The sigma form of the second Painlevé hierarchy

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THE SIGMA FORM OF THE SECOND PAINLEVÉ HIERARCHY.

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Abstract. In this paper we study the so-called sigma form of the second Painlevé hierarchy. To obtain this form, we use some properties of the Hamiltonian structure of the second Painlevé hierarchy and of the Lenard operator.

Contents

1. Introduction

The Painlevé differential equations were discovered more than a hundred years ago and since the eighties have appeared in many branches on mathematics and physics, including several of Dubrovin's seminal papers on Frobenius manifolds.

The reason behind the ubiquitous appearance of these equations is that they are innately linked to the Toda hierarchy. In [6], Dubrovin and Zhang proved that the tau function of a generic solution to the extended Toda hierarchy is annihilated by some combinations of the Virasoro operators. It is such Virasoro constraints that regulate the correlation functions of many systems in random matrix theory, in string theory and topological field theory. For example in [7], expressions for the genus $g \geq 1$ total Gromov–Witten potential were obtained via the genus zero quantities derived from the Virasoro constraints.

The link between Toda–type systems and Painlevé equations becomes explicit when the latter are re-formulated in the so called *sigma form* introduced in [10, 9, 8] as the equation satisfied by the logarithmic derivative of the isomonodromic τ–function. An other approach to obtain the sigma form was proposed by Okamoto who developed the Hamiltonian theory of the Painlevé differential equations and showed that all Bäcklund transformations can be obtained as natural affine Weyl groups actions on the sigma form ([14], [15], [13], [16]).

In this paper, we present the sigma form for the second Painlevé hierarchy, an infinite sequence of non linear ODEs containing

$$\mathrm{P_{II}}: \qquad w''(z) = 2w(z)^3 + z\,w(z) + \alpha_1,$$

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as its simplest equation. Further in the paper, we will omit the argument z and use $'$ as the derivative w.r.t. z.

The n-th element is of order $2n$, and depends on n parameters denoted by $t_1, \dots, t_{n-1}$ and α_n:

(1)
$$\mathrm{P}_{\mathrm{II}}^{(n)}: \left(\frac{\mathrm{d}}{\mathrm{d}z} + 2w\right)\mathcal{L}_n\left[w' - w^2\right] + \sum_{l=1}^{n-1} t_l \left(\frac{\mathrm{d}}{\mathrm{d}z} + 2w\right)\mathcal{L}_l\left[w' - w^2\right] = zw + \alpha_n,\ n \geq 1,$$

where $\mathcal{L}_n$ is the operator defined by the recursion relation

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{L}_{n+1} = \left(\frac{\mathrm{d}^3}{\mathrm{d}z^3} + 4\left(w' - w^2\right)\frac{\mathrm{d}}{\mathrm{d}z} + 2\left(w' - w^2\right)'\right)\mathcal{L}_n, \quad \mathcal{L}_0\left[w' - w^2\right] = \tfrac{1}{2}, \tag{2}$$

with boundary condition

$$\mathcal{L}_n[0] := 0, \quad \forall n \geq 1. \tag{3}$$

The second Painlevé hierarchy is often presented with $t_1 = \dots = t_{n-1} = 0$ [4, 11, 3]. We will not fix these parameters as it was considered in [12].

The Hamiltonian form of the second Painlevé hierarchy was produced in [12] where the authors gave canonical coordinates $P_1, \dots, P_n, Q_1, \dots, Q_n$ and a Hamiltonian function $\mathcal{H}^{(n)}$ such that $\mathrm{P}_{\mathrm{II}}^{(n)}$ is equivalent to

$$\frac{\partial Q_i}{\partial z} = \frac{\partial \mathcal{H}^{(n)}}{\partial P_i}, \qquad \frac{\partial P_i}{\partial z} = -\frac{\partial \mathcal{H}^{(n)}}{\partial Q_i}, \quad i = 1, \dots, n. \tag{4}$$

In particular $\mathcal{H}^{(n)}$ is a polynomial in $P_1, \dots, P_n, Q_1, \dots, Q_n$ and that the Hamiltonian equations satisfy the Painlevé property.

The sigma function is by definition the evaluation of the Hamiltonian on solutions, namely

$$\sigma_n(z) := \mathcal{H}^{(n)}\left(P_1(z), \dots, P_n(z), Q_1(z), \dots, Q_n(z)\right). \tag{5}$$

Our main result in this paper is the following

Theorem 1.1. *Consider the Lenard operators* $\widehat{\mathcal{L}}_k$ *defined by*

$$\frac{\mathrm{d}}{\mathrm{d}z}\widehat{\mathcal{L}}_{k+1}\left[\sigma_n' - \frac{t_{n-1}}{2}\right] = \left(\frac{\mathrm{d}^3}{\mathrm{d}z^3} + 2\left(2\sigma_n' - t_{n-1}\right)\frac{\mathrm{d}}{\mathrm{d}z} + 2\sigma_n''\right)\widehat{\mathcal{L}}_k\left[\sigma_n' - \frac{t_{n-1}}{2}\right], \tag{6}$$
$$\widehat{\mathcal{L}}_0\left[\sigma_n' - \frac{t_{n-1}}{2}\right] = \tfrac{1}{2}, \quad t_0 = -z,$$

with the boundary condition

$$\widehat{\mathcal{L}}_k[0] := \left(-\frac{t_{n-1}}{2}\right)^k \frac{1}{k}\begin{pmatrix} 2k \\ k \end{pmatrix}, \tag{7}$$

and define

$$f_n = \sum_{l=1}^{n} t_l \widehat{\mathcal{L}}_l\left[\sigma_n'(z) - \frac{t_{n-1}}{2}\right], \quad t_n = 1. \tag{8}$$

Then, for $n > 1$, the n-th element of the second Painlevé hierarchy (1) *is equivalent to*

(9)
$$-f_n' + (f_n')^2 + (z - 2f_n)\left(\sum_{l=1}^{n} t_l\left(\widehat{\mathcal{L}}_l'' - \widehat{\mathcal{L}}_{l+1}\right) + 2f_1 f_n + \sigma_n - \frac{1}{2}t_{n-1}\, z\right) = \alpha_n\left(\alpha_n - 1\right).$$

Remark 1.1. We note that an other equation of order $2n+1$ for the sigma function of the n-the element of the second Painlevé hierarchy was produced by Stuart Andrew, a former master student of the second author in integral form [1]. The formula (9) in this paper is an explicit ODE of order $2n$ as expected (see Lemma 3.3 in Section 3).

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2. Hamiltonian structure of the second Painlevé hierarchy

In this section we resume some results in [12] that turn out to be useful in our proof of Theorem 1.1.

Let us consider the isomonodromic deformation problem for the $\mathrm{P}_{\mathrm{II}}^{(n)}$ hierarchy

$$\frac{\partial\Psi}{\partial z}=\mathcal{B}\Psi=\begin{pmatrix}-\lambda & w\\ w & \lambda\end{pmatrix}\Psi,$$

$$\frac{\partial\Psi}{\partial\lambda}=\mathcal{A}^{(n)}\Psi=\frac{1}{\lambda}\left[\begin{pmatrix}-\lambda z & -\alpha_n\\ -\alpha_n & \lambda z\end{pmatrix}+M^{(n)}+\sum_{l=1}^{n-1}t_l M^{(l)}\right],$$

$$(2k+1)\frac{\partial\Psi}{\partial t_k}=\left(M^{(k)}-\begin{pmatrix}0 & (\partial_z+2w)\,\mathcal{L}_k\left[w'-w^2\right]\\ (\partial_z+2w)\,\mathcal{L}_k\left[w'-w^2\right] & 0\end{pmatrix}\right)\Psi,$$

where the matrix $M^{(l)}$ is defined as

$$M^{(l)}=\begin{pmatrix}\displaystyle\sum_{j=1}^{2l+1}A_j^{(l)}\lambda^j & \displaystyle\sum_{j=1}^{2l}B_j^{(l)}\lambda^j\\ \displaystyle\sum_{j=1}^{2l}C_j^{(l)}\lambda^j & \displaystyle-\sum_{j=1}^{2l+1}A_j^{(l)}\lambda^j\end{pmatrix},$$

with

(10)

$$A_{2l+1}^{(l)}=4^l;\qquad A_{2k}^{(l)}=0,\quad k=0,\dots,l;$$

$$A_{2k+1}^{(l)}=\frac{4^{k+1}}{2}\left\{\mathcal{L}_{l-k}\left[w'-w^2\right]-\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}}{\mathrm{d}z}+2w\right)\mathcal{L}_{l-k-1}\left[w'-w^2\right]\right\},\quad k=0,\dots l-1;$$

$$B_{2k+1}^{(l)}=\frac{4^{k+1}}{2}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}}{\mathrm{d}z}+2w\right)\mathcal{L}_{l-k-1}\left[w'-w^2\right],\quad k=0,\dots,l-1;$$

$$B_{2k}^{(l)}=-4^k\left(\frac{\mathrm{d}}{\mathrm{d}z}+2w\right)\mathcal{L}_{l-k}\left[w'-w^2\right],\quad k=1,\dots,l.$$

The compatibility condition

$$\frac{\partial\mathcal{A}^{(n)}}{\partial z}-\frac{\partial\mathcal{B}}{\partial\lambda}=\left[\mathcal{B},\mathcal{A}^{(n)}\right]$$

gives the n-th member of the second Painlevé hierarchy (1).

It is convenient to introduce new notations to define the matrix $\mathcal{A}^{(n)}$. Let us set

$$\begin{aligned} a_{2k+1}^{(n)} &= \sum_{l=1}^{n} t_l A_{2k+1}^{(l)}, \quad k=1,\dots,n; \qquad a_1^{(n)} = \sum_{l=1}^{n} t_l A_1^{(l)} - z; \\ b_{2k+1}^{(n)} &= \sum_{l=1}^{n} t_l B_{2k+1}^{(l)}, \quad k=1,\dots,n-1; \\ b_{2k}^{(n)} &= \sum_{l=1}^{n} t_l B_{2k}^{(l)}, \quad k=1,\dots,n; \qquad b_0^{(n)} = -\alpha_n, \quad t_n = 1. \end{aligned} \tag{11}$$

Therefore, $\mathcal{A}^{(n)}$ can be represented in the following form

$$\mathcal{A}^{(n)} = \begin{pmatrix} \sum_{k=0}^{n} a_{2k+1}^{(n)} \lambda^{2k} & \sum_{k=0}^{2n} b_k^{(n)} \lambda^{k-1} \\ \sum_{k=0}^{2n} (-1)^k b_k^{(n)} \lambda^{k-1} & -\sum_{k=0}^{n} a_{2k+1}^{(n)} \lambda^{2k} \end{pmatrix}.$$

The canonical coordinates are given by relations

$$P_k = \Pi_{2k} = \frac{a_{2(n-k)+1}^{(n)} + b_{2(n-k)+1}^{(n)}}{a_{2n+1}^{(n)}}, \quad Q_k = \sum_{j=1}^{n} \frac{1}{2j} b_{2j}^{(n)} \frac{\partial S_{2j}}{\partial \Pi_{2k}}, \quad k=1,\dots,n;$$

$$S_k = \sum_{j=1}^{2n} q_j^k, \quad k=1,\dots,2n;$$

$$\Pi_1 = q_1 + \dots + q_{2n}, \quad \Pi_2 = \sum_{1 \le j \le 2n} q_j q_k, \quad \dots, \quad \Pi_{2n} = q_1 q_2 \dots q_{2n},$$

where q_j are solutions of the following equation

$$\sum_{k=0}^{n-1} \left(b_{2k+1}^{(n)} + a_{2k+1}^{(n)} \right) q_j^{2k} + a_{2n+1}^{(n)} q_j^{2n} = 0,$$

and $p_j = \sum_{k=0}^{n} b_{2k}^{(n)} q_j^{2k-1}$.

The coordinates $P_1, \dots, P_n, Q_1, \dots, Q_n$ are canonical with the Poisson structure

$$\{P_i, P_j\} = \{Q_i, Q_j\} = 0, \quad \{P_i, Q_j\} = \delta_{ij}, \quad i,j = 1,\dots,n.$$

The corresponding Hamiltonian in terms of the canonical coordinates is

$$\begin{aligned} \mathcal{H}^{(n)}(P_1,\dots,P_n,Q_1,\dots,Q_n,z) = -\frac{1}{4^n} \Bigg(& \sum_{l=0}^{n-1} a_{2l+1}^{(n)} a_{2(n-l)-1}^{(n)} - \sum_{l=0}^{n-1} b_{2l+1}^{(n)} b_{2(n-l)-1}^{(n)} \\ & + \sum_{l=0}^{n} b_{2l}^{(n)} b_{2(n-l)}^{(n)} \Bigg) + \frac{Q_n}{4^n}, \end{aligned} \tag{12}$$

where the coefficients of $\mathcal{A}^{(n)}$ can be expressed as polynomials in the canonical coordinates by theorem 6.1 in [12].

We conclude this section be reminding a useful formula valid both for the Lenard operators $\mathcal{L}_n$ and $\widehat{\mathcal{L}}_n$ [12]:

$$\begin{aligned}\mathcal{L}_{n+1} &= \mathcal{L}_n'' + 3\mathcal{L}_n\mathcal{L}_1 \\ &\quad + \sum_{j=1}^{n-1}\left(\mathcal{L}_{n-j}\left(4\mathcal{L}_1\mathcal{L}_j - \mathcal{L}_{j+1} + 2\mathcal{L}_j''\right) - \mathcal{L}_j'\mathcal{L}_{n-j}'\right), \quad n > 1.\end{aligned} \tag{13}$$

3. Proof of main theorem

Firstly, the following correlation between the sigma coordinates and the solution of the $\mathrm{P}_{\mathrm{II}}^{(n)}$ equation was proved in [1]:

Lemma 3.1. *The sigma function σ_n of definition* (5) *is related to the solution w_n of* (1) *by the following*

$$w_n' - w_n^2 = \sigma_n' - \frac{t_{n-1}}{2},$$

where $t_0 = -z$.

Proof. By definition (5) of the sigma-coordinates, we have

$$\sigma_n(z) := \mathcal{H}^{(n)}\left(P_1(z), \dots, P_n(z), Q_1(z), \dots, Q_n(z)\right).$$

where $P_1, \dots, P_n$, $Q_1, \dots, Q_n$ are canonical coordinates. Its first derivative is

$$\sigma_n'(z) = \left\{\mathcal{H}^{(n)}, \mathcal{H}^{(n)}\right\} + \frac{\partial \mathcal{H}^{(n)}}{\partial z} = \frac{\partial \mathcal{H}^{(n)}}{\partial z}.$$

By formula (12), the only term in $\mathcal{H}^{(n)}$ that depends explicitly on z is

$$-\frac{1}{2^{2n-1}} a_1^{(n)} a_{2n-1}^{(n)}.$$

Hence, using formulas (10) and (11), $\sigma_n'(z)$ is calculated as

$$\begin{aligned}\sigma_n'(z) = \frac{\partial \mathcal{H}^{(n)}}{\partial z} &= -\frac{1}{2^{2n-1}}\partial_z\left(a_1^{(n)} a_{2n-1}^{(n)}\right) \\ &= -\frac{1}{2^{2n-1}}\partial_z\left(\left(\sum_{l=1}^{n} t_l A_1^{(l)} - z\right)\sum_{l=1}^{n} t_l A_{2n-1}^{(l)}\right) \\ &= \frac{1}{2^{2n-1}}\sum_{l=1}^{n} t_l A_{2n-1}^{(l)} = \frac{1}{2^{2n-1}}\left(t_n A_{2n-1}^{(n)} + t_{n-1} A_{2n-1}^{(n-1)}\right) \\ &= \mathcal{L}_1\left[w_n' - w_n^2\right] + \frac{t_{n-1}}{2} = w_n' - w_n^2 + \frac{t_{n-1}}{2}.\end{aligned}$$

□

Note that thanks to Lemma 3.1, we can define all Lenard operators in terms of $\sigma_n(z)$ rather than $w(z)$. However, we need to be careful in the integration step to extract $\mathcal{L}_{n+1}\left[w' - w^2\right]$ from formula (2). The fact that the integrand is exact was proved in [17], and the choice of the integration constant depends on the condition we impose on $\mathcal{L}_n[0]$.

Lemma 3.2. *For the n-th member of the second Pailevé hierarchy, $n > 1$, the Lenard operators defined by* (2)*,* (3) *coincide with the operators defined recursively by relations* (6) *with boundary conditions* (7)*, or in other words*

$$\mathcal{L}_k\left[w_n' - w_n^2\right] = \widehat{\mathcal{L}}_k\left[\sigma_n' - \frac{t_{n-1}}{2}\right],$$

where w_n denotes the solution of (1)*.*

Proof. We prove this statement by induction. We know that

$$\mathcal{L}_1\left[w_n' - w_n^2\right] = w_n' - w_n^2.$$

On the other side, using (6) we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\widehat{\mathcal{L}}_1\left[\sigma_n' - \frac{t_{n-1}}{2}\right] = \sigma_n'',$$

so that

$$\widehat{\mathcal{L}}_1\left[\sigma_n' - \frac{t_{n-1}}{2}\right] = \sigma_n' + \mathrm{const},$$

and by imposing the boundary condition (7), we obtain

$$\widehat{\mathcal{L}}_1\left[\sigma_n' - \frac{t_{n-1}}{2}\right] = \sigma_n' - \frac{t_{n-1}}{2},$$

that due to Lemma 3.1 gives $\mathcal{L}_1\left[w' - w^2\right] = \widehat{\mathcal{L}}_1\left[\sigma_n' - \frac{t_{n-1}}{2}\right]$.

Let us now assume that the statement is true for $k = l$ and prove it for $k = l+1$.

Let us call c_{l+1} the constant term of $\widehat{\mathcal{L}}_{l+1}\left[\sigma_n' - \frac{t_{n-1}}{2}\right]$. By formula (13), we have that c_{l+1} is defined as

$$c_{l+1} = 3c_1c_l + \sum_{j=1}^{l-1} c_{l-j}\left(4c_1c_j - c_{j+1}\right).$$

This discrete equation is solved by $c_k = \left(-\frac{t_{n-1}}{2}\right)^k \frac{1}{k}\begin{pmatrix} 2k \\ k \end{pmatrix}$ as we wanted to prove. □

Remark 3.1. When $n = 1$, $t_{n-1} = t_0 = -z$ is no longer constant. In this there is no change of boundary condition and the operators $\widehat{\mathcal{L}}_k$ are simply the standard operators $\mathcal{L}_k$ defined by (2), (3), applied to $\sigma_1' + z/2$. In this proof of theorem 1.1 we consider $n \geq 1$ and prove (9) as well as the following (valid for $n = 1$)

$$-f_1' + \left(f_1'\right)^2 + \left(z - 2f_1\right)\left(\left(\widehat{\mathcal{L}}_1'' - \widehat{\mathcal{L}}_2\right) + 2f_1^2 + \sigma_1 + \frac{z^2}{4}\right) = \alpha_1\left(\alpha_1 - 1\right), \tag{14}$$

Now we can proceed to the proof of theorem 1.1.

Proof of theorem 1.1. Suppose that $z - 2f_n \neq 0$, i.e. $\alpha_n \notin \frac{1}{2}\mathbb{Z}$. Then $w\left(z\right)$ is expressed from (1) as

$$w = \frac{f_n' - \alpha_n}{z - 2f_n}. \tag{15}$$

From lemma 3.1 and (15) we obtain

$$-f_n' + (f_n')^2 + (z-2f_n)\, f_n'' - \underbrace{\left(\sigma_n' - \frac{t_{n-1}}{2}\right)}_{\widehat{\mathcal{L}}_1\left[\sigma_n' - \frac{t_{n-1}}{2}\right]} (z-2f_n)^2 = \alpha_n\,(\alpha_n - 1)\,. \tag{16}$$

This equation involves derivatives of $\sigma_n(z)$ of order $2n+1$. Since we are looking for an ODE of order $2n$, we need to remove these. To this aim, we differentiate (16) obtaining

$$(z-2f_n)\left(f_n''' + 4\widehat{\mathcal{L}}_1 f_n' + 2\widehat{\mathcal{L}}_1' f_n - 2\widehat{\mathcal{L}}_1 - z\widehat{\mathcal{L}}_1'\right) = 0, \tag{17}$$

and use the Lenard recursion relation (2) obtaining

$$(z-2f_n)\,\frac{d}{dz}\left(\sum_{l=1}^{n} t_l \widehat{\mathcal{L}}_{l+1} - z\widehat{\mathcal{L}}_1 - \sigma_n + h_n(z) + \text{const}\right) = 0, \tag{18}$$

where we understand that $\widehat{\mathcal{L}}_k$ is applied to $\sigma_n' - \frac{t_{n-1}}{2}$ and

$$h_n(z) = \frac{1}{2} z\left(t_{n-1}\,(1-\delta_{n,1}) - \frac{1}{2} z\,\delta_{n,1}\right).$$

By our assumption, $z - 2f_n \neq 0$. Thus, (18) becomes

$$\sum_{l=1}^{n} t_l \widehat{\mathcal{L}}_{l+1} - z\widehat{\mathcal{L}}_1 - \sigma_n + h_n(z) = 0, \tag{19}$$

where we have absorbed the integration constant in sigma (constant shifts in sigma do not change the dynamics).

So we have two equations, (16) and (19) that both contain derivatives of $\sigma_n(z)$ of order $2n+1$. We can replace f_n'' in (16) by $f_n'' - \left(\sum_{l=1}^{n} t_l \widehat{\mathcal{L}}_{l+1} - z\widehat{\mathcal{L}}_1 - \sigma_n + h_n(z)\right)$ thus obtaining (9). □

Lemma 3.3. *The sigma form* (9) *is an ODE of order* $2n$.

Proof. By using formula (13) we see immediately that (9) is equivalent to

$$\begin{aligned} &- f_n' + (f_n')^2 + (z-2f_n)\left(\sigma_n - h_n(z) - \left(\sigma_n' - \frac{t_{n-1}}{2}\right) f_n \right.\\ &\quad \left. - \left(\sum_{l=1}^{n} t_l \sum_{j=1}^{l-1}\left(\widehat{\mathcal{L}}_{l-j}\left(4\widehat{\mathcal{L}}_1\widehat{\mathcal{L}}_j - \widehat{\mathcal{L}}_{j+1} + 2\widehat{\mathcal{L}}_j''\right) - \widehat{\mathcal{L}}_j'\widehat{\mathcal{L}}_{l-j}'\right)\right)\right) = \alpha_n\,(\alpha_n-1)\,, \end{aligned}$$

which is an ODE of order $2n$. □

To demonstrate how our theorem 1.1 works, we give two examples for cases $n=1$ and $n=2$.

Example 3.1. For $n=1$ we use (14) to obtain

$$-\left(\sigma_1'' + \frac{1}{2}\right) + \left(\sigma_1'' + \frac{1}{2}\right)^2 - 2\sigma_1'\left(\sigma_1 + \frac{1}{4}z^2 - \left(\sigma_1' + \frac{z}{2}\right)^2\right) = \alpha_1\,(\alpha_1 - 1)\,,$$

$$(\sigma_1'')^2 - 2\sigma_1\sigma_1' + 2z\,(\sigma_1')^2 + 2\,(\sigma_1')^3 = \left(\alpha_1 - \frac{1}{2}\right)^2.$$

Remark 3.2. If we consider the following map of the Okamoto Hamiltonian in [13]

$$\mathrm{H}_{\mathrm{II}}\left(p,q\right)=\frac{1}{2}p\left(p-2q^2-z\right)-\left(\alpha+\frac{1}{2}\right)q \quad\mapsto\quad 2\mathrm{H}_{\mathrm{II}}\left(2p,\frac{1}{4}q\right)+\frac{1}{2}q,$$

the Okamoto sigma form for the second Painlevé equation in [13] coincides with our sigma form.

Example 3.2. Set $n=2$ in (9):

$$-f_2'+\left(f_2'\right)^2+\left(z-2f_2\right)\left(t_1\left(\widehat{\mathcal{L}}_1''-\widehat{\mathcal{L}}_2\right)+\left(\widehat{\mathcal{L}}_2''-\widehat{\mathcal{L}}_3\right)\right.$$
$$\left.+2f_1f_2+\sigma_2\left(z\right)-\frac{1}{2}t_1z\right)=\alpha_2\left(\alpha_2-1\right),$$

where

$$f_1=\widehat{\mathcal{L}}_1\left[\sigma_2'-\frac{t_1}{2}\right],\quad f_2=t_1\widehat{\mathcal{L}}_1\left[\sigma_2'-\frac{t_1}{2}\right]+\widehat{\mathcal{L}}_2\left[\sigma_2'-\frac{t_1}{2}\right],$$

with the Lenard operators

$$\widehat{\mathcal{L}}_1\left[\sigma_2'-\frac{t_1}{2}\right]=\sigma_2'-\frac{t_1}{2},\qquad \widehat{\mathcal{L}}_2\left[\sigma_2'-\frac{t_1}{2}\right]=\sigma_2'''+3\left(\sigma_2'\right)^2-3t_1\left(\sigma_2'-\frac{1}{4}t_1\right),$$

$$\widehat{\mathcal{L}}_3\left[\sigma_2'-\frac{t_1}{2}\right]=\sigma_2^{(\mathrm{v})}+10\sigma_2'\sigma_2'''+5\left(\sigma_2''\right)^2+10\left(\sigma_2'\right)^3$$
$$-5t_1\left(\sigma_2'''+3\left(\sigma_2'\right)^2\right)+\frac{5}{2}t_1^2\left(3\sigma_2'-\frac{1}{2}t_1\right).$$

Then the sigma form for $\mathrm{P}_{\mathrm{II}}^{(2)}$ is

$$\left(\sigma_2^{(\mathrm{iv})}\right)^2+12\sigma_2'\sigma_2''\sigma_2^{(\mathrm{iv})}-4t_1\sigma_2''\sigma_2^{(\mathrm{iv})}-\sigma_2^{(\mathrm{iv})}+4\sigma_2'\sigma_2'''-2t_1\left(\sigma_2'''\right)^2$$
$$-2\left(\sigma_2''\right)^2\sigma_2'''+20\left(\sigma_2'\right)^3\sigma_2'''-24t_1\left(\sigma_2'\right)^2\sigma_2'''+9t_1^2\sigma_2'\sigma_2'''-2z\sigma_2'\sigma_2'''$$
$$-t_1^3\sigma_2'''+2zt_1\sigma_2'''-2\sigma_2\sigma_2'''+30\left(\sigma_2'\right)^2\left(\sigma_2''\right)^2-20t_1\sigma_2'\left(\sigma_2''\right)^2+\frac{7}{2}t_1^2\left(\sigma_2''\right)^2$$
$$+z\left(\sigma_2''\right)^2-6\sigma_2'\sigma_2''+2t_1\sigma_2''+24\left(\sigma_2'\right)^5-46t_1\left(\sigma_2'\right)^4+34t_1^2\left(\sigma_2'\right)^3-4z\left(\sigma_2'\right)^3$$
$$-12t_1^3\left(\sigma_2'\right)^2+8zt_1\left(\sigma_2'\right)^2-6\sigma_2\left(\sigma_2'\right)^2+2t_1^4\sigma_2'-4zt_1^2\sigma_2'+4t_1\sigma_2\sigma_2'$$
$$-\frac{1}{8}t_1^5+\frac{1}{2}t_1^3z-\frac{1}{2}t_1^2\sigma_2-\frac{1}{2}t_1z^2+z\sigma_2=\alpha_2\left(\alpha_2-1\right).$$

4. Bäcklund transformations

The Bäcklund transformations of (1) have two generators [13, 5, 3, 2]

$$s:\qquad \tilde{w}_n\left(z,t;\tilde{\alpha}_n\right)=w_n\left(z,t;\alpha_n\right)-\frac{2\alpha_n-1}{2\sum_{l=0}^{n}t_l\mathcal{L}_l\left[w_n'-w_n^2\right]},\quad \tilde{\alpha}_n=1-\alpha_n$$

$$r:\qquad w_n\left(z,t;-\alpha_n\right)=-w_n\left(z,t;\alpha_n\right).$$

Theorem 4.1. *The Bäcklund transformations of the sigma forms* (9), (14) *act on the sigma function*

$$s\left(\sigma_n\right)=\sigma_n,\qquad\qquad r\left(\sigma_n\right)=\sigma_n-2w_n.$$

Proof. Since the right hand side of the sigma form (9), (14) is invariant under the s-action, σ_n is also invariant under this action. Regarding the r-action, we have

$$r\left(w_n' - w^2\right) = -w_n' - w_n^2 = \sigma_n' - 2w_n'.$$

After integration w.r.t. z we obtain the final formula.

□

References

[1] Stuart J Andrew. Sigma form of the second Painleve hierarchy. Master's thesis, Loughborough University, 2014.

[2] Irina Bobrova. On symmetries of the non-stationary $P_{II}^{(n)}$ hierarchy and their applications. *arXiv preprint:2010.10617*, 2020.

[3] Peter A Clarkson, Andrew N W Hone and Nalini Joshi. Hierarchies of Difference Equations and Bäcklund Transformations. *Journal of Nonlinear Mathematical Physics*, 10:13-26, 2003.

[4] Peter A Clarkson, Nalini Joshi, and Marta Mazzocco. The Lax pair for the mKdV hierarchy. *Théories asymptotiques et équations de Painlevé*, 14:53–64, 2006.

[5] Peter A Clarkson, Nalini Joshi, and Andrew Pickering. Bäcklund transformations for the second Painlevé hierarchy: a modified truncation approach. *Inverse problems*, 15(1):175, 1999.

[6] Boris Dubrovin and Youjin Zhang. Virasoro symmetries of the extended Toda hierarchy. *Comm. Math. Phys.*, 250:161–193, 2004.

[7] Boris Dubrovin and Youjin Zhang. Normal forms of hierarchies of integrable PDEs. *Frobenius manifolds and Gromov-Witten invariants, a new*, 2005.

[8] Hermann Flaschka and Alan C Newell. Monodromy-and spectrum-preserving deformations I. *Communications in Mathematical Physics*, 76(1):65–116, 1980.

[9] Michio Jimbo and Tetsuji Miwa. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II. *Physica D: Nonlinear Phenomena*, 2(2):407–448, 1981.

[10] Michio Jimbo, Tetsuji Miwa, and Kimio Ueno. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients: I. General theory and τ-function. *Physica D: Nonlinear Phenomena*, 2(2):306–352, 1981.

[11] Nalini Joshi. The second Painlevé hierarchy and the stationary KdV hierarchy. *Publications of the Research Institute for Mathematical Sciences*, 40(3):1039–1061, 2004.

[12] Marta Mazzocco and Man Yue Mo. The Hamiltonian structure of the second Painlevé hierarchy. *Nonlinearity*, 20(12):2845, 2007.

[13] Kazuo Okamoto. Studies on the Painlevé Equations. III. Second and Fourth Painlevé Equations PII and PIV. *Mathematische Annalen*, 275:221–256, 1986.

[14] Kazuo Okamoto. Studies on the Painlevé Equations. I. Sixth Painlevé equation PIV. *Ann. Mat. Pura Appl.*, 146(4):337–?381, 1987.

[15] Kazuo Okamoto. Studies on the Painlevé equations. II. Fifth Painlevé equation PV. *Japanese journal of mathematics. New series*, 13(1):47–76, 1987.

[16] Kazuo Okamoto. Studies on the Painlevé equations. IV. Third Painlevé equation PIII. *Funkcial. Ekvac*, 30(2-3):305–332, 1987.

[17] Lax Peter. Almost periodic solutions of the kdv equation. *SIAM Rev.*, 18:351–375, 1976.

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