.

2. Notation and a Chernoff type bound

Given a graph or digraph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. We also write $|G|$ for the number of vertices in G.

If G is an undirected graph, we write $\delta(G)$ for the minimum degree of G and $\Delta(G)$ for its maximum degree. Given $X, Y \subseteq V(G)$, we denote the number of all edges of G with one endvertex in X and the other in Y by $e_G(X, Y)$. An *r-factor* of a graph G is a spanning subgraph H of G in which every vertex has degree r.

If G is a digraph, we write xy for an edge which is directed from x to y. The *outneighbourhood* $N_G^+(x)$ of a vertex x is the set of all those vertices y for which $xy \in E(G)$. The *inneighbourhood* $N_G^-(x)$ of x is the set of all those vertices y for which $yx \in E(G)$. We write $d_G^+(x) := |N_G^+(x)|$ for the *outdegree of* x and $d_G^-(x) := |N_G^-(x)|$ for the *indegree of* x. We write $\delta^+(G) := \min_{x \in V(G)} d_G^+(x)$ for the *minimum outdegree* of G, $\delta^-(G) := \min_{x \in V(G)} d_G^-(x)$ for its *minimum indegree*, $\delta^0(G) := \min\{\delta^+(G), \delta^-(G)\}$ for its *minimum semidegree* and $\Delta^0(G) := \max\{\delta^+(G), \delta^-(G)\}$ for the *maximum semidegree* of G. Given $X, Y \subseteq V(G)$, we denote the number of all edges of G with initial vertex in X and final vertex Y by $e_G(X, Y)$. In all these definitions, we omit the subscript G if the graph or digraph G is clear from the context. An *r-factor* of a digraph G is a spanning subdigraph H of G with $d_H^+(x) = r = d_H^-(x)$ for every $x \in V(G)$.

The constants in the hierarchies used to state our results have to be chosen from right to left. More precisely, if we claim that a result holds whenever $0 < 1/n \ll a \ll b \ll c \leq 1$ (where n is the order of the graph or digraph), then this means that there are non-decreasing functions $f : (0, 1] \to (0, 1]$, $g : (0, 1] \to (0, 1]$ and $h : (0, 1] \to (0, 1]$ such that the result holds for all $0 < a, b, c \leq 1$ and all $n \in \mathbb{N}$ with $b \leq f(c)$, $a \leq g(b)$ and $1/n \leq h(a)$. We will not calculate these functions explicitly.

We will often use the following Chernoff type bound (see e.g. Theorem 2.1 and Corollary 2.2 in [25]).

Proposition 2.1. *Suppose X has binomial distribution and $0 < a < 1$. Then*

$$\mathbb{P}(X \leq (1-a)\mathbb{E}X) \leq e^{-\frac{a^2}{3}\mathbb{E}X} \quad \textit{and} \quad \mathbb{P}(X \geq (1+a)\mathbb{E}X) \leq e^{-\frac{a^2}{3}\mathbb{E}X}.$$

3. Edge-disjoint Hamilton cycles in undirected graphs

The purpose of this section is to prove Theorem 1.2. We will then use it to derive Theorems 1.3 and 1.5.

3.1. Regular orientations and Theorem 1.2.

To prove Theorem 1.2, we will show that the edges of every r-regular (undirected) robust expander can be oriented such that the oriented graph G^{orient} thus obtained from G is a $r/2$-regular robust outexpander (see Lemma 3.6). Theorem 1.1 then implies that G^{orient} has a Hamilton decomposition, which clearly corresponds to a Hamilton decomposition of G.

In order to prove Lemma 3.6, we first show that there is some orientation G' of G such that G' is a robust outexpander (see Lemma 3.1). It is not hard to

check that a random orientation will satisfy this property. However, it will only be almost regular, and not regular. We then show that every robust outexpander (and thus also G') contains a sparse *regular* spanning subdigraph G^* which is still a robust outexpander (see Lemma 3.4). It now remains to find a regular orientation $G^\diamond$ of $G \setminus E(G^*)$. Then $G^{\text{orient}} := G^* \cup G^\diamond$ is a $r/2$-regular orientation of G and, since G^* is a robust outexpander, G^{orient} is still a robust outexpander.

The proofs of Lemmas 3.1 and 3.4 (and thus of Lemma 3.6) are not algorithmic. In Section 6.2 we will also prove an algorithmic version of Lemma 3.6 (which leads to an algorithmic proof of Theorem 1.2) based on Szemerédi's regularity lemma. However, this does not mean that the corresponding part of the current subsection can be omitted: Lemma 3.4 (and thus also Lemmas 3.2 and 3.3) will be used again in our proof of Theorem 1.6. Moreover, Lemmas 3.1 and 3.4 are also crucial ingredients in [34]. So it is only the very short derivation of Lemma 3.6 itself that might be considered redundant. We have included it for completeness, as the underlying observation that addition of edges preserves robust expansion seems to be very useful (as is also illustrated in [34, 15]).

As described above, we first show that every (undirected) robust expander has an almost regular orientation which is a robust outexpander.

Lemma 3.1. *Suppose that* $0 < 1/n \ll \eta \ll \nu, \tau, \alpha < 1$. *Suppose that* G *is a robust* (ν, τ)*-expander on* n *vertices with* $\delta(G) \geq \alpha n$. *Then one can orient the edges of* G *in such a way that the oriented graph* G' *thus obtained from* G *satisfies the following properties:*

(i) G' *is a robust* $(\nu/4, \tau)$*-outexpander.*
(ii) $d^+_{G'}(x) = (1 \pm \eta) d_G(x)/2$ *and* $d^-_{G'}(x) = (1 \pm \eta) d_G(x)/2$ *for every vertex* x *of* G.

Proof. Orient each edge xy of G randomly (where the probability for each of the two possible directions is $1/2$), independently of all other edges. Let G' denote the oriented graph obtained in this way. Using Proposition 2.1 it is easy to show that (ii) fails with probability at most $1/4$.

So consider any $S \subseteq V(G)$ with $\tau n \leq |S| \leq (1-\tau)n$. Let $U' := RN^+_{\nu/4, G'}(S)$ and $U := RN_{\nu, G}(S)$. Let U^* be any subset of U of size $\nu n/4$. Consider any $u \in U^*$ and let $X := N^-_{G'}(u) \cap (S \setminus U^*)$. Note that

$$\mathbb{E}X \geq \frac{\nu n - |U^*|}{2} = \frac{3\nu n}{8}.$$

So Proposition 2.1 implies that

$$\mathbb{P}(X \leq \nu n/4) \leq \mathbb{P}(X \leq 2\mathbb{E}X/3) \leq e^{-\mathbb{E}X/27} \leq e^{-\nu n/100}.$$

Thus the probability that all vertices in U^* have at most $\nu n/4$ inneighbours in $S \setminus U^*$ (in the oriented graph G') is at most $e^{-\nu^2 n^2/400}$ (note these events are independent for different vertices in U^* as we are only considering the inneighbours in $S \setminus U^*$ rather than all inneighbours in S). But if $|U'| \leq |S| + \nu n/4$, then there exists a set $U^* \subseteq U$ of size $\nu n/4$ such that every vertex in U^* has at most $\nu n/4$

inneighbours in $S \setminus U^*$. So the probability that U' has size at most $|S| + \nu n/4$ is at most

$$\binom{|U|}{\nu n/4} e^{-\nu^2 n^2/400} \leq 2^n e^{-\nu^2 n^2/400} \leq e^{-\nu^2 n^2/401}.$$

Summing over all possible sets S shows that the probability that (i) fails is also at most 1/4. □

The next lemma shows that the edges of any robust outexpander can be split in such a way that the two digraphs thus obtained are still robust outexpanders.

Lemma 3.2. *Suppose that $0 < 1/n \ll \eta \ll \nu, \tau, \alpha, \lambda, 1-\lambda < 1$. Let G be a digraph on n vertices with $\delta^0(G) \geq \alpha n$ which is a robust (ν, τ)-outexpander. Then G can be split into two edge-disjoint spanning subdigraphs G_1 and G_2 such that the following two properties hold.*

(i) *$d^+_{G_1}(x) = (1 \pm \eta)\lambda d^+_G(x)$ and $d^-_{G_1}(x) = (1 \pm \eta)\lambda d^-_G(x)$ for every $x \in V(G)$.*
(ii) *G_1 is a robust $(\lambda\nu/2, \tau)$-outexpander and G_2 is a robust $((1-\lambda)\nu/2, \tau)$-outexpander.*

Proof. Consider a random partition of the edges of G into two subdigraphs G_1 and G_2 where an edge is included into G_1 with probability λ and into G_2 with probability $1-\lambda$ (independently of all other edges). Proposition 2.1 immediately implies that the probability that (i) fails is at most 1/4.

The argument for (ii) is similar to the proof of Lemma 3.1. Consider any $S \subseteq V(G)$ with $\tau n \leq |S| \leq (1-\tau)n$. Let $U := RN^+_{\nu,G}(S)$, $U_1 := RN^+_{\lambda\nu/2,G_1}(S)$ and $U_2 := RN^+_{(1-\lambda)\nu/2,G_2}(S)$. Consider any $u \in U$. The expected number of edges from S to u in G_1 is at least $\nu\lambda n$. So Proposition 2.1 implies that the probability that u does not lie in U_1 is at most $e^{-\nu\lambda n/12}$. Let U' be any subset of U of size $\nu n/2$. Then the probability that no vertex of U' lies in U_1 is at most $e^{-\nu^2\lambda n^2/24}$ (since these events are independent for different vertices of U'). So the probability that U_1 has size at most $|S| + \nu n/2$ is at most

$$\binom{|U|}{\nu n/2} e^{-\nu^2\lambda n^2/24} \leq 2^n e^{-\nu^2\lambda n^2/24} \leq e^{-\nu^2\lambda n^2/25}.$$

One can use a similar argument to show that the probability that U_2 has size at most $|S| + \nu n/2$ is at most $e^{-\nu^2(1-\lambda)n^2/25}$. Summing over all possible sets S shows that the probability that (ii) fails is also at most 1/4. □

The next lemma (which was proved in [39]) will be used in the proof of Lemma 3.4 to turn a sparse digraph G_1 into a regular one by adding a digraph G'_2 whose degree sequence complements that of G_1. The lemma guarantees the existence of such a digraph G'_2 in any robust outexpander, provided that the degrees of G'_2 are within a certain range.

Lemma 3.3. *Suppose that $0 < 1/n \ll \varepsilon, \xi \ll \nu \leq \tau \ll \alpha < 1$. Let G be a digraph on n vertices with $\delta^0(G) \geq \alpha n$ which is a robust (ν, τ)-outexpander. For every vertex x of G let $n^+_x, n^-_x \in \mathbb{N}$ be such that $(1-\varepsilon)\xi n \leq n^+_x, n^-_x \leq (1+\varepsilon)\xi n$ and*

such that $\sum_{x\in V(G)} n_x^+ = \sum_{x\in V(G)} n_x^-$. *Then* G *contains a spanning subdigraph* G' *such that* $d^+_{G'}(x) = n_x^+$ *and* $d^-_{G'}(x) = n_x^-$ *for every* $x \in V(G)$.

In order to prove Lemma 3.4, we first apply Lemma 3.2 to split the edges of G into two digraphs G_1 and G_2 which are both still robust outexpanders, where G_1 is very sparse. We then use Lemma 3.3 to find a subdigraph G_2' inside G_2 whose degree sequence complements that of G_1. So $G_1 \cup G_2'$ will then be a *regular* robust outexpander.

Lemma 3.4. *Suppose that* $0 < 1/n \ll \nu' \ll \xi \ll \nu \leq \tau \ll \alpha < 1$. *Let* G *be a robust* (ν,τ)*-outexpander on* n *vertices with* $\delta^0(G) \geq \alpha n$. *Then* G *contains a* ξn*-factor which is still a robust* (ν',τ)*-outexpander.*

Proof. Choose new constants η, λ such that $0 < 1/n \ll \eta, \nu' \ll \lambda \ll \xi$. Apply Lemma 3.2 to partition G into edge-disjoint spanning subdigraphs G_1 and G_2 such that the following properties are satisfied:

- $d^+_{G_1}(x) = (1 \pm \eta)\lambda d^+_G(x)$ and $d^-_{G_1}(x) = (1 \pm \eta)\lambda d^-_G(x)$ for every $x \in V(G)$.
- G_1 is a robust $(\lambda\nu/2, \tau)$-outexpander and G_2 is a robust $((1-\lambda)\nu/2, \tau)$-outexpander.

For every $x \in V(G)$, let $n_x^\pm := \xi n - d^\pm_{G_1}(x)$. Note that

$$(1-\sqrt{\lambda})\xi n \leq \xi n - 2\lambda n \leq n_x^+, n_x^- \leq \xi n.$$

Moreover, $\delta^0(G_2) \geq \alpha n - (1+\eta)\lambda n \geq \alpha n/2$. Thus we can apply Lemma 3.3 to G_2 (with $\sqrt{\lambda}$, $(1-\lambda)\nu/2$, $\alpha/2$ playing the roles of ε, ν, α) to obtain a spanning subdigraph G_2' of G_2 such that $d^\pm_{G_2'}(x) = n_x^\pm$ for every $x \in V(G)$. Then $G^* := G_1 \cup G_2'$ is a ξn-factor of G. Moreover, G^* is a robust (ν',τ)-outexpander since it contains the robust $(\lambda\nu/2,\tau)$-outexpander G_1 (and since $\nu' \leq \lambda\nu/2$). □

We also use the following classical result of Petersen.

Theorem 3.5. *Every regular graph of positive even degree contains a* 2*-factor.*

We can now combine Lemmas 3.1 and 3.4 as well as Theorem 3.5 to prove the main ingredient for our proof of Theorem 1.2.

Lemma 3.6. *Suppose that* $0 < 1/n \ll \nu' \ll \nu \leq \tau \ll \alpha < 1$. *Let* G *be an* r*-regular graph on* n *vertices such that* $r \geq \alpha n$ *is even and* G *is a robust* (ν,τ)*-expander. Then one can orient the edges of* G *in such a way that the oriented graph* G^{orient} *thus obtained from* G *is an* $r/2$*-regular robust* (ν',τ)*-outexpander.*

Proof. Choose a new constant ξ such that $\nu' \ll \xi \ll \nu$. Apply Lemma 3.1 to find an orientation G' of G such that G' is a robust $(\nu/4,\tau)$-outexpander and such that $\delta^0(G') \geq \alpha n/3$ (say). Now apply Lemma 3.4 to find a ξn-factor G^* of G' which is still a robust (ν',τ)-outexpander. Let H be the undirected graph obtained from G by deleting all the edges in G^*. Then H is $(r-2\xi n)$-regular. Thus Petersen's theorem (Theorem 3.5) implies that H has a decomposition into edge-disjoint 2-factors. Orient each cycle in these 2-factors consistently. This gives an orientation of H which is $(r-2\xi n)/2$-regular. Adding the edges of G^* gives an orientation

G^{orient} of G which is $r/2$-regular. Moreover, G^{orient} is a robust (ν', τ)-outexpander since it contains the robust (ν', τ)-outexpander G^*. □

We can now prove an undirected analogue of Theorem 1.1 on Hamilton decompositions of robust expanders.

Proof of Theorem 1.2. Let $\tau^* := \tau(\alpha)$, where $\tau(\alpha)$ is as defined in Theorem 1.1. Choose a new constant τ such that $0 < \tau \ll \alpha, \tau^*$. Note that whenever $\nu' \leq \nu$, every robust (ν, τ)-outexpander is also a robust (ν', τ)-outexpander. So we may assume that $0 \ll \nu \ll \tau$. Now choose $n_0 \in \mathbb{N}$ and ν' such that $1/n_0 \ll \nu' \ll \nu$. (τ and n_0 will be the constants returned by Theorem 1.2.) Now let G be a graph satisfying the conditions of Theorem 1.2 with these parameters, i.e. G is an r-regular robust (ν, τ)-expander on $n \geq n_0$ vertices such that $r \geq \alpha n$ is even. Apply Lemma 3.6 to obtain an $r/2$-regular orientation G^{orient} of G such that G^{orient} is a robust (ν', τ)-outexpander (and thus also a robust (ν', τ^*)-outexpander). Now apply Theorem 1.1 to find a Hamilton decomposition of G^{orient} (with ν' playing the role of ν in Theorem 1.1). Clearly, this Hamilton decomposition corresponds to a Hamilton decomposition of G. □

3.2. **Theorems 1.3 and 1.5.** Our next aim is to derive Theorem 1.3 from Theorem 1.2. For this, it suffices to show that every graph of minimum degree a little larger than $n/2$ is a robust expander. To prove this, we will use the following analogue of this result for regular digraphs, which was proved as Lemma 13.2 in [39]. Its proof follows easily from the definition of robust outexpansion. We will also use Lemma 3.7 in the proof of Theorem 1.6.

Lemma 3.7. *Suppose that $0 < \nu \leq \tau \leq \varepsilon < 1$ are such that $\varepsilon \geq 2\nu/\tau$. Let G be a digraph on n vertices with minimum semidegree $\delta^0(G) \geq (1/2 + \varepsilon)n$. Then G is a robust (ν, τ)-outexpander.*

Lemma 3.8. *Suppose that $0 < \nu \leq \tau \leq \varepsilon < 1$ are such that $\varepsilon \geq 2\nu/\tau$. Let G be a graph on n vertices with minimum degree $\delta(G) \geq (1/2 + \varepsilon)n$. Then G is a robust (ν, τ)-expander.*

Proof. Let G' be the digraph obtained from G by replacing every (undirected) edge xy of G by two directed edges xy and yx. Then $\delta^0(G') \geq (1/2 + \varepsilon)n$ and so Lemma 3.7 implies that G' is a robust (ν, τ)-outexpander. But this implies that G is a robust (ν, τ)-expander. □

Theorem 1.3 (which guarantees a Hamilton decomposition of dense regular graphs) is now an immediate consequence of Lemma 3.8 and Theorem 1.2.

Proof of Theorem 1.3. Let $\tau^* := \tau(1/2)$, where $\tau(1/2)$ is as defined in Theorem 1.2. Choose new constants $n_0 \in \mathbb{N}$ and ν, τ such that $0 < 1/n_0 \ll \nu \ll \tau \leq \varepsilon, \tau^*$. Now let G be a graph satisfying the conditions of Theorem 1.3, i.e. G is an r-regular graph on $n \geq n_0$ vertices, where $r \geq (1/2 + \varepsilon)n$ is even. Then

Lemma 3.8 implies that G is a robust (ν,τ)-expander (and so also a robust (ν,τ^*)-expander). Thus Theorem 1.2 implies that G has a Hamilton decomposition. □

Finally, we can use Theorem 1.3 to derive Theorem 1.5 (which concerns edge-disjoint Hamilton cycles in graphs of given minimum degree).

Proof of Theorem 1.5. Choose $n_0 \in \mathbb{N}$ and an additional constant ε' such that $1/n_0 \ll \varepsilon' \ll \varepsilon$. Consider any graph G on $n \geq n_0$ vertices with minimum degree $\delta \geq (2-\sqrt{2}+\varepsilon)n$. Then Theorem 1.4 implies that

$$\mathrm{reg}_{\mathrm{even}}(G) \geq \mathrm{reg}_{\mathrm{even}}(n,\delta) \geq g_{\mathrm{even}}(n,\delta).$$

(Recall that $g_{\mathrm{even}}(n,\delta)$ is the largest even integer r with $r \leq (\delta+\sqrt{n(2\delta-n)})/2$.) Note that the only solution to $a+\sqrt{2a-1}=1$ with $0 \leq a \leq 1$ is $a = 2-\sqrt{2}$. So the lower bound on δ implies that $g_{\mathrm{even}}(n,\delta) \geq (1/2+\varepsilon')n$. Let G' be a regular subgraph of G of degree $\mathrm{reg}_{\mathrm{even}}(G)$. Since $\mathrm{reg}_{\mathrm{even}}(G) \geq g_{\mathrm{even}}(n,\delta) \geq (1/2+\varepsilon')n$, Theorem 1.3 implies that G' has a decomposition into $\mathrm{reg}_{\mathrm{even}}(G)/2$ edge-disjoint Hamilton cycles. □

4. Edge-disjoint Hamilton cycles in digraphs of large minimum semidegree

The purpose of this section is to prove Theorem 1.6. To do this, we will first determine $\mathrm{reg}^{\mathrm{dir}}(n,\delta)$ for all $n/2 \leq \delta < n$, i.e. the largest number r such that every digraph on n vertices of minimum semidegree δ contains an r-factor. Recall that $\mathrm{reg}(G)$ denotes the largest degree of a regular spanning subdigraph of G.

Lemma 4.1. *Suppose that $\delta, n \in \mathbb{N}$ are such that $n/2 \leq \delta < n$. Let*

$$r^* := \frac{\delta+\sqrt{n(2\delta-n)+\mathbf{1}_{n\not\equiv\delta}}}{2}, \quad \textit{where} \quad \mathbf{1}_{n\not\equiv\delta} := \begin{cases} 0 & \textit{if } n \equiv \delta \mod 2 \\ 1 & \textit{if } n \not\equiv \delta \mod 2. \end{cases}$$

(i) *Let G be a digraph on n vertices with minimum semidegree δ. Then $\mathrm{reg}(G) \geq \lfloor r^* \rfloor$.*
(ii) *There is a digraph G on n vertices with minimum semidegree δ such that $\mathrm{reg}(G) = \lfloor r^* \rfloor$.*

In particular, $\mathrm{reg}^{\mathrm{dir}}(n,\delta) = \lfloor r^* \rfloor$.

Proof. Let $r := \lfloor r^* \rfloor$. In order to prove (i), we have to show that G contains an r-factor. To do this, our aim is to apply the Max-Flow-Min-Cut theorem. So let H be the (unoriented) bipartite graph whose vertex classes A and B are both copies of $V(G)$ and in which $a \in A$ is joined to $b \in B$ if ab is a (directed) edge of G. Give every edge of H capacity 1. Add a source u^* which is joined to every vertex $a \in A$ with an edge of capacity r. Add a sink w^* which is joined to every vertex $b \in B$ with an edge of capacity r. Note that an integer-valued rn-flow corresponds to the desired r-factor of G. Thus by the Max-Flow-Min-Cut theorem it suffices to show that every cut has capacity at least rn.

So consider a minimal cut $\mathcal{C}$. Let U be the set of all those vertices $a \in A$ for which $u^*a \notin \mathcal{C}$. Similarly, let W be the set of all those vertices $b \in B$ for which $bw^* \notin \mathcal{C}$. We only consider the case when $|W| \geq |U|$ (the other case is similar). Let $U' := A \setminus U$ and $W' := B \setminus W$. Thus the capacity of $\mathcal{C}$ is

$$c := r|U'| + e_B(U, W) + r|W'|.$$

But

$$e_B(U,W) = e_G(U,W) \geq \delta|U| - e_G(U,W') \geq \delta|U| - |U||W'| = r|U| + |U|(\delta - r - |W'|).$$

Thus

$$c \geq r|U'| + r|U| + |U|(\delta - r - |W'|) + r|W'| = rn + |U|(\delta - r - |W'|) + r|W'|.$$

So it suffices to show that

$$0 \leq |U|(\delta - r - |W'|) + r|W'| = |W'|(r - |U|) + |U|(\delta - r). \tag{2}$$

This holds if $|U| \leq r$. So we may assume that $|U| > r$. Since $|W| \geq |U|$ we have that $|W'| \leq |U'|$. So (2) holds if

$$\begin{aligned} 0 &\leq |U'|(r - |U|) + |U|(\delta - r) = (n - |U|)(r - |U|) + |U|(\delta - r) \\ &= nr + |U|^2 - |U|(n - \delta + 2r) \\ &= \left(|U| - \frac{n - \delta + 2r}{2}\right)^2 - \frac{(n - \delta + 2r)^2}{4} + nr. \end{aligned} \tag{3}$$

But writing

$$f(x) := nx - \frac{(n - \delta + 2x)^2}{4},$$

we have that $f(r^*) = -\mathbf{1}_{n \not\equiv \delta}/4$. Moreover, $f(x)$ is decreasing for all $x \geq \delta/2$ and $\delta/2 \leq r \leq r^*$. (To check the latter, it is easiest to consider the cases when $\delta = n/2$ and when $\delta > n/2$ separately.) Altogether this shows that $f(r) \geq -\mathbf{1}_{n \not\equiv \delta}/4$.

On the other hand, $r \in \mathbb{N}$ implies that

$$\left(|U| - \frac{n - \delta + 2r}{2}\right)^2 \geq \begin{cases} 0 & \text{if } n \equiv \delta \mod 2 \\ 1/4 & \text{if } n \not\equiv \delta \mod 2 \end{cases}.$$

Altogether this shows that (3) holds. This proves (i).

The proof of (ii) is similar to the proof of its analogue for undirected graphs (see Section 2 in [12]). If $\delta = n - 1$, then $r^* = n - 1$ and so we can take G to be the complete digraph on n vertices. Thus we may assume that $\delta \leq n - 2$. Let

$$\Delta := \left\lceil \frac{n + \sqrt{n(2\delta - n) + \mathbf{1}_{n \not\equiv \delta}}}{2} \right\rceil.$$

It is easy to check that $\delta < \Delta \leq n - 1$. (Here we use that $(2\delta - n)^2 < n(2\delta - n)$ for the lower bound and $\delta \leq n - 2$ for the upper bound.) Let A be an empty digraph on $n - \Delta$ vertices and let B be a $(\delta + \Delta - n)$-regular digraph on Δ vertices. Let G be obtained from the disjoint union of A and B by adding all edges from A to

B and all edges from B to A. Thus $\delta^0(G) = \delta$ and $\Delta^0(G) = \Delta$. We claim that $\mathrm{reg}(G) \leq \frac{\Delta(\delta+\Delta-n)}{2\Delta-n}$. Indeed, given any r-factor G' of G, we have

$$r\Delta = \sum_{x\in B} d^-_{G'}(x) \leq \Delta(\delta+\Delta-n) + e_{G'}(A,B) = \Delta(\delta+\Delta-n) + r(n-\Delta),$$

which implies the claim. But

$$\frac{\Delta(\delta+\Delta-n)}{2\Delta-n} = \frac{\delta}{2} + \frac{n\delta/2}{2\Delta-n} - \frac{\Delta(n-\Delta)}{2\Delta-n}.$$

Let η be such that $\Delta = \frac{n+\sqrt{n(2\delta-n)+\mathbf{1}_{n\not\equiv\delta}}}{2} + \eta$. So $0 \leq \eta < 1$. Then

$$\begin{aligned}\Delta(n-\Delta) &= \left(\frac{n}{2} + \left(\frac{\sqrt{n(2\delta-n)+\mathbf{1}_{n\not\equiv\delta}}}{2} + \eta\right)\right)\left(\frac{n}{2} - \left(\frac{\sqrt{n(2\delta-n)+\mathbf{1}_{n\not\equiv\delta}}}{2} + \eta\right)\right)\\ &= \frac{n^2}{4} - \frac{n(2\delta-n)+\mathbf{1}_{n\not\equiv\delta}}{4} - \eta\sqrt{n(2\delta-n)+\mathbf{1}_{n\not\equiv\delta}} - \eta^2\\ &= \frac{n^2-n\delta}{2} - \frac{\mathbf{1}_{n\not\equiv\delta}}{4} - \eta\sqrt{n(2\delta-n)+\mathbf{1}_{n\not\equiv\delta}} - \eta^2.\end{aligned}$$

Thus

$$\frac{\Delta(\delta+\Delta-n)}{2\Delta-n} = \frac{\delta}{2} + \frac{n(2\delta-n)+2\eta\sqrt{n(2\delta-n)+\mathbf{1}_{n\not\equiv\delta}}}{2(2\Delta-n)} + \frac{\mathbf{1}_{n\not\equiv\delta}}{4(2\Delta-n)} + \frac{\eta^2}{2\Delta-n}.$$

Since also $(2\Delta-n)\sqrt{n(2\delta-n)+\mathbf{1}_{n\not\equiv\delta}} = n(2\delta-n)+\mathbf{1}_{n\not\equiv\delta}+2\eta\sqrt{n(2\delta-n)+\mathbf{1}_{n\not\equiv\delta}}$, we deduce that

$$\begin{aligned}\mathrm{reg}(G) \leq \frac{\Delta(\delta+\Delta-n)}{2\Delta-n} &= \frac{\delta+\sqrt{n(2\delta-n)+\mathbf{1}_{n\not\equiv\delta}}}{2} - \frac{\mathbf{1}_{n\not\equiv\delta}}{4(2\Delta-n)} + \frac{\eta^2}{2\Delta-n}\\ &= r^* + \frac{\eta^2-\mathbf{1}_{n\not\equiv\delta}/4}{2\Delta-n}.\end{aligned}$$

If $\eta^2-\mathbf{1}_{n\not\equiv\delta}/4 \leq 0$, this implies that $\mathrm{reg}(G) \leq \lfloor r^*\rfloor$, as required. So we may assume that $\eta^2-\mathbf{1}_{n\not\equiv\delta}/4 > 0$. Now recall that $\Delta > \delta$ and hence $2\Delta-n \geq 2(\delta+1)-n \geq 2$. This means that we may assume that

$$\mathrm{reg}(G) \leq r^* + \eta^2/2 - \mathbf{1}_{n\not\equiv\delta}/8. \tag{4}$$

Let η_* be such that $\lceil r^*\rceil = r^* + \eta_*$. So $0 \leq \eta_* < 1$ and it is easy to see that

$$\eta = \begin{cases}\eta_* & \text{if } n \equiv \delta \mod 2\\ \eta_* \pm 1/2 & \text{if } n \not\equiv \delta \mod 2\end{cases}.$$

Thus if $n \equiv \delta \mod 2$ then $\eta^2/2 - \mathbf{1}_{n\not\equiv\delta}/8 = \eta^2/2 = \eta_*^2/2 < \eta_*$. If $n \not\equiv \delta \mod 2$ then $\eta^2/2 - \mathbf{1}_{n\not\equiv\delta}/8 \leq \eta_*^2/2 + \eta_*/2 < \eta_*$. This shows that in both cases the right hand side of (4) is strictly less than $r^* + \eta_* = \lceil r^*\rceil$, which in turn implies that $\mathrm{reg}(G) \leq \lfloor r^*\rfloor$. Together with (i) this shows that $\mathrm{reg}(G) = \lfloor r^*\rfloor$ in all cases, as required. □

We are now in a position to find (nearly) optimal packings of edge-disjoint Hamilton cycles in digraphs of given minimum semidegree δ.

Proof of Theorem 1.6. We first prove (i). Lemma 4.1 implies that $\mathrm{reg}^{\mathrm{dir}}(n,\delta) = f(n,\delta)$ whenever $n/2 \leq \delta < n$. Now suppose that $\delta \geq (1/2+\varepsilon)n$, where n is sufficiently large compared with $1/\varepsilon$. To show that $\mathrm{ham}^{\mathrm{dir}}(n,\delta) \geq f(n,\delta) - \varepsilon n$, we let $\tau^* := \tau(1/5)$, where $\tau(1/5)$ is as defined in Theorem 1.1. Choose new constants ν', ν, τ, ξ such that

$$1/n_0 \ll \nu' \ll \xi \ll \nu \ll \tau \leq \varepsilon, \tau^*.$$

Consider any digraph G on n vertices with minimum semidegree δ. Then Lemma 3.7 implies that G is a robust (ν,τ)-outexpander. Apply Lemma 3.4 to obtain a ξn-factor G_1 of G such that G_1 is still a robust (ν',τ)-outexpander (and thus also a robust (ν',τ^*)-outexpander). Let $G_2 := G \setminus E(G_1)$ and let G_2' be a spanning subdigraph of G_2 which is $\mathrm{reg}(G_2)$-regular. Since $\delta^0(G_2) = \delta^0(G) - \xi n = \delta - \xi n$, Lemma 4.1(i) implies that $\mathrm{reg}(G_2) \geq f(n,\delta) - \varepsilon n$. Thus $G_1 \cup G_2'$ is regular of degree $\xi n + \mathrm{reg}(G_2) \geq f(n,\delta) - \varepsilon n \geq n/5$ (say). Moreover, $G_1 \cup G_2'$ is still a robust (ν',τ^*)-outexpander (since it contains the robust (ν',τ^*)-outexpander G_1). So we may apply Theorem 1.1 to $G_1 \cup G_2'$ (with ν' playing the role of ν) to obtain a Hamilton decomposition of $G_1 \cup G_2'$. Thus $\mathrm{ham}(G) \geq f(n,\delta) - \varepsilon n$. This implies that $\mathrm{ham}^{\mathrm{dir}}(n,\delta) \geq f(n,\delta) - \varepsilon n$.

The proof of (ii) is identical to the proof of Theorem 1.5, except that instead of applying Theorem 1.3 we apply Theorem 1.7. □

5. Edge-disjoint Hamilton cycles in random tournaments, random and quasi-random graphs

The purpose of this section is to prove Theorems 1.9, 1.10 and 1.11, which give optimal packings of edge-disjoint Hamilton cycles in random tournaments, in very dense random graphs and in quasi-random graphs respectively.

5.1. Optimal packings of Hamilton cycles in random tournaments.

To prove Theorem 1.9, we will apply the Max-Flow-Min-Cut theorem to show that a.a.s. a random tournament T contains a $\delta^0(T)$-factor (see Lemma 5.3). Since a.a.s. $\delta^0(T)$ is almost $|T|/2$, Theorem 1.9 will then follow from Theorem 1.8. For our proof of Lemma 5.3 we need some bounds on $\delta^0(T)$ as well as on the number of edges between any two sufficiently large subsets of vertices of T, which we will prove in Lemma 5.2. To do this, we in turn need the following notation and some well-known facts about the binomial distribution. Given $X \sim Bin(n-1, 1/2)$, we write

- $b(r) := \mathbb{P}(X = r) = \binom{n-1}{r}(1/2)^{n-1}$,
- $B(m_1, m_2) := \mathbb{P}(m_1 \leq X \leq m_2)$, and
- $B(m) := \mathbb{P}(X \leq m)$.

$b'(r)$, $B'(m_1, m_2)$ and $B'(m)$ are defined similarly for $X \sim Bin(n-2, 1/2)$.

Proposition 5.1. *Suppose that $r \geq n/2 - \sqrt{2n \log n}$ and $0 < h \leq n^{3/5}$. Then*

(i) $\frac{b'(r)}{b(r)} \leq 1 + 1/\log n$;
(ii) $b(n/2 - h) \geq \frac{1}{2\sqrt{n}} e^{-2h^2/n - 4h^3/n^2}$;
(iii) $B(n/2 - h) \leq \frac{\sqrt{n}}{h} e^{-2h^2/n}$.

(i) is a special case of Lemma 11(v) in [30]. (ii) is a special case of Theorem 1.5 in [8]. (iii) is a special case of the de Moivre-Laplace Theorem (Theorem 1.6(ii) in [8]). The bounds in (ii) and (iii) are weaker than those in [8], which allows us to replace appearances of the term $n-1$ (which would normally appear e.g. in the exponent of (iii)) by n.

Lemma 5.2. *Suppose that $0 < \varepsilon < 1$ is fixed. Let T be a random tournament obtained from K_n by orienting each edge of K_n randomly (where the probability for each of the two possible directions is $1/2$), independently of all other edges. Then a.a.s. the following conditions hold:*
(i) $\delta^0(T) \geq n/2 - (1+\varepsilon)\sqrt{n(\log n)/2}$.
(ii) $\delta^0(T) \leq n/2 - (1-\varepsilon)\sqrt{n(\log n)/2}$.
(iii) *All vertex sets $A, B \subseteq V(T)$ satisfy $|e_T(A,B) - |A||B|/2| \leq 9n^{3/2}$.*

Assertions (i) and (ii) are very similar to those for the minimum degree of a binomial random graph $G_{n,1/2}$ on n vertices with edge probability $1/2$. However, we cannot just quote these, as the correlations between vertex degrees in $G_{n,1/2}$ is slightly different than in a tournament. For our purposes, it will suffice to apply Lemma 5.2 with $\varepsilon = 1/10$, say.

Proof. First we prove (i). Consider any vertex x. Let $h := (1+\varepsilon)\sqrt{n(\log n)/2}$. Then Proposition 5.1(iii) implies that

$$\mathbb{P}(d_T^+(x) \leq n/2 - h) = B(n/2 - h) \leq \frac{\sqrt{n}}{h} e^{-2h^2/n} \leq n^{-(1+\varepsilon)^2} \leq n^{-1-2\varepsilon}.$$

The analogue holds for the indegree of x. So taking a union bound shows that the probability that T does not satisfy (i) is at most $2n \cdot n^{-1-2\varepsilon} \leq n^{-\varepsilon}$.

Next we prove (ii). Let $h_1 := (1-\varepsilon/2)\sqrt{n(\log n)/2}$ and $m_1 := n/2 - h_1$. Let $h_2 := (1-\varepsilon)\sqrt{n(\log n)/2}$ and $m_2 := n/2 - h_2$. Let Y be the number of vertices $x \in V(T)$ such that $m_1 \leq d_T^+(x) \leq m_2$. Then Proposition 5.1(ii) implies that

$$\begin{aligned}\mathbb{E}(Y) &\geq n(h_2 - h_1)b(m_1) \geq n \cdot \frac{\varepsilon}{2}\sqrt{n(\log n)/2} \cdot \frac{1}{2\sqrt{n}} e^{-2h_1^2/n - 4h_1^3/n^2} \\ &\geq n \cdot (\log n)^{1/3} \cdot n^{-(1-\varepsilon/2)^2} \geq n^{\varepsilon/2}.\end{aligned}$$

Also

$$\mathbb{E}_2(Y) = n(n-1) \cdot 2 \cdot (1/2) \cdot B'(m_1 - 1, m_2 - 1)B'(m_1, m_2) \leq n^2 B'(m_1, m_2)^2.$$

Hence Proposition 5.1(i) implies that

$$\frac{\sqrt{\mathbb{E}_2(Y)}}{\mathbb{E}(Y)} \leq \frac{\sum_{r=m_1}^{m_2} nb'(r)}{\sum_{r=m_1}^{m_2} nb(r)} = \frac{\sum_{r=m_1}^{m_2} nb(r)\frac{b'(r)}{b(r)}}{\sum_{r=m_1}^{m_2} nb(r)} \leq 1 + \frac{1}{\log n}. \tag{5}$$

So

$$\begin{aligned} Var(Y) &= \mathbb{E}_2(Y) + \mathbb{E}(Y) - \mathbb{E}(Y)^2 \leq (1 + 1/\log n)^2 \mathbb{E}(Y)^2 + \mathbb{E}(Y) - \mathbb{E}(Y)^2 \\ &= (2/\log n + 1/(\log n)^2)\mathbb{E}(Y)^2 + \mathbb{E}(Y). \end{aligned} \tag{6}$$

So by Chebyshev's inequality,

$$\mathbb{P}(Y = 0) \leq \frac{Var(Y)}{\mathbb{E}(Y)^2} \leq \frac{2}{\log n} + \frac{1}{(\log n)^2} + \frac{1}{\mathbb{E}(Y)} \leq \frac{3}{\log n}.$$

Since the indegrees of the vertices satisfy the analogous properties, it follows that (ii) fails with probability at most $6/\log n$.

Let us now check (iii). We will first prove the following claim.

Claim. *Let $S, T \subseteq V(T)$ be such that $S \cap T = \emptyset$. Then*

$$\mathbb{P}(|e_T(S,T) - |S||T|/2| \geq 4n^{3/2}) \leq 2e^{-2n}.$$

To prove the claim, suppose first that least one of S, T has size at most $4n^{1/2}$. Then $e_T(S,T) \leq 4n^{3/2}$ and so trivially $\mathbb{P}(|e_T(S,T) - |S||T|/2| \geq 4n^{3/2}) = 0$. So we may assume that $|S|, |T| \geq 4n^{1/2}$. Let $a := (12n/|S||T|)^{1/2}$. Then $a < 1$ and $a|S||T|/2 \leq 4n^{3/2}$. Thus Proposition 2.1 implies that

$$\begin{aligned} \mathbb{P}(|e_T(S,T) - |S||T|/2| \geq 4n^{3/2}) &\leq \mathbb{P}(|e_T(S,T) - |S||T|/2| \geq a|S||T|/2) \\ &\leq 2e^{-a^2|S||T|/6} = 2e^{-2n}, \end{aligned}$$

which proves the claim.

Now consider any $A, B \subseteq V(T)$. Then

$$\begin{aligned} e_T(A,B) &= e_T(A, B \setminus A) + e_T(A \setminus B, A \cap B) + e_T(A \cap B, A \cap B) \\ &= e_T(A, B \setminus A) + e_T(A \setminus B, A \cap B) + \binom{|A \cap B|}{2}. \end{aligned}$$

Our claim implies that with probability at least $1 - 4e^{-2n}$ both

$$|e_T(A, B \setminus A) - |A||B \setminus A|/2| \leq 4n^{3/2}$$

and

$$|e_T(A \setminus B, A \cap B) - |A \setminus B||A \cap B|/2| \leq 4n^{3/2}$$

hold. But

$$\begin{aligned} \frac{|A||B \setminus A|}{2} + \frac{|A \setminus B||A \cap B|}{2} + \binom{|A \cap B|}{2} + \frac{|A \cap B|}{2} &= \\ &= \frac{1}{2}\left(|A||B \setminus A| + (|A \setminus B| + |A \cap B|)|A \cap B|\right) \\ &= \frac{1}{2}\left(|A||B \setminus A| + |A||A \cap B|\right) = \frac{|A||B|}{2}. \end{aligned}$$

Altogether this shows that with probability at least $1 - 4e^{-2n}$ we have

$$|e_T(A,B) - |A||B|/2| \leq 8n^{3/2} + |A \cap B|/2 \leq 9n^{3/2}.$$

So taking a union bound shows that (iii) fails with probability at most $2^{2n} \cdot 4e^{-2n}$.

Altogether this shows that with probability at most $n^{-\varepsilon}+6/\log n+2^{2n}\cdot 4e^{-2n} \leq 1/2$ at least one of (i), (ii), (iii) fails. □

Lemma 5.3. *Suppose that* $0 < 1/n \ll \varepsilon \ll 1$. *Let* T *be a tournament on* n *vertices which satisfies conditions (i)–(iii) of Lemma 5.2. Then* T *contains a* $\delta^0(T)$*-factor.*

Proof. The proof is similar to that of Lemma 4.1(i). Let $\delta := \delta^0(T)$. As in the proof of Lemma 4.1(i), our aim is to apply the Max-Flow-Min-Cut theorem. So let H be the (unoriented) bipartite graph whose vertex classes A and B are both copies of $V(T)$ and in which $a \in A$ is joined to $b \in B$ if ab is a (directed) edge of T. Give every edge of H capacity 1. Add a source u^* which is joined to every vertex $a \in A$ with an edge of capacity δ. Add a sink w^* which is joined to every vertex $b \in B$ with an edge of capacity δ. Note that an integer-valued δn-flow corresponds to the desired spanning subdigraph of T. Thus by the Max-Flow-Min-Cut theorem it suffices to show that every cut has capacity at least δn.

So consider a minimal cut $\mathcal{C}$. Let U be the set of all those vertices $a \in A$ for which $u^*a \notin \mathcal{C}$. Similarly, let W be the set of all those vertices $b \in B$ for which $bw^* \notin \mathcal{C}$. Let $U' := A \setminus U$ and $W' := B \setminus W$. Thus the capacity of $\mathcal{C}$ is

$$c := \delta|U'| + e_B(U,W) + \delta|W'|.$$

Note that $e_B(U,W) = e_T(U,W) \geq (\delta - |U'|)|W|$. Thus if $|W| \leq \delta$, then

$$c = \delta|U'| + e_B(U,W) + \delta|W'| \geq \delta|U'| - |U'||W| + \delta(|W| + |W'|) \geq \delta n,$$

as required.

The argument in the case when $|U| \leq \delta$ is similar. So it remains to consider the case when $|U|, |W| \geq \delta$. Note that Lemma 5.2(i),(ii) implies that

$$n/2 - \sqrt{n \log n} \leq \delta \leq n/2 - \sqrt{n \log n}/2. \tag{7}$$

In particular, this means

$$n/3 \leq \delta \leq |U| \leq n. \tag{8}$$

So Lemma 5.2(iii) implies that

$$\begin{aligned}
c &\geq \delta|U'| + \frac{|U||W|}{2} - 9n^{3/2} + \delta|W'| \\
&= \delta|U'| + \frac{|U|}{2}(n - |W'|) - 9n^{3/2} + \delta|W'| \\
&\overset{(7)}{\geq} \delta|U'| + \frac{|U|}{2}(2\delta + \sqrt{n \log n}) - \frac{|U||W'|}{2} - 9n^{3/2} + \delta|W'| \\
&= \delta(|U'| + |U|) + \frac{|U|}{2}\sqrt{n \log n} - 9n^{3/2} + |W'|\left(\delta - \frac{|U|}{2}\right) \\
&\overset{(8)}{\geq} \delta n + \frac{n}{7}\sqrt{n \log n} + |W'|\left(\delta - \frac{|U|}{2}\right) && (9) \\
&\overset{(7),(8)}{\geq} \delta n + \frac{n}{7}\sqrt{n \log n} - |W'|\sqrt{n \log n}. && (10)
\end{aligned}$$

Now suppose first that $|W'| \leq n-2\delta \leq 2\sqrt{n\log n}$ (where the last inequality follows from (7)). Then (10) implies that

$$c \geq \delta n + \frac{n}{7}\sqrt{n\log n} - 2n\log n \geq \delta n,$$

as required. So we are left with the case when $|U|, |W| \geq \delta$ and $|W'| \geq n - 2\delta$. But then $|W| \leq 2\delta$. Moreover, interchanging the roles of U and W shows that instead of (9) one also has that

$$c \geq \delta n + \frac{n}{7}\sqrt{n\log n} + |U'|\left(\delta - \frac{|W|}{2}\right) \geq \delta n + \frac{n}{7}\sqrt{n\log n} \geq \delta n,$$

as required. □

Theorem 1.9 is now an immediate consequence of Theorem 1.8 and Lemmas 5.2 and 5.3.

Proof of Theorem 1.9. Consider a random tournament T obtained from K_n by orienting each edge of K_n randomly (where the probability for each of the two possible directions is 1/2), independently of all other edges. Note that each tournament on n vertices occurs equally likely in this model. So this is equivalent to choosing a tournament T on n vertices uniformly at random. Lemmas 5.2 and 5.3 together imply that a.a.s. T contains a $\delta^0(T)$-factor T'. Since a.a.s. $\delta^0(T) \geq n/2 - \sqrt{n\log n}$ by Lemma 5.2(i), we can apply Theorem 1.8 to obtain a Hamilton decomposition of T'. The Hamilton cycles in this Hamilton decomposition are as required in Theorem 1.9. □

5.2. **Optimal packings of Hamilton cycles in very dense random graphs.** Recall from Section 1 that Theorem 1.10 is only open in the case when p tends to 1 rather quickly, more precisely when $p \geq 1 - n^{-1/4}(\log n)^9$. The proof of this case of Theorem 1.10 is similar to that of Theorem 1.9. The main step is to prove Lemma 5.4 below. Since a.a.s. $\delta(G_{n,p}) \geq 7n/12$ (say) if $p \geq 2/3$, Theorem 1.3 and Lemma 5.4 together immediately imply that Theorem 1.10 holds if $p = p(n) \geq 2/3$. It would not be difficult to replace the '2/3' by an arbitrary constant, though in this case one would have to apply Theorem 1.2 rather than Theorem 1.3. Below, we say that a matching in a graph G is *optimal* if it covers all but at most one vertex of G.

Lemma 5.4. *Suppose that $p = p(n) \geq 2/3$ is monotone. Then a.a.s. $G_{n,p}$ satisfies the following properties:*

(i) *If $\delta(G_{n,p})$ is even then $G_{n,p}$ contains a $\delta(G_{n,p})$-factor.*
(ii) *If $\delta(G_{n,p})$ is odd then there is an optimal matching M in $G_{n,p}$ such that $G_{n,p} - M$ contains a $(\delta(G_{n,p}) - 1)$-factor.*

Note that Lemma 5.4 allows us to deduce not just Theorem 1.10 (in the range when $p \geq 2/3$) but the following stronger *property* $\mathcal{H}$, where a graph G has property $\mathcal{H}$ if G contains $\lfloor \delta(G)/2 \rfloor$ edge-disjoint Hamilton cycles, together with

an additional edge-disjoint optimal matching if $\delta(G)$ is odd. Property $\mathcal{H}$ was also verified for the relevant range of p in [30], but not in [32].

The next lemma guarantees a $\delta(G)$-factor or a $(\delta(G)-1)$-factor together with an optimal matching in any graph G whose minimum degree is close to n and which satisfies some very weak conditions on the number of vertices of degree $\delta(G)$ and on the size of the gap $\Delta(G)-\delta(G)$.

Lemma 5.5. *Let $s,t,n \in \mathbb{N}$. Suppose that G is a graph on n vertices such that $\delta(G) \geq n-t$ and let s denote the number of vertices of degree $\delta(G)$ in G. Suppose that $n \geq s+3t+2t(\Delta(G)-\delta(G))$. Then the following properties hold.*

- *If $\delta(G)$ is even then G contains a $\delta(G)$-factor.*
- *If $\delta(G)$ is odd then there is an optimal matching M in G such that $G-M$ contains a $(\delta(G)-1)$-factor.*

Proof. Let $\delta := \delta(G)$. In our proof we will often use the following observation:

Whenever G' is a spanning subgraph of G with $\delta(G')=\delta$ and $X \subseteq V(G)$ is a set of at least $2t$ vertices, then $G'[X]$ contains an optimal matching. Moreover, if $|X|$ is odd then any vertex can be chosen as the one not covered by this matching. $(*)$

(To see that $(*)$ holds, note that $\delta(G'[X]) \geq |X|-t \geq |X|/2$. So $(*)$ follows from Dirac's theorem.) Let $x_1,\dots,x_n$ be an enumeration of the vertices of G such that $d(x_1) \geq \dots \geq d(x_n)$. Let $X^{\min}$ be the set of all vertices of degree δ in G. Let $d := d(x_{2t})$. Note that $s < n-2t$, so $d > \delta$. For each $i < 2t$ we pick a set N_i of $d(x_i)-d$ neighbours of x_i in $V(G) \setminus (X^{\min} \cup \{x_1,\dots,x_{2t}\})$ such that these sets N_i are pairwise disjoint. To see that this can be done, suppose that we have already chosen $N_1,\dots,N_i$ for some $i < 2t-1$ and that we now wish to choose N_{i+1}. Since $X^{\min} \cup \{x_1,\dots,x_{2t}\} \cup N_1 \cup \dots \cup N_i$ has size

$$s+2t+\sum_{j=1}^{i}|N_j| \leq s+2t+(2t-2)(\Delta(G)-\delta) \leq \delta-(\Delta(G)-\delta),$$

it follows that there are at least $\Delta(G)-\delta \geq d(x_i)-d$ possible vertices for N_{i+1}. So we can choose N_{i+1}. Let G_0 be the graph obtained from G by deleting the edges between x_i and N_i for each $i < 2t$. So $\Delta(G_0)=d$ and $\delta(G_0)=\delta$. Let $X_0^{\max}$ be the set of all those vertices which have degree d in G_0. Thus all of $x_1,\dots,x_{2t}$ lie in $X_0^{\max}$ and so $|X_0^{\max}| \geq 2t$. Let $X_0^{\min}$ be the set of all those vertices which have degree δ in G_0. Thus $X_0^{\min} \subseteq X^{\min} \cup \bigcup_{i=1}^{2t-1} N_i$ and so

$$|X_0^{\min}| \leq s+\sum_{i=1}^{2t-1}|N_i| \leq s+(2t-1)(\Delta(G)-\delta) =: s'. \tag{11}$$

We will now prove the following claim.

Claim. *Let $0 \leq i < d-\delta$. Suppose that G_i is a spanning subgraph of G with $\Delta(G_i)=d-i$ and $\delta(G_i)=\delta$. Suppose that the set $X_i^{\max}$ of all vertices of degree*

$\Delta(G_i)$ in G_i has size at least $2t$ and that the set $X_i^{\min}$ of all vertices of degree δ in G_i has size at most $s'+i$. Then the following holds:

(i) *If $i \leq d-\delta-2$ then G_i contains a spanning subgraph G_{i+1} with $\Delta(G_{i+1}) = d-i-1$, $\delta(G_i)=\delta$ and such that the set $X_{i+1}^{\max}$ of all vertices of degree $\Delta(G_{i+1})$ in G_{i+1} has size at least $2t$ and the set $X_{i+1}^{\min}$ of all vertices of degree δ in G_{i+1} has size at most $s'+i+1$.*

(ii) *If $i=d-\delta-1$ and if δ is even then G_i contains a δ-factor.*

(iii) *If $i=d-\delta-1$ and if δ is odd there is an optimal matching M in G_i such that G_i-M contains a $(\delta-1)$-factor.*

Clearly, if we apply the claim $d-\delta$ times, starting with G_0 then this gives a δ-factor or an optimal matching and a $(\delta-1)$-factor which are as required in the lemma. So it suffices to prove the claim. First apply $(*)$ to find an optimal matching M' in $G_i[X_i^{\max}]$. If $|X_i^{\max}|$ is even, let G_{i+1} be obtained from G_i by deleting the edges in M'. If $i \leq d-\delta-2$ then G_{i+1} satisfies (i). If $i=d-\delta-1$ then G_{i+1} is a δ-factor. So if δ is even then (ii) holds. Moreover, if δ is odd, this implies that n must be even and we can apply $(*)$ to find a perfect matching M in G_{i+1}. Deleting M gives a $(\delta-1)$-factor. So (iii) holds.

So we may assume that $|X_i^{\max}|$ is odd. Let $y \in X_i^{\max}$ be the vertex not covered by M'. If $i \leq d-\delta-2$, choose a neighbour y' of y such that $y' \notin X_i^{\min}$ and if $|X_i^{\max}|=2t$ then $y' \notin X_i^{\max}$. This is possible since

$$\delta \geq 2t+s+2t(\Delta(G)-\delta) \overset{(11)}{=} 2t+s'+\Delta(G)-\delta \geq 2t+(s'+d-\delta) > 2t+|X_i^{\min}|.$$

Then the graph G_{i+1} obtained from G_i by deleting M' and the edge yy' satisfies (i). Note that if $i=d-\delta-1$ then δ must be odd (since then $|X_i^{\max}|$ is odd and G_i has precisely $|X_i^{\max}|$ vertices of degree $\delta+1$ while all other vertices have degree δ). Choose any neighbour y' of y. Let G_i' be the graph obtained from G_i by deleting M' and yy'. Then y' has degree $\delta-1$ in G_i' and all other vertices have degree δ. But this implies that n is odd and so we can apply $(*)$ to find a matching M in G_i' which covers all vertices apart from y'. Deleting M from G_i' yields a $(\delta-1)$-factor. So (iii) holds. □

Similarly as at the beginning of Section 5.1, given $X \sim Bin(n-1,p)$, we write

- $b(r) := \mathbb{P}(X=r) = \binom{n-1}{r}p^r(1-p)^{n-1-r}$, and
- $B(m) := \mathbb{P}(X \leq m)$.

$b'(r)$ and $B'(m)$ are defined analogously for $X \sim Bin(n-2,p)$. The next lemma will be used to show that if $p \geq 1-n^{-2/3}$ then a.a.s. $G_{n,p}$ satisfies the assumptions of Lemma 5.5.

Lemma 5.6. *Suppose that $0<p=p(n) \leq n^{-2/3}$ is monotone. There exists an $\varepsilon>0$ such that a.a.s $G_{n,p}$ satisfies the following properties:*

(i) $\Delta(G_{n,p}) \leq 2n^{1/3}$.

(ii) *Either at most $(1-\varepsilon)n$ vertices in $G_{n,p}$ have degree $\Delta(G_{n,p})$ or at least εn vertices of $G_{n,p}$ are isolated (or both).*

Obviously (ii) is very crude, but we were not able to find an explicit statement in the literature which implies (ii), so we include a proof for completeness.

Proof. Condition (i) is an extremely weak version of Corollary 3.4 in [8]. In order to check (ii), let $q := 1-p$. Suppose first that $pqn \to \infty$ as $n \to \infty$. Let Y denote the number of vertices of degree at most pn in $G_{n,p}$. Then the de Moivre-Laplace theorem implies that we have $\mathbb{E}(Y) \in [n/3, 2n/3]$. Note that

$$\mathbb{E}_2(Y) = n(n-1)(pB'(np-1)^2 + qB'(np)^2) \leq n^2 B'(np)^2.$$

Moreover, a straightforward calculation shows that $b'(r)/b(r) \leq 1+2p$. Similarly as in (5) one can use this to show that

$$\sqrt{\mathbb{E}_2(Y)}/\mathbb{E}(Y) \leq 1+2p.$$

Similarly as in (6) it follows that $Var(Y) \leq 5p\mathbb{E}(Y)^2 + \mathbb{E}(Y)$. Then Chebyshev's inequality implies that

$$\mathbb{P}(Y \neq (1 \pm 1/6)\mathbb{E}(Y)) \leq \frac{36Var(Y)}{\mathbb{E}(Y)^2} \leq 36 \cdot 5p + \frac{36}{\mathbb{E}(Y)} \to 0,$$

as $n \to \infty$. But if $Y = (1 \pm 1/6)\mathbb{E}(Y)$, then $Y \in [5n/18, 14n/18]$. This implies that there are at most $n - Y \leq 13n/18$ vertices of maximum degree.

So suppose next that $pqn \not\to \infty$. Let Z denote the number of isolated vertices in $G_{n,p}$. Thus there is a constant ε with $0 < \varepsilon < 1$ so that

$$\mathbb{E}(Z) = n(1-p)^{n-1} \geq ne^{-(p-p^2)n} \geq ne^{-pn/2} \geq 2\varepsilon n.$$

Moreover,

$$\mathbb{E}_2(Z) = n(n-1)(1-p)b'(0)^2 \leq n^2 b'(0)^2 \leq n^2(1+2p)^2 b(0)^2 = (1+2p)^2\mathbb{E}(Z)^2.$$

As before, we can use Chebyshev's inequality to show that the probability that at most εn vertices are isolated (i.e. that $Z \leq \varepsilon n$) tends to 0, as $n \to \infty$. □

We can now combine Lemmas 5.5 and 5.6 to prove Lemma 5.4.

Proof of Lemma 5.4. If $pn(1-p)/(\log n)^6 \to \infty$, then Lemma 5.4 is a special case of Lemma 21 in [30] (see the remarks after the proof of that lemma to see that its conditions are satisfied for such p). Thus we may assume that $p \geq 1 - n^{-2/3}$ (with room to spare). Let $\delta := \delta(G_{n,p})$. Applying Lemma 5.6 to the complement of $G_{n,p}$ shows that a.a.s. $\delta \geq n - 1 - 2n^{1/3} \geq n - 3n^{1/3}$ and there exists an $\varepsilon > 0$ such that at least one of the following conditions hold:

- At most $(1-\varepsilon)n$ vertices of $G_{n,p}$ have degree δ.
- At least εn vertices of $G_{n,p}$ have degree $n-1$.

Thus either $G_{n,p}$ is complete (in which case the lemma holds) or at most $(1-\varepsilon)n =: s$ vertices have degree δ. Let $t := 3n^{1/3}$. Then

$$s + 3t + 2t(\Delta(G_{n,p}) - \delta) \leq (1-\varepsilon)n + 9n^{1/3} + 6n^{1/3}(n - (n - 3n^{1/3})) < n.$$

Thus Lemma 5.5 implies a.a.s. $G_{n,p}$ satisfies (i) or (ii) of Lemma 5.4. □

5.3. **Hamilton decompositions of quasi-random graphs.** Our aim of this section is to observe that if λ is sufficiently small then every (n,d,λ)-graph is a robust outexpander. We need the following notation. Given sets A,B of vertices of a graph G, we let $e'(A,B)$ denote the set of ordered pairs of vertices a,b such that $a\in A$, $b\in B$ and ab is an edge of G. (So if $A=B$, we have $e'(A,A)=2e(A,A)$.) The following well-known result (see e.g. Corollary 2.5 in Chapter 9 of [4]) shows that for (n,d,λ)-graphs with small λ, $e'(A,B)$ is close to its 'expected value'.

Theorem 5.7. *Suppose that G is an (n,d,λ)-graph. Then for every pair A,B of nonempty sets of vertices we have*

$$\left|\frac{e'(A,B)}{|A||B|}-\frac{d}{n}\right|\leq\frac{\lambda}{\sqrt{|A||B|}}.$$

Lemma 5.8. *Suppose that $1/n\ll\varepsilon\ll\nu\ll\tau,\alpha<1$. Suppose that G is an (n,d,λ)-graph with $\lambda\leq\varepsilon n$ and $d\geq\alpha n$. Then G is a robust (ν,τ)-expander.*

Proof. Consider any set S with $\tau n\leq|S|\leq(1-\tau)n$. Let $RN=RN_\nu(S)$. Then

$$e'(S,RN)\geq|S|d-\nu n^2\geq(1-\tau/4)|S|d.$$

Since clearly $e'(S,RN)\leq|S||RN|$ it follows that $|RN|\geq d/2$. Thus $\lambda/\sqrt{|S||RN|}\leq\sqrt{\varepsilon}$. Together with Theorem 5.7 this implies that

$$e'(S,RN)\leq(d/n+\sqrt{\varepsilon})|S||RN|\leq(d+\sqrt{\varepsilon}n)(1-\tau)|RN|\leq(1-\tau/2)d|RN|.$$

Altogether, this implies that $|RN|\geq(1+\tau/4)|S|\geq|S|+\nu n$, as required. □

Proof of Theorem 1.11. Let $\tau:=\tau(\alpha)$, where $\tau(\alpha)$ is as defined in Theorem 1.2. Choose new constants $n_0\in\mathbb{N}$ and ν,ε such that $0<1/n_0\ll\varepsilon\ll\nu\ll\tau,\alpha$. Consider any (n,d,λ)-graph G on $n\geq n_0$ vertices with $\lambda\leq\varepsilon n$ and such that $d\geq\alpha n$ is even. Then Lemma 5.8 implies that G is a robust (ν,τ)-expander. Thus Theorem 1.2 implies that G has a Hamilton decomposition. □

6. Algorithmic aspects and the Regularity lemma

In this section, we use Szemerédi's regularity lemma [48] to provide an algorithm which checks robust expansion and to give an algorithmic proof of Theorem 1.2.

6.1. **An algorithm for checking robust expansion.** The purpose of this subsection is to prove Theorem 1.15, i.e. our aim is to show that one can decide in polynomial time whether a digraph G is a robust outexpander. More precisely, there is a polynomial algorithm which either decides that G is not a robust (ν,τ)-outexpander or that G is a robust outexpander with slightly worse parameters.

Roughly speaking, in order to prove Theorem 1.15 we apply the regularity lemma to G to obtain a reduced digraph R and check whether R is a robust outexpander. The latter can be done in constant time. Lemma 6.3 shows that if R is a robust outexpander, then G is a robust outexpander (with slightly worse parameters). Lemma 6.2 shows that if G is a robust outexpander, then R is one as well.

Before we can state the regularity lemma for digraphs, we need the following notation. If G is an undirected bipartite graph with vertex classes X and Y, then the *density* of G is defined as

$$d(X,Y) := \frac{e(X,Y)}{|X||Y|}.$$

Given $\varepsilon > 0$, we say that G is *ε-regular* if for any $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \geq \varepsilon|X|$ and $|Y'| \geq \varepsilon|Y|$ we have $|d(X',Y') - d(X,Y)| < \varepsilon$. G is *(ε,d)-regular* if G is ε-regular and has density $d \pm \varepsilon$.

Given disjoint vertex sets X and Y in a digraph G, we use $G[X,Y]$ to denote the bipartite subdigraph of G whose vertex classes are X and Y and whose edges are all the edges of G directed from X to Y. We often view $G[X,Y]$ as an undirected bipartite graph. In particular, we say $G[X,Y]$ is *ε-regular with density d* if this holds when $G[X,Y]$ is viewed as an undirected graph.

Next we state the degree form of the regularity lemma for (di-)graphs. A regularity lemma for digraphs was proven by Alon and Shapira [3]. The degree form follows from this in the same way as the undirected version (see [37] for a sketch of the latter). An algorithmic version of the (undirected) regularity lemma was proved in [1]. An algorithmic version of the directed version can be proved in essentially the same way (see [11] for a sketch of the argument proving a similar statement).

Lemma 6.1 (Regularity lemma for (di-)graphs)**.** *For any ε, M' there exist M, n_0 such that if G is a digraph on $n \geq n_0$ vertices and $d \in [0,1]$, then there exists a partition of $V(G)$ into $V_0, \dots, V_k$ and a spanning subdigraph G' of G such that the following conditions hold:*

(i) *$M' \leq k \leq M$.*
(ii) *$|V_0| \leq \varepsilon n$.*
(iii) *$|V_1| = \dots = |V_k| =: m$.*
(iv) *$d^+_{G'}(x) > d^+_G(x) - (d+\varepsilon)n$ for all vertices $x \in V(G)$.*
(v) *$d^-_{G'}(x) > d^-_G(x) - (d+\varepsilon)n$ for all vertices $x \in V(G)$.*
(vi) *For all $i \in [k]$ the digraph $G'[V_i]$ is empty.*
(vii) *For all $1 \leq i,j \leq k$ with $i \neq j$ the pair $G'[V_i,V_j]$ is ε-regular and either has density 0 or density at least d.*

The analogue also holds when G is an undirected graph, in which case (iv) and (v) are replaced by the condition that $d_{G'}(x) > d_G(x) - (d+\varepsilon)n$ for all vertices $x \in V(G)$.

Note that in the directed case the densities of the pairs $G'[V_i,V_j]$ and $G'[V_j,V_i]$ might be different from each other. We refer to V_0 as the *exceptional set* and to $V_1, \dots, V_k$ as *clusters*. $V_0, V_1, \dots, V_k$ as above is also called a *regularity partition* and G' is called the *pure (di-)graph*. Given a digraph G on n vertices, we form the *reduced digraph R of G with parameters ε, d and M'* by applying the regularity lemma with these parameters to obtain $V_0, \dots, V_k$. R is then the digraph whose vertices are the clusters $V_1, \dots, V_k$ and whose edges are those (ordered) pairs V_iV_j

of clusters for which $G'[V_i, V_j]$ is nonempty. The *reduced graph* R of an undirected graph G is defined in a similar way.

The next result from [46] implies that the property of a digraph G being a robust outexpander is 'inherited' by the reduced digraph R of G. It also shows that this even holds for spanning subdigraphs R' of R whose edges correspond to all those pairs which have density a little larger than d. The latter statement will be used in Section 6.2. In [46] the lemma is only stated in the directed case, but the argument for the undirected case is identical. (A weaker version of this lemma was already proved in [40].)

Lemma 6.2. *Suppose that $1/n \ll 1/M' \ll \varepsilon \ll d \leq d' \leq \nu \leq \tau < 1$ and $d' \leq \nu/20$. Let G be a digraph on n vertices which is a robust (ν, τ)-outexpander. Let R be the reduced digraph of G with parameters ε, d and M' and let G' be the pure digraph. Let R' be the spanning subdigraph of R such that $E(R')$ consists of all those edges $V_iV_j \in E(R)$ for which $G'[V_i, V_j]$ has density at least d'. Then R' is a robust $(\nu/4, 3\tau)$-outexpander. The analogue also holds if G is an undirected robust (ν, τ)-expander.*

The following lemma gives a converse to this (but with a weaker bound).

Lemma 6.3. *Suppose that $1/n \ll \varepsilon, 1/k \ll d, \nu, \tau < 1$. Let G be a digraph on n vertices. Let $V_0, V_1, \ldots, V_k$ be a partition of $V(G)$ with $|V_0| \leq \varepsilon n$ and $|V_1| = \cdots = |V_k| =: m$. Suppose that R is a digraph whose vertices are $V_1, \ldots, V_k$ such that for every edge V_iV_j of R the bipartite graph $G[V_i, V_j]$ is ε-regular of density at least d. If R is a robust (ν, τ)-outexpander, then G is a robust $(d\nu^2/8, 2\tau)$-outexpander. Similarly, if G is undirected and R is a robust (ν, τ)-expander, then G is a robust $(d\nu^2/8, 2\tau)$-expander.*

Proof. We only prove the directed version, the argument for the undirected case is similar. Consider any set $S \subseteq V(G)$ with $2\tau n \leq |S| \leq (1-2\tau)n$. Let S_R be the set of all those clusters V_i for which $|V_i \cap S| \geq \nu m/5$. Note that

$$|S_R| \geq (|S| - \nu mk/5 - |V_0|)/m \geq |S|/m - \nu k/4. \tag{12}$$

So in particular $|S_R| \geq \tau k$. Let $RN(S_R) := RN^+_{\nu,R}(S_R)$. Since R is a robust (ν, τ)-outexpander, it follows that at least one of the following two properties hold:

(a) $|RN(S_R)| \geq |S_R| + \nu k$.
(b) $|S_R| > (1-\tau)k$ and therefore $|RN(S_R)| \geq (1-\tau+\nu)k$.

(The second part of (b) follows by considering the robust outneighbourhood of a subset of S_R of size $(1-\tau)k$.) Let N_G denote the set of vertices of G lying in clusters of $RN(S_R)$. If (a) holds then

$$|N_G| \geq |S_R|m + \nu km \overset{(12)}{\geq} |S| - \nu n/4 + 3\nu n/4 = |S| + \nu n/2.$$

If (b) holds then

$$|N_G| \geq (1-\tau+\nu)km \geq (1-\tau)n \geq |S| + \nu n/2.$$

Now let $N^* := N_G \backslash RN^+_{d\nu^2/8,G}(S)$. We will show that $|N^*| \leq \nu n/4$. (Together with the bound on $|N_G|$ this then implies the lemma.) So suppose that $|N^*| \geq \nu n/4$. Then there is a cluster V_i which contains at least $\nu m/4$ vertices of N^*. (So $V_i \in RN(S_R)$.) Let $V_i^* := V_i \cap N^*$. Since no vertex in V_i^* lies in the robust outneighbourhood $RN^+_{d\nu^2/8,G}(S)$ of S, the number of edges from S to V_i^* in G is at most $d\nu^2 n|V_i^*|/8$. On the other hand, since $V_i \in RN(S_R)$, the number of edges in R from S_R to V_i is at least νk. Let S_R^* denote the initial vertices (i.e. the initial clusters) of these edges. Since $S_R^* \subseteq S_R$, every $V_j \in S_R^*$ satisfies $|V_j \cap S| \geq \nu m/5$. Together with the fact that the edge $V_jV_i \in E(R)$ corresponds to an (ε, d)-regular pair $G[V_i, V_j]$ in G and $|V_i^*| \geq \nu m/4 \geq \varepsilon m$, it follows that the number of edges in G from $V_j \cap S$ to V_i^* is at least $5d|V_j \cap S||V_i^*|/6 \geq d\nu m|V_i^*|/6$. Summing over all clusters $V_j \in S_R^*$ shows that the number of edges in G from S to V_i^* is at least $\nu k \cdot d\nu m|V_i^*|/6 \geq d\nu^2 n|V_i^*|/7$, a contradiction. □

Proof of Theorem 1.15. Let $d := \nu/20$. Choose $\varepsilon > 0$ and $n_0, M' \in \mathbb{N}$ such that $1/n_0 \ll \varepsilon, 1/M' \ll d$. We may assume that $n \geq n_0$ (since otherwise we can solve the problem by complete enumeration). Apply the regularity lemma (Lemma 6.1) with parameters ε, d, M' to G obtain a reduced digraph R and a pure digraph G'. Now check whether R is a robust $(\nu/4, 3\tau)$-outexpander (this can be done in constant time). If it is not a robust $(\nu/4, 3\tau)$-outexpander, then Lemma 6.2 (with R and d playing the roles of R' and d') implies that G is not a robust (ν, τ)-outexpander. If R is a robust $(\nu/4, 3\tau)$-outexpander, then Lemma 6.3 (with G' playing the role of G) implies that G' is a robust $(\nu^3/2560, 6\tau)$-outexpander. Since $G' \subseteq G$, this shows that G is a robust $(\nu^3/2560, 6\tau)$-outexpander. The algorithm for (undirected) robust expanders is similar. □

6.2. **Regular orientations – algorithmic proof of Lemma 3.6.** The purpose of this subsection is to give an algorithmic proof of the following version of Lemma 3.6.

Lemma 6.4. *Suppose that $0 < 1/n \ll \nu' \ll \nu \leq \tau \ll \alpha < 1$. Let G be an r-regular graph on n vertices such that $r \geq \alpha n$ is even and G is a robust (ν, τ)-expander. Then one can orient the edges of G in such a way that the oriented graph G^{orient} thus obtained from G is an $r/2$-regular robust $(\nu', 6\tau)$-outexpander. Moreover, this orientation G^{orient} can be found in time polynomial in n.*

Note that Lemma 3.6 guarantees that G^{orient} is robust (ν', τ)-outexpander, instead of just a robust $(\nu', 6\tau)$-outexpander. But this does not matter for the applications. In particular, using a similar argument as before, Lemma 6.4 immediately gives an algorithmic proof of Theorem 1.2, as the only other ingredient is the (algorithmic) Theorem 1.1.

The proof idea in this case is to consider a suitable 'fractional' and 'almost regular' orientation of the reduced graph R of G obtained from the regularity lemma. This orientation is 'lifted' to G and minor 'irregularities' are eliminated by re-orienting a small proportion of the edges (via Lemma 6.6).

We will need the following observations, which follow immediately from the definition of robust outexpansion.

Proposition 6.5. *Suppose that $0 < 1/n \ll \beta \ll \nu \leq \tau \ll \alpha < 1$ and that G is a robust (ν, τ)-outexpander on n vertices.*

(i) *Suppose that G' is obtained from G by reversing the orientations of at most βn^2 edges. Then G' is a robust $(\nu/2, \tau)$-outexpander.*
(ii) *Suppose that $\delta^0(G) \geq \alpha n$. Then G has diameter at most $1/\nu$.*

For a vertex x in an oriented graph G, let $\mathrm{disc}(x, G) := |d^+_G(x) - d^-_G(x)|$. For an oriented graph G, let $\mathrm{disc}(G) := \sum_{x \in V(G)} \mathrm{disc}(x, G)$. So G is regular if and only if $\mathrm{disc}(G) = 0$.

Lemma 6.6. *Suppose that $0 < 1/n \ll \beta \ll \nu \leq \tau \ll \alpha < 1$ and that $\alpha n \in \mathbb{N}$. Let G be a $2\alpha n$-regular graph on n vertices which has an orientation G' which is a robust (ν, τ)-outexpander with $\mathrm{disc}(G') \leq \beta n^2$. Then G has an orientation G'' which is an αn-regular robust $(\nu/2, \tau)$-outexpander. Moreover, this orientation can be found in time polynomial in n.*

The idea of the proof is to repeatedly apply the following argument: as long as there is a vertex x whose outdegree is less than αn, then there must also be a vertex y whose indegree is greater than αn. We find a short directed path P from x to y and reverse the orientation of the edges on P.

Proof. We say that a vertex x in an oriented graph H is *H-balanced* if $\mathrm{disc}(x, H) \leq \alpha n$. Let B be the set of G'-balanced vertices. Note that $\mathrm{disc}(G') \leq \beta n^2$ implies that

$$|B| \geq (1 - \sqrt{\beta})n. \tag{13}$$

Let $G_0 := G'$ and consider the following statements, where $0 \leq i \leq \beta n^2/2$:

(a_i) If $i \geq 1$, then G_i is an oriented graph obtained from G_{i-1} by reversing the orientations of the edges of a directed path P_{i-1} of length at most $4/\nu$ in G_{i-1}.
(b_i) $\mathrm{disc}(x, G_i) \leq \mathrm{disc}(x, G_0)$ for all $x \in V(G)$.
(c_i) $\mathrm{disc}(G_i) \leq \beta n^2 - 2i$.

Note that (a_0),(b_0) vacuously hold and (c_0) holds by assumption.

Suppose inductively that for some $i \geq 0$ we have constructed $G_0, \ldots, G_i$ such that (a_i)–(c_i) hold. Suppose also that there is a vertex x with $\mathrm{disc}(x, G_i) > 0$. Without loss of generality, we have $d^+_{G_i}(x) > d^-_{G_i}(x)$. So $d^+_{G_i}(x) > \alpha n$ and thus x has an outneighbour $x' \in B$ (in G_i). If $x \in B$, let $x^* := x$, otherwise, let $x^* := x'$. There must also be a vertex y with $d^-_{G_i}(y) > d^+_{G_i}(y)$. So $d^-_{G_i}(y) > \alpha n$ and y has an inneighbour $y' \in B$ (in G_i). If $y \in B$, let $y^* := y$, otherwise, let $y^* := y'$.

Note that (b_i) implies that any $b \in B$ is G_i-balanced. So it follows from (13) that $\delta^0(G_i[B]) \geq \alpha n/3 \geq \alpha|B|/3$. Moreover, (13) implies that $G_i[B]$ is a robust $(\nu/2, 2\tau)$-outexpander. So by Proposition 6.5(ii), $G_i[B]$ contains a (directed) x^*y^*-path P'_i of length at most $2/\nu$. We now extend P'_i into an xy-path P_i in G_i of length at most $2/\nu + 2 \leq 4/\nu$ (where we may have $P'_i = P_i$). Let

G_{i+1} be obtained from G_i by reversing the orientations of the edges along P_i. Then $\mathrm{disc}(x, G_{i+1}) < \mathrm{disc}(x, G_i)$, $\mathrm{disc}(y, G_{i+1}) < \mathrm{disc}(y, G_i)$ and $\mathrm{disc}(z, G_{i+1}) = \mathrm{disc}(z, G_i) \leq \mathrm{disc}(z, G_0)$ for any vertex $z \neq x, y$. So (a_{i+1})–(c_{i+1}) hold.

So we must have $\mathrm{disc}(G_s) = 0$ for some $s \leq \beta n^2/2$. Let $G'' := G_s$. By definition, G'' is αn-regular. Moreover, G'' is obtained from G' by reversing the orientations of at most $2\beta n^2/\nu \leq \sqrt{\beta} n^2$ edges. Proposition 6.5(i) now implies that G' is a robust $(\nu/2, \tau)$-outexpander, as required. Moreover, each P_i (and thus G'') can be found in polynomial time, since the shortest path problem can be solved in polynomial time. □

The following lemma follows immediately from Lemma 4.10(iii) in [39]. As remarked after its proof in [39], the proof of this lemma is algorithmic.

Lemma 6.7. *Suppose that $0 < 1/m \ll \varepsilon \ll d' \leq d \leq 1$. Let G be an ε-regular bipartite graph of density d with vertex classes of size m. Then G contains an $(\varepsilon^{1/12}, d')$-regular spanning subgraph J. Moreover, if $x \in V(G)$ satisfies $d_G(x) = (d \pm \varepsilon)m$, then $d_J(x) = (d' \pm \sqrt{\varepsilon})m$.*

Proof of Lemma 6.4. Choose $\varepsilon, d, d_1 > 0$ and $M' \in \mathbb{N}$ such that $1/n \ll 1/M' \ll \varepsilon \ll d \ll \nu' \ll d_1 \ll \nu$. Apply the undirected version of the regularity lemma (Lemma 6.1) with parameters ε, d, M' to G obtain a reduced graph R and a pure graph G'. For each edge XY of R, let d_{XY} denote the density of $G'[X, Y]$ and let $d'_{XY} := d_{XY}/2$. Let R^* be the digraph obtained from R by replacing each (undirected) edge XY of R by the directed edges XY and YX. Apply Lemma 6.7 with $G'[X, Y]$, d_{XY} and d'_{XY} playing the roles G, d and d' to obtain an $(\varepsilon^{1/12}, d'_{XY})$-regular spanning subgraph J_{XY} of $G'[X, Y]$. We say that $x \in X \cup Y$ is *XY-useful* if $d_{G'[XY]}(x) = (d_{XY} \pm \varepsilon)m$. So if x is XY-useful, then by Lemma 6.7 we have $d_{J_{XY}}(x) = (d'_{XY} \pm \sqrt{\varepsilon})m$.

Let $J_{YX} := G'[X, Y] \setminus E(J_{XY})$. Note that the above implies that J_{YX} is $(2\varepsilon^{1/12}, d'_{XY})$-regular and that if x is XY-useful, then $d_{J_{YX}}(x) = (d'_{XY} \pm 2\sqrt{\varepsilon})m$. For each edge XY of R, orient all edges in J_{XY} from X to Y and orient all edges in J_{YX} from Y to X.

We say that an edge of G is *good* if it lies in $G'[X, Y]$ for some edge XY of R and *bad* otherwise. The above gives an orientation of all good edges of G. Orient the bad edges of G arbitrarily to obtain an oriented graph G^*. Let R_1 be the spanning subgraph of R such that $E(R_1)$ consists of all those edges $XY \in E(R)$ for which $d_{XY} \geq d_1$. By the undirected version of Lemma 6.2, R_1 is a robust $(\nu/4, 3\tau)$-expander. Let R_1^* be the digraph obtained from R_1 by replacing each (undirected) edge XY of R_1 by the directed edges XY and YX. Clearly, it follows that R_1^* is a robust $(\nu/4, 3\tau)$-outexpander. Moreover, note that for any edge XY of R_1^*, the density of the corresponding pair $G^*[X, Y]$ is at least $d_1/3$. Thus we can apply the directed version of Lemma 6.3 with $G^*, R_1^*, d_1/3$ playing the roles of G, R, d to see that G^* is a robust $(d_1\nu^2/(3 \cdot 128), 6\tau)$-outexpander and thus also a robust $(2\nu', 6\tau)$-outexpander.

We claim that $\mathrm{disc}(G^*) \leq 3dn^2$. To prove the claim, for any vertex $x \in V(G)$, let $b(x)$ denote the number of bad edges incident to x. Note that Lemma 6.1

implies that

$$\sum_{x\in V(G)} b(x) \leq n(\varepsilon+d)n + |V_0|n \leq 2dn^2. \tag{14}$$

Let G^*_{good} be the oriented subgraph of G^* consisting of (orientations of) the good edges. Suppose that $x \in X$ for some cluster X and that x is XY-useful. Then our construction implies that

$$\left| |N^+_{G^*_{\text{good}}}(x) \cap Y| - |N^-_{G^*_{\text{good}}}(x) \cap Y \right| \leq 3\sqrt{\varepsilon}m. \tag{15}$$

Note that for a given edge XY of R, the number of vertices $x \in X$ which are not XY-useful is at most $2\varepsilon m$. We say that a vertex $x \in X$ is *useful* if it is XY-useful for at least at least $(1-\sqrt{\varepsilon})|R|$ clusters Y. Suppose that x is useful. Then (15) implies that

$$\text{disc}(x, G^*) \leq b(x) + 3\sqrt{\varepsilon}m|R| + \sqrt{\varepsilon}m|R| \leq b(x) + 4\sqrt{\varepsilon}n. \tag{16}$$

Note that the total number of vertices which are not useful is at most $|V_0| + 2\sqrt{\varepsilon}|R|m \leq 3\sqrt{\varepsilon}n$. Also, if x is not useful, we have $\text{disc}(x, G^*) \leq n$. So (14) and (16) together imply that $\text{disc}(G^*) \leq 2dn^2 + 4\sqrt{\varepsilon}n^2 + 3\sqrt{\varepsilon}n^2 \leq 3dn^2$. So we can now apply Lemma 6.6 with $G^*, 3d, 2\nu', 6\tau, r/2n$ playing the roles of $G', \beta, \nu, \tau, \alpha$ to obtain an $r/2$-regular orientation of G^{orient} of G which is a robust $(\nu', 6\tau)$-outexpander. □

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