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AN UPPER BOUND ON THE CHEBOTAREV INVARIANT
OF A FINITE GROUP*

BY

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ABSTRACT

A subset $\{g_1, \dots, g_d\}$ of a finite group G invariably generates G if the set $\{g_1^{x_1}, \dots, g_d^{x_d}\}$ generates G for every choice of $x_i \in G$. The Chebotarev invariant $C(G)$ of G is the expected value of the random variable n that is minimal subject to the requirement that n randomly chosen elements of G invariably generate G . The first author recently showed that $C(G) \leq \beta\sqrt{|G|}$ for some absolute constant β . In this paper we show that, when G is soluble, then β is at most $5/3$. We also show that this is best possible. Furthermore, we show that, in general, for each $\epsilon > 0$ there exists a constant c_ϵ such that $C(G) \leq (1 + \epsilon)\sqrt{|G|} + c_\epsilon$.

1. Introduction

Following [8] and [5], we say that a subset $\{g_1, g_2, \dots, g_d\}$ of a group G *invariably generates* G if $\{g_1^{x_1}, g_2^{x_2}, \dots, g_d^{x_d}\}$ generates G for every d -tuple $(x_1, x_2, \dots, x_d) \in G^d$. The Chebotarev invariant $C(G)$ of G is the expected value of the random variable n that is minimal subject to the requirement that n randomly chosen elements of G invariably generate G .

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In [9], Kowalski and Zywna conjectured that $C(G) = O(\sqrt{|G|})$ for every finite group G . Progress on the conjecture was first made in [8], where it was shown that $C(G) = O(\sqrt{|G|} \log |G|)$ (here, and throughout this paper, “log” means log to base 2). The conjecture was confirmed by the first author in [10]; more precisely, [10, Theorem 1] states that *there exists an absolute constant β such that $C(G) \leq \beta\sqrt{|G|}$ whenever G is a finite group*.

In this paper, we use a different approach to the problem. In doing so, we show that one can take $\beta = 5/3$ when G is soluble, and that this is best possible. Furthermore, we show that for each $\epsilon > 0$, there exists a constant c_ϵ such that $C(G) \leq (1 + \epsilon)\sqrt{|G|} + c_\epsilon$. From [9, Proposition 4.1], one can see that this is also (asymptotically) best possible.

Our main result is as follows

THEOREM 1: *Let G be a finite group.*

- (i) *For any $\epsilon > 0$, there exists a constant c_ϵ such that $C(G) \leq (1 + \epsilon)\sqrt{|G|} + c_\epsilon$;*
- (ii) *If G is a finite soluble group, then $C(G) \leq \frac{5}{3}\sqrt{|G|}$, with equality if and only if $G = C_2 \times C_2$.*

We also derive an upper bound on $C(G)$, for a finite soluble group G , in terms of the set of *crowns* for G . Before stating this result, we require the following notation: Let G be a finite soluble group. Given an irreducible G -module V which is G -isomorphic to a complemented chief factor of G , let $\delta_V(G)$ be the number of complemented factors in a chief series of G which are G -isomorphic to V . Then set $\theta_V(G) = 0$ if $\delta_V(G) = 1$, and $\theta_V(G) = 1$ otherwise. Also, let $q_V(G) := |\text{End}_G(V)|$, let $n_V(G) := \dim_{\text{End}_G(V)} V$, and let $H_V(G) := G/C_G(V)$ (we will suppress the G in this notation when the group is clear from the context). Also, let $\sigma := 2.118456563\dots$ be the constant appearing in [11, Corollary 2]. The afore mentioned upper bound can now be stated as follows.

THEOREM 2: *Let G be a finite soluble group, and let A [respectively B] be a set of representatives for the irreducible G -modules which are G -isomorphic to a non-central [resp. central] complemented chief factor of G . Then*

$$C(G) \leq \sum_{V \in A} \min \left\{ (\delta_V \cdot \theta_V + c_V) |V|, \left(\left\lceil \frac{\delta_V \cdot \theta_V}{n_V} \right\rceil + \frac{q_V^{n_V}}{q_V^{n_V} - 1} \right) |H_V| \right\} + \max_{V \in B} \delta_V + \sigma$$

where $c_V := q_V/(q_V - 1) \leq 2$.

The layout of the paper is as follows. In Section 2 we recall the notion of a *crown* in a finite group. In Section 3 we prove Theorem 2 and deduce a number of consequences, while Section 4 is reserved for the proof of Theorem 1 Part (i). Finally, we prove Theorem 1 Part (ii) in Section 5.

2. Crowns in finite groups

In Section 2, we recall the notion and the main properties of crowns in finite groups. Let L be a monolithic primitive group and let A be its unique minimal normal subgroup. For each positive integer k , let L^k be the k -fold direct product of L . The crown-based power of L of size k is the subgroup L_k of L^k defined by

$$L_k = \{(l_1, \dots, l_k) \in L^k \mid l_1 \equiv \dots \equiv l_k \pmod{A}\}.$$

Equivalently, $L_k = A^k \text{Diag } L^k$.

Following [7], we say that two irreducible G -groups V_1 and V_2 are G -equivalent and we put $V_1 \sim_G V_2$, if there are isomorphisms $\phi : V_1 \rightarrow V_2$ and $\Phi : V_1 \rtimes G \rightarrow V_2 \rtimes G$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & V_1 & \longrightarrow & V_1 \rtimes G & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow \phi & & \downarrow \Phi & & \parallel & & \\ 1 & \longrightarrow & V_2 & \longrightarrow & V_2 \rtimes G & \longrightarrow & G & \longrightarrow & 1. \end{array}$$

Note that two G -isomorphic G -groups are G -equivalent. In the particular case where V_1 and V_2 are abelian the converse is true: if V_1 and V_2 are abelian and G -equivalent, then V_1 and V_2 are also G -isomorphic. It is proved (see for example [7, Proposition 1.4]) that two chief factors V_1 and V_2 of G are G -equivalent if and only if either they are G -isomorphic between them or there exists a maximal subgroup M of G such that $G/\text{Core}_G(M)$ has two minimal normal subgroups N_1 and N_2 G -isomorphic to V_1 and V_2 respectively. For example, the minimal normal subgroups of a crown-based power L_k are all L_k -equivalent.

Let $V = X/Y$ be a chief factor of G . A complement U to V in G is a subgroup U of G such that $UV = G$ and $U \cap X = Y$. We say that $V = X/Y$ is a Frattini chief factor if X/Y is contained in the Frattini subgroup of G/Y ; this is equivalent to saying that V is abelian and there is no complement to V in G . The number $\delta_V(G)$ of non-Frattini chief factors G -equivalent to V in

any chief series of G does not depend on the series. Now, we denote by L_V the monolithic primitive group associated to V , that is

$$L_V = \begin{cases} V \rtimes (G/C_G(V)) & \text{if } V \text{ is abelian,} \\ G/C_G(V) & \text{otherwise.} \end{cases}$$

If V is a non-Frattini chief factor of G , then L_V is a homomorphic image of G . More precisely, there exists a normal subgroup N of G such that $G/N \cong L_V$ and $\text{soc}(G/N) \sim_G V$. Consider now all the normal subgroups N of G with the property that $G/N \cong L_V$ and $\text{soc}(G/N) \sim_G V$: the intersection $R_G(V)$ of all these subgroups has the property that $G/R_G(V)$ is isomorphic to the crown-based power $(L_V)_{\delta_V(G)}$. The socle $I_G(V)/R_G(V)$ of $G/R_G(V)$ is called the V -crown of G and it is a direct product of $\delta_V(G)$ minimal normal subgroups G -equivalent to V .

LEMMA 3: [1, Lemma 1.3.6] *Let G be a finite group with trivial Frattini subgroup. There exists a chief factor V of G and a non trivial normal subgroup U of G such that $I_G(V) = R_G(V) \times U$.*

LEMMA 4: [4, Proposition 11] *Assume that G is a finite group with trivial Frattini subgroup and let $I_G(V), R_G(V), U$ be as in the statement of Lemma 3. If $KU = KR_G(V) = G$, then $K = G$.*

3. Crown-based powers with abelian socle

The aim of this section is to prove Theorem 2. For a finite group G and an irreducible G -group V , we write $\Omega_{G,V}$ for the set of maximal subgroups M of G such that either $\text{soc}(G/\text{Core}_G(M)) \sim_G V$ or $\text{soc}(G/\text{Core}_G(M)) \sim_G V \times V$. Also, for $M \in \Omega_{G,V}$, we write \widetilde{M} for the union of the G -conjugates of M . We will also say that the elements $g_1, g_2, \dots, g_k \in G$ satisfy the V -property in G if $g_1, g_2, \dots, g_k \in \widetilde{M}$ for some $M \in \Omega_V$. Finally, let $P_{G,V}^*(k)$ denote the probability that k randomly chosen elements of G satisfy the V -property in G .

Suppose now that V is abelian, and consider the faithful irreducible linear group $H := G/C_G(V)$. We will denote by $\text{Der}(H, V)$ the set of the derivations from H to V (i.e. the maps $\zeta : H \rightarrow V$ with the property that $\zeta(h_1 h_2) = \zeta(h_1)^{h_2} + \zeta(h_2)$ for every $h_1, h_2 \in H$). If $v \in V$ then the map $\zeta_v : H \rightarrow V$ defined by $\zeta_v(h) = [h, v]$ is a derivation, called an *inner derivation* from H to V . The set $\text{InnDer}(H, V) = \{\zeta_v \mid v \in V\}$ of the inner derivations from H to V is a subgroup

of $\text{Der}(V, H)$ and the factor group $H^1(H, V) = \text{Der}(H, V)/\text{InnDer}(H, V)$ is the first cohomology group of H with coefficients in V .

PROPOSITION 5: *Let H be a group acting faithfully and irreducibly on an elementary abelian p -group V . For a positive integer u , we consider the semidirect product $G = V^u \rtimes H$ where the action of H is diagonal on V^u ; that is, H acts in the same way on each of the u direct factors. Assume also that $u = \delta_V(G)$. View V as a vector space over the field $F = \text{End}_H(V)$. Let $h_1, \dots, h_k \in H$, and $w_1, \dots, w_k \in V^u$, and write $w_i = (w_{i,1}, w_{i,2}, \dots, w_{i,u})$. Assume that $h_1 w_1, h_2 w_2, \dots, h_k w_k$ satisfy the V -property in G . Then for $1 \leq j \leq u$, the vectors*

$$r_j := (w_{1,j}, w_{2,j}, \dots, w_{k,j})$$

of V^k are linearly dependent modulo the subspace $W + D$, where

$$W := \{(y_1, y_2, \dots, y_k) : y_i \in [h_i, V] \text{ for } 1 \leq i \leq k\}, \text{ and}$$

$$D := \{(\zeta(h_1), \zeta(h_2), \dots, \zeta(h_k)) \in V^k : \zeta \in \text{Der}(H, V)\}.$$

Proof. Let M be a maximal subgroup of G such that $M \in \Omega_V$, and $h_1 w_1, \dots, h_k w_k \in \widetilde{M}$. Since $u = \delta_V(G)$, M cannot contain V^u , and hence $MV^u = G$. Thus, $M/M \cap V^u \cong H$, and hence there exists an integer $t \geq 0$ and elements $h_{k+1} w_{k+1}, \dots, h_{k+t} w_{k+t} \in M$ such that $h_1, \dots, h_k, h_{k+1}, \dots, h_{k+t}$ invariably generate H . But then, [10, Proposition 6] implies, in particular, that $r_1, \dots, r_u \in V^k$ are linearly dependent modulo $W + D$, as needed. ■

Before proceeding to the proof of Theorem 2, we require the following easy result from probability theory.

PROPOSITION 6: *Write $B(k, p)$ for the binomial random variable with k trials and probability $0 < p \leq 1$. Fix $l \geq 0$. Then*

$$\sum_{k=l}^{\infty} P(B(k, p) = l) \leq \frac{1}{p}.$$

Proof. Note first that

$$\binom{k}{l} x^{k-l} = \frac{1}{l!} \frac{d^l}{dx^l} x^k$$

where $\frac{d^l}{dx^l}x^k$ denotes the l -th derivative of x^k . Let $x = 1 - p$. By definition, $P(B(k, p) = l) = \binom{k}{l}(1 - x)^l x^{k-l}$. Thus

$$\begin{aligned} \sum_{k=l}^{\infty} P(B(k, p) = l) &= (1 - x)^l \sum_{k=l}^{\infty} \binom{k}{l} x^{k-l} \\ &= \frac{(1 - x)^l}{l!} \sum_{k=l}^{\infty} \frac{d^l}{dx^l} x^k \\ &= \frac{(1 - x)^l}{l!} \frac{d^l}{dx^l} \sum_{k=l}^{\infty} x^k \\ &\leq \frac{(1 - x)^l}{l!} \frac{d^l}{dx^l} \frac{1}{1 - x} \\ &= \frac{(1 - x)^l}{l!} \frac{l!}{(1 - x)^{(l+1)}} = \frac{1}{1 - x} = \frac{1}{p} \end{aligned}$$

as needed. (Note that the third equality above follows since the series $\sum_{k=l}^{\infty} x^k$ is convergent.) ■

We shall also require the following. We remark that since $P_{G,V}^*(k) \leq \sum_{\widetilde{M} \in \Omega_V} \left(\frac{|\widetilde{M}|}{|G|}\right)^k$ and $\frac{|\widetilde{M}|}{|G|} < 1$, $\sum_{k=0}^{\infty} P_{G,V}^*(k)$ converges.

PROPOSITION 7: *Let G be a finite group, and let A [respectively B] be a set of representatives for the irreducible G -groups which are G -equivalent to a non-central [resp. central] non-Frattini chief factor of G . Then*

- (1) $C(G) \leq \sum_{V \in A} \sum_{k=0}^{\infty} P_{G,V}^*(k) + \max_{V \in B} \delta_V + \sigma$, and;
- (2) If $\text{Frat}(G) = 1$ and U and V are as in Lemma 3, then $C(G) \leq C(G/U) + \sum_{k=0}^{\infty} P_{G,V}^*(k)$.

Proof. By definition, $C(G) = \sum_{k=0}^{\infty} (1 - P_I(G, k))$, where $P_I(G, k)$ denotes the probability that k randomly chosen elements of G invariably generate G . Let $P_{G,G/G'}(k)$ denote the probability that k randomly chosen elements g_1, \dots, g_k of G satisfy $\langle G'g_1, \dots, G'g_k \rangle = G$. Then it is easy to see that

$$(3.1) \quad 1 - P_I(G, k) \leq 1 - P_{G,G/G'}(k) + \sum_{V \in A} P_{G,V}^*(k).$$

Clearly $P_{G,G/G'}(k)$ is the probability that a random k -tuple of elements from G/G' generates G/G' . Hence, $C(G/G') = \sum_{k=0}^{\infty} (1 - P_{G,G/G'}(k))$ is at most $d(G/G') + \sigma$ by [11, Corollary 2] (here, for a group X , $d(X)$ denotes the minimal

number of elements required to generate X). Since $d(G/G') \leq \max_{V \in B} \delta_V$, it follows from (3.1) that $C(G) \leq \max_{V \in B} \delta_V + \sigma + \sum_{V \in A} \sum_{k=0}^{\infty} P_{G,V}^*(k)$, and Part (i) follows.

Assume that $\text{Frat}(G) = 1$, and let U and V be as in Lemma 3. Then

$$(3.2) \quad 1 - P_I(G, k) \leq 1 - P_I(G/U, k) + \sum_W P_{G,W}^*(k)$$

where the sum in the second term goes over all complemented chief factors W of G not containing U . Now, if M is a maximal subgroup of G not containing U , then M contains $R_G(V)$, by Lemma 4. Hence, $\text{Core}_G(M)$ contains $R_G(V)$, so $M \in \Omega_{G,V}$. Since $C(G) = \sum_{k=0}^{\infty} (1 - P_I(G, k))$, Part (ii) now follows immediately from (3.2), and this completes the proof. ■

The proof of Theorem 2 will follow as a corollary of the proof of the next proposition. For a finite group G , and an abelian chief factor V of G , set $H_V = H_V(G) := G/C_G(V)$, $m = m_V = m_V(G) := \dim_{\text{End}_G(V)} H^1(H_V, V)$, and write $p = p_V = p_V(G)$ for the probability that a randomly chosen element h of H_V fixes a non zero vector in V . Also, let $\delta_V = \delta_V(G)$ be the number of complemented factors in a chief series of G which are G -isomorphic to V , and set $\theta_V = \theta_V(G) = 0$ if $\delta_V = 1$, and $\theta_V = 1$ otherwise. Finally, let $q_V = q_V(G) := |\text{End}_G(V)|$ and $n_V = n_V(G) := \dim_{\text{End}_G(V)} V$.

PROPOSITION 8: *Let G be a finite group with trivial Frattini subgroup, and let U, V and $R = R_G(V)$ be as in Lemma 3. If V is nonabelian, then set $\alpha_U := \sum_{k=0}^{\infty} P_{G,V}^*(k)$. If V is abelian, then write $q = q_V, n = n_V$ and $H = H_V, p = p_V$ and $m = m_V$. Also, set $\delta = \delta_V$ and define $\theta = 0$ if $\delta = 1, \theta = 1$ otherwise, and set*

$$\alpha_U := \begin{cases} \sum_{0 \leq i \leq \delta-1} \frac{q^{\delta-i}}{q^{\delta-i} - q^i} \leq \delta + \frac{q}{(q-1)^2} & \text{if } H = 1, \\ \min \left\{ \left(\delta \cdot \theta + m + \frac{q}{q-1} \right) \frac{1}{p}, \left(\lceil \frac{\delta \cdot \theta}{n} \rceil + \frac{q^n}{q^n - 1} \right) |H| \right\} & \text{otherwise.} \end{cases}$$

Then

$$C(G) \leq C(G/U) + \alpha_U.$$

Proof. By Proposition 7 Part (ii), we have

$$(3.3) \quad C(G) \leq C(G/U) + \sum_{k=0}^{\infty} P_{G,V}^*(k).$$

Thus, we just need to prove that $\sum_{k=0}^{\infty} P_{G,V}^*(k) \leq \alpha_U$. Therefore, we may assume that V is abelian. Writing bars to denote reduction modulo $R_G(V)$, note that if M is a maximal subgroup of G with $M \in \Omega_{G,V}$, then $R_G(V) \leq M$ and $\overline{M} \in \Omega_{\overline{G},V}$. Hence, $P_{G,V}^*(k) \leq P_{\overline{G},V}^*(k)$, so we may assume that $R_G(V) = 1$. Thus, $G \cong V^\delta \rtimes H$, where H acts faithfully and irreducibly on V , and diagonally on V^δ .

Suppose first that $|H| = 1$. Then $G = V^\delta \cong (C_r)^\delta$, for some prime r , and $P_{G,V}^*(k)$ is the probability that k randomly chosen elements of G fail to generate G . Hence, $\sum_{k=0}^{\infty} P_{G,V}^*(k)$ is the expected number of random elements to generate $(C_r)^\delta$, which is well known to be

$$\sum_{i=0}^{\delta-1} \frac{r^\delta}{r^\delta - r^i}.$$

See, for instance, [11, top of page 193].

So we may assume that $|H| > 1$. Let $F = \text{End}_H V$, so that $|F| = q$, $\dim_F V = n$, and $|V| = q^n$. Fix elements x_1, x_2, \dots, x_k in G , and for $i \in \{1, \dots, k\}$, let $x_i = w_i h_i$ with $w_i \in V^\delta$ and $h_i \in H$. For $t \in \{1, \dots, \delta\}$ let

$$r_t = (\pi_t(w_1), \dots, \pi_t(w_k)) \in V^k.$$

where π_t denotes projection onto the t -th direct factor of V^δ . Moreover let

$$W := \{(u_1, u_2, \dots, u_k) : u_i \in [h_i, V] \text{ for } 1 \leq i \leq k\}, \text{ and}$$

$$D := \{(\zeta(h_1), \zeta(h_2), \dots, \zeta(h_k)) \in V^k : \zeta \in \text{Der}(H, V)\}.$$

By Proposition 5, $P_{G,V}^*(k)$ is at most the probability that r_1, \dots, r_δ are linearly dependent modulo $W + D$. Also, for an f -tuple $J := (j_1, j_2, \dots, j_f)$ of distinct elements j_i of $\{1, \dots, k\}$, set

$$r_{t,J} := (\pi_t(w_{j_1}), \pi_t(w_{j_2}), \dots, \pi_t(w_{j_f})) \in V^f$$

for $t \in \{1, \dots, \delta\}$, and set

$$W_J := \{(u_{j_1}, u_{j_2}, \dots, u_{j_f}) \in V^f : u_i \in [h_{j_i}, V] \text{ for } 1 \leq i \leq f\}, \text{ and}$$

$$D_J := \{(\zeta(h_{j_1}), \zeta(h_{j_2}), \dots, \zeta(h_{j_f})) \in V^f : \zeta \in \text{Der}(H, V)\}.$$

Notice that: (*) If J is fixed and r_1, \dots, r_δ are F -linearly dependent modulo $W + D$, then the vectors $r_{1,J}, \dots, r_{\delta,J}$ of V^f are F -linearly dependent modulo $W_J + D_J$.

We will prove first that

$$(3.4) \quad \sum_{k=0}^{\infty} P_{G,V}^*(k) \leq (\delta \cdot \theta + m + c_V) \frac{1}{p},$$

where c_V is as in the statement of Theorem 2. To this end, let Δ_l be the subset of H^k consisting of the k -tuples (h_1, \dots, h_k) with the property that $C_V(h_i) \neq 0$ for precisely l different choices of $i \in \{1, \dots, k\}$. If $(h_1, \dots, h_k) \in \Delta_l$, then, by [10, Lemma 7], $W + D$ is a subspace of $V^k \cong F^{nk}$ of codimension at least $l - m$: so the probability that r_1, \dots, r_δ are F -linearly dependent modulo $W + D$ is at most

$$\begin{aligned} p_l &= 1 - \left(\frac{q^{nk} - q^{nk-l+m}}{q^{nk}} \right) \cdots \left(\frac{q^{nk} - q^{nk-l+m+\delta-1}}{q^{nk}} \right) \\ &= 1 - \left(1 - \frac{1}{q^{l-m}} \right) \cdots \left(1 - \frac{q^{\delta-1}}{q^{l-m}} \right) \\ &\leq \min \left\{ 1, \left(\frac{q^\delta - 1}{q - 1} \right) \frac{1}{q^{l-m}} \right\} \leq \min \{ 1, 1/q^{l-m-\delta \cdot \theta} \}. \end{aligned}$$

Hence, we have

$$\begin{aligned} \sum_{k=0}^{\infty} P_{G,V}^*(k) &\leq \sum_{k=0}^{\infty} \sum_{l=0}^k P(B(k,p) = l) \min \{ 1, q^{\delta \cdot \theta + m - l} \} \\ &\leq \sum_{k=0}^{\infty} P(B(k,p) < \delta \cdot \theta + m) + \sum_{k=0}^{\infty} \sum_{l=\delta \cdot \theta + m}^k P(B(k,p) = l) q^{\delta \cdot \theta + m - l} \\ &\leq \sum_{k=0}^{\infty} P(B(k,p) < \delta \cdot \theta + m) + \sum_{l=0}^{\infty} q^{-l} \sum_{k=l+\delta \cdot \theta + m}^{\infty} P(B(k,p) = l + \delta \cdot \theta + m) \\ &\leq \frac{\delta \cdot \theta + m + c_V}{p} \end{aligned}$$

where $c_V = \frac{q}{q-1}$. Note that the last step above follows from Proposition 6.

Thus, all that remains is to show that

$$(3.5) \quad \sum_{k=0}^{\infty} P_{G,V}^*(k) \leq \left(\left\lceil \frac{\delta \cdot \theta}{n} \right\rceil + \frac{q^n}{q^n - 1} \right) |H|.$$

For this, we define Ω_l to be the subset of H^k consisting of the k -tuples (h_1, \dots, h_k) with the property that $h_i = 1$ for precisely l different choices of $i \in \{1, \dots, k\}$. Suppose that $(h_1, \dots, h_k) \in \Omega_l$, and set $J := (j_1, j_2, \dots, j_l)$, where $j_1 < j_2 < \dots < j_l$ and $\{j_1, j_2, \dots, j_l\} = \{i \mid 1 \leq i \leq k, h_i = 1\}$. Then, by (*), the probability p'_l that $r_1, r_2, \dots, r_\delta$ are F -linearly dependent modulo $W + D$ is at

most the probability that the vectors $r_{1,J}, r_{2,J}, \dots, r_{\delta,J} \in V^l$ are F -linearly dependent modulo $W_J + D_J$. But $W_J + D_J = 0$, by the definition of J . Thus we have

$$\begin{aligned}
 p'_l &\leq 1 - \left(\frac{q^{nl} - 1}{q^{nl}}\right) \dots \left(\frac{q^{nl} - q^{nl-\delta-1}}{q^{nl}}\right) \\
 &= 1 - \left(1 - \frac{1}{q^{nl}}\right) \dots \left(1 - \frac{q^{\delta-1}}{q^{nl}}\right) \leq \min \left\{1, \left(\frac{q^\delta - 1}{q - 1}\right) \frac{1}{q^{nl}}\right\} \leq \min \left\{1, \frac{1}{q^{nl-\delta\cdot\theta}}\right\}.
 \end{aligned}$$

Hence, if $\alpha := \lceil \frac{\delta\cdot\theta}{n} \rceil$, and $p' = 1/|H|$ is the probability that a randomly chosen element of H is the identity, then we have

$$\begin{aligned}
 \sum_{k=0}^{\infty} P_{G,V}^*(k) &\leq \sum_{k=0}^{\infty} P(B(k, p') < \alpha) + \sum_{k=0}^{\infty} \sum_{l=\alpha}^k P(B(k, p') = l) q^{\delta\cdot\theta-nl} \\
 &\leq \sum_{k=0}^{\infty} P(B(k, p') < \alpha) + \sum_{l=0}^{\infty} q^{-nl-n\alpha+\delta\cdot\theta} \sum_{k=l+\alpha}^{\infty} P(B(k, p') = l + \alpha) \\
 &\leq \sum_{k=0}^{\infty} P(B(k, p') < \alpha) + \sum_{l=0}^{\infty} q^{-nl} \sum_{k=l+\alpha}^{\infty} P(B(k, p') = l + \alpha) \\
 &\leq \frac{1}{p'} \left(\alpha + \frac{q^n}{q^n - 1}\right)
 \end{aligned}$$

Note that the last step above again follows from Proposition 6. Since $p' = 1/|H|$, (3.5) follows, whence the result. ■

We are now ready to prove Theorem 2.

Proof of Theorem 2. By Proposition 7 Part (i), we have

$$C(G) \leq \max_{V \in \mathcal{B}} \delta_V + \sigma + \sum_{k=0}^{\infty} \sum_{V \in \mathcal{A}} P_{G,V}^*(k)$$

Thus, it will suffice to prove that

$$(3.6) \quad \sum_{k=0}^{\infty} P_{G,V}^*(k) \leq \min \left\{ (\delta_V + c_V) q_V^{n_V}, \left(\left\lceil \frac{\delta_V}{n_V} \right\rceil + \frac{q_V^{n_V}}{q_V^{n_V} - 1} \right) |H_V| \right\}$$

for each non-central complemented chief factor V of G .

However, since $H^1(H, V) = 0$ by [12, Lemma 1], and since $p_V \leq |H_v|/|H| \leq 1/|V|$ (for any non-zero vector $v \in V$), this follows immediately from the proof of Proposition 8. ■

COROLLARY 9: *Let G be a finite soluble group, and let A and B be as in Theorem 2. Then*

$$C(G) \leq d(G) \sum_{V \in A} \left(1 + \frac{q_V^{n_V} |H_V|}{q_V^{n_V} - 1} \right) + \sigma.$$

Proof. For $V \in A \cup B$, set $\gamma_V := \lceil \delta_V / n_V \rceil$, and $p'_V = 1/|H_V|$. Note also that $n_V = |H_V| = 1$ when $V \in B$. Arguing as in the last paragraph of the proof of Proposition 8, we have

$$\begin{aligned} C(G) &\leq \sum_{V \in A} \sum_{k=0}^{\infty} \sum_{l=0}^k \min\{q_V^{-n_V l + \delta_V}, 1\} P(B(k, p'_V) = l) + \max_{V \in B} \delta_V + \sigma \\ &\leq \sum_{V \in A} \sum_{k=0}^{\infty} P(B(k, p'_V) < \gamma_V) + \sum_{V \in A} \sum_{l=0}^{\infty} q_V^{-n_V l} \sum_{k=l+\gamma_V}^{\infty} P(B(k, p'_V) = l + \gamma_V) + \\ &\quad \max_{V \in B} \delta_V + \sigma \\ &\leq \sum_{V \in A} \gamma_V / p'_V + \sum_{V \in A} \frac{q_V^{n_V}}{p'_V (q_V^{n_V} - 1)} + \max_{V \in B} \delta_V + \sigma \\ &\leq \left(\max_{V \in A \cup B} \gamma_V \right) \sum_{V \in A} \left(1 + \frac{q_V^{n_V}}{q_V^{n_V} - 1} \right) |H_V| + \sigma. \end{aligned}$$

We remark that the third inequality above follows from Proposition 6. Finally, [3, Theorem 1.4 and paragraph after the proof of Theorem 2.7] imply that $d(G) = \max_{V \in A \cup B} \left\{ 1 + a_V + \left\lfloor \frac{\delta_V - 1}{n_V} \right\rfloor \right\}$, where $a_V = 0$ if $V \in B$, and $a_V = 1$ otherwise. In particular, $d(G) \geq \max_{V \in A \cup B} \gamma_V$, and the result follows. ■

4. Proof of Theorem 1 Part (i)

Before proceeding to the proof of Part (i) of Theorem 1, we require the following result, which follows immediately from the arguments used in [10, Proof of Proposition 10].

PROPOSITION 10: [10, Proof of Proposition 10] *Let H be a finite group acting faithfully and irreducibly on an elementary abelian group V , and denote by p the probability that a randomly chosen element h of H centralises a non zero vector of V . Also, write $m := \dim_{\mathbb{E}nd_H(V)} H^1(H, V)$. Assume that $H^1(H, V)$ is nontrivial and that $|H| \geq |V|$. Then there exists an absolute constant C such that $p|H| \geq 2(m + 1)^2$ if $|H| \geq C$.*

Proof of Theorem 1 Part (i). Since $C(G) = C(G/\text{Frat}(G))$, we may assume that $\text{Frat}(G) = 1$. Thus, Proposition 8 applies: adopting the same notation as used therein, we have

$$(4.1) \quad C(G) \leq C(G/U) + \alpha_U.$$

Using (4.1), the proof of the theorem reduces to proving that

$$(4.2) \quad \alpha_U \leq (1 + \beta_U)\sqrt{|G|}$$

where $\beta_U \rightarrow 0$ as $|U| \rightarrow \infty$. Indeed, suppose that (4.2) holds, fix $\epsilon > 0$, and suppose that Theorem 1 holds for groups of order less than $|G|$. Then since $|U| > 1$, there exists a constant c_ϵ such that $C(G/U) \leq (1 + \epsilon)\sqrt{|G/U|} + c_\epsilon$. Hence, by (4.1) and (4.2) we have $C(G) \leq (1 + \beta_U + \frac{1+\epsilon}{\sqrt{|U|}})\sqrt{|G|} + c_\epsilon$. It is now clear that by choosing $|U|$ to be large enough, we have $C(G) \leq (1 + \epsilon)\sqrt{|G|} + c_\epsilon$, as needed.

Assume first that U is nonabelian. By [10, Proof of Lemma 13], there exist absolute constants c_1 and c_2 such that

$$P_{G,V}^*(k) \leq \min \left\{ 1, c_1\sqrt{|G|^3}(1 - c_2/\log |G|)^k \right\}.$$

Also, there exists a constant c_3 such that if $k \geq c_3(\log |G|)^2$, then $c_1\sqrt{|G|^3}(1 - c_2/\log |G|)^k$ tends to 0 as $|G|$ tends to ∞ . It follows that

$$\begin{aligned} \alpha_U &= \sum_{k=0}^{\infty} P_{G,V}^*(k) \\ &\leq \lceil c_3(\log |G|)^2 \rceil + c_1\sqrt{|G|^3}(1 - c_2/\log |G|)^{\lceil c_3(\log |G|)^2 \rceil} \sum_{k=0}^{\infty} (1 - c_2/\log |G|)^k \\ &= \lceil c_3(\log |G|)^2 \rceil + \frac{c_1}{c_2}\sqrt{|G|^3} \log |G| (1 - c_2/\log |G|)^{\lceil c_3(\log |G|)^2 \rceil} \end{aligned}$$

and (4.2) holds.

So we may assume that U is abelian, and hence $|G| \geq |V|^\delta |H|$. The inequality (4.2) then follows easily from the definition of α_U , except when $\delta = 1$ and $|H| \geq |V|$. Indeed, if $|H| \leq |V|$ and $\delta = 1$, then $\frac{q^n}{q^n - 1} \rightarrow 1$ as $|U| = q^n \rightarrow \infty$; if $|H| \leq |V|$ and $\delta > 1$, then

$$\left(\left\lceil \frac{\delta}{n} \right\rceil + \frac{q^n}{q^n - 1} \right) |H| \leq \frac{\left\lceil \frac{\delta}{n} \right\rceil + \frac{q^n}{q^n - 1}}{|V|^{\frac{\delta-1}{2}}} \sqrt{|G|}$$

which clearly gives us what we need, since $|U| = |V|^\delta$ is tending to ∞ . The other cases are similar.

So assume that $\delta = 1$ and $|H| \geq |V|$. We distinguish two cases:

- (1) $m \neq 0$ and $|V| \leq |H| \leq (m + 1)^2|V|$. Denote by p the probability that a randomly chosen element h of H centralizes a non-zero vector of V : By Proposition 10, there exists an absolute constant C such that $p|H| \geq 2(m + 1)^2$ if $|H| \geq C$. Thus,

$$\alpha_U \leq \left(m + \frac{q}{q-1}\right) \frac{1}{p} \leq (m + 2) \frac{|H|}{2(m + 1)^2} \leq \frac{|H|}{m + 1} \leq \sqrt{|H||V|}$$

if $|H| \geq C$, from which (4.2) follows.

- (2) $|H| \geq |V|(m + 1)^2$. We remark first that, for any fixed nonzero vector v in V , we have $p \geq \frac{|H_v|}{|H|}$, where H_v denotes the stabiliser of v in H . If H is not a transitive linear group, then there is an orbit Ω for the action of H on $V \setminus \{0\}$ with $|\Omega| \leq q^n/2$. Choose $v \in \Omega$: we have

$$\frac{1}{p} \leq \frac{|H|}{|H_v|} \leq \frac{q^n}{2},$$

hence

$$\alpha_U \leq \frac{m + 2}{p} \leq (m + 1)q^n \leq \sqrt{|H||V|}.$$

We remain with the case when H is a transitive linear group. There are four infinite families:

- (a) $H \leq \Gamma L(1, q^n)$;
- (b) $SL(a, r) \trianglelefteq H$, where $r^a = q^n$;
- (c) $Sp(2a, r) \trianglelefteq H$, where $a \geq 2$ and $r^{2a} = q^n$;
- (d) $G_2(r) \trianglelefteq H$, where q is even, and $q^n = r^6$.

Furthermore, H and m are exhibited in [2, Table 7.3]: in each case, we have $m \leq 1$. Furthermore, we have $|H| = (q^n - 1)\rho$, where ρ is the order of a point stabiliser. Hence, if $\rho \geq 9$, then

$$\alpha_U \leq \frac{m + 2}{p} \leq \frac{3}{p} \leq 3|V| \leq \sqrt{|H||V|}.$$

So we may assume that $\rho \leq 8$. Suppose first that (a) holds. Then H is soluble, so $m = 0$. Also, $\rho = |H_v| \leq 8$ implies that $n \leq 8$. Hence, as $q^n \leq q^8$ approaches ∞ , $\frac{q}{q-1}$ approaches 1, and (4.2) follows since

$$\alpha_U \leq \frac{q}{q-1}|V| \leq \frac{q}{q-1}\sqrt{|H||V|}.$$

So we may assume that (a) does not hold. In particular, if (b) or (c) holds then $a \geq 2$. It follows (in either of the cases (b), (c) or (d)) that if q^n is large enough, then $|H| \geq 9q^n$, and so

$$\alpha_U \leq \frac{(m+2)}{p} \leq 3q^n \leq \sqrt{|H||V|}.$$

This gives us what we need, and completes the proof. ■

5. Proof of Theorem 1 Part (ii)

In this section, we prove Part (ii) of Theorem 1 in a number of steps. The first is as follows:

LEMMA 11: *Let G be a finite soluble group with trivial Frattini subgroup, and let U and V be as in Lemma 3. Assume that V is abelian and non-central in G , and let $H = H_V$. Then*

$$\frac{\alpha_U}{|G|^{1/2}} < \frac{5}{3} \left(\frac{|U|^{1/2} - 1}{|U|^{1/2}} \right)$$

except when $|H| < |V|$ and one of the following cases occur:

- (1) $\delta = 2, q^n = 4$ and $|R_G(V)| = 1$.
- (2) $\delta = 2, q^n = 3$ and $|R_G(V)| \leq 2$.
- (3) $\delta = 1, 4 \leq q^n \leq 7$ and $|R_G(V)| = 1$.
- (4) $\delta = 1, q^n = 3$ and $|R_G(V)| \leq 3$.

Proof. Note that $m = 0$ since H is soluble. We distinguish the following cases: Case 1) $|H| < |V|$ and $\delta \neq 1$. Since, $|G| = \lambda|H||V|^\delta$ for some positive integer λ it suffices to prove

$$(5.1) \quad \frac{3 \left(\delta + \frac{q^n}{q^n - 1} \right) \left(\frac{q^{n\delta/2}}{q^{n\delta/2} - 1} \right) |H|}{5\lambda^{1/2}|H|^{1/2}q^{n\delta/2}} \leq \frac{3}{5\lambda^{1/2}} \left(\delta + \frac{q^n}{q^n - 1} \right) \left(\frac{(q^n - 1)^{1/2}}{q^{n\delta/2} - 1} \right) < 1.$$

If $\delta \geq 3$ then

$$\frac{3}{5\lambda^{1/2}} \left(\delta + \frac{q^n}{q^n - 1} \right) \left(\frac{(q^n - 1)^{1/2}}{q^{n\delta/2} - 1} \right) \leq \frac{3}{5\lambda^{1/2}} \left(3 + \frac{q^n}{q^n - 1} \right) \left(\frac{(q^n - 1)^{1/2}}{q^{3n/2} - 1} \right) < 1.$$

Suppose $\delta = 2$. If $q^n \geq 5$, then

$$\frac{3}{5\lambda^{1/2}} \left(\delta + \frac{q^n}{q^n - 1} \right) \left(\frac{(q^n - 1)^{1/2}}{q^{n\delta/2} - 1} \right) \leq \frac{3}{5\lambda^{1/2}} \left(2 + \frac{q^n}{q^n - 1} \right) \left(\frac{(q^n - 1)^{1/2}}{q^n - 1} \right) < 1.$$

Suppose $\delta = 2$ and $q^n = 4$. We have $|H| = 3$ so if $\lambda \neq 1$, then

$$\frac{3 \left(\delta + \frac{q^n}{q^n-1} \right) \left(\frac{q^{n\delta/2}}{q^{n\delta/2}-1} \right) |H|}{5\lambda^{1/2}|H|^{1/2}q^{n\delta/2}} \leq \frac{2 \cdot 3^{1/2}}{3 \cdot \lambda^{1/2}} < 1$$

Suppose $\delta = 2$ and $q^n = 3$. We have $|H| = 2$ so if $\lambda > 2$, then

$$\frac{3 \left(\delta + \frac{q^n}{q^n-1} \right) \left(\frac{q^{n\delta/2}}{q^{n\delta/2}-1} \right) |H|}{5\lambda^{1/2}|H|^{1/2}q^{n\delta/2}} \leq \frac{21 \cdot 2^{1/2}}{20 \cdot \lambda^{1/2}} < 1.$$

Case 2) $|H| \geq |V|$ (and consequently $n \neq 1$) and $\delta \neq 1$. It suffices to prove that

$$(5.2) \quad \frac{3 \left(\delta + \frac{q}{q-1} \right) \left(\frac{q^{n\delta/2}}{q^{n\delta/2}-1} \right) q^n}{5|H|^{1/2}q^{n\delta/2}} < 1.$$

Suppose $q^n \neq 4$.

$$\frac{3 \left(\delta + \frac{q}{q-1} \right) \left(\frac{q^{n\delta/2}}{q^{n\delta/2}-1} \right) q^n}{5|H|^{1/2}q^{n\delta/2}} \leq \frac{3 \left(2 + \frac{q}{q-1} \right) \left(\frac{q^n}{q^n-1} \right)}{5q^{n/2}} < 1.$$

Suppose $q^n = 4$. We have $H = \text{GL}(2, 2) \cong \text{Sym}(3)$, and consequently $|H| = 6$ and $p = 2/3$. Hence

$$\frac{\alpha_U}{|G|^{1/2}} \frac{5}{3} \left(\frac{|U|^{1/2}}{|U|^{1/2}-1} \right) \leq \frac{(\delta+2) \cdot \frac{1}{p} \cdot \frac{3}{5} \cdot \frac{4}{3}}{|H|^{1/2} \cdot 2^\delta} \leq \frac{6}{5\sqrt{6}} < 1.$$

Case 3) $|H| < |V|$ and $\delta = 1$. Since, $|G| = \lambda|H||V|^\delta$ for some positive integer λ it suffices to prove

$$(5.3) \quad \frac{3 \left(\frac{q^n}{q^n-1} \right) \left(\frac{q^{n/2}}{q^{n/2}-1} \right) |H|^{1/2}}{5\lambda^{1/2}q^{n/2}} < 1.$$

If $q^n \geq 8$, or $7 \geq q^n \geq 4$ and $\lambda \neq 1$, or $q^n = 3$ and $\lambda > 3$, then

$$\frac{3 \left(\frac{q^n}{q^n-1} \right) \left(\frac{q^{n/2}}{q^{n/2}-1} \right) |H|^{1/2}}{5 \cdot \lambda^{1/2} \cdot q^{n/2}} \leq \frac{3 \left(\frac{q^n}{q^n-1} \right) \left(\frac{(q^n-1)^{1/2}}{q^{n/2}-1} \right)}{5 \cdot \lambda^{1/2}} = \frac{3 \left(\frac{q^n}{(q^n-1)^{1/2}(q^{n/2}-1)} \right)}{5 \cdot \lambda^{1/2}} < 1.$$

Case 4) $|H| \geq |V|$ (and consequently $n \neq 1$) and $\delta = 1$. It suffices to prove that

$$(5.4) \quad \frac{3 \left(\frac{q}{q-1} \right) \left(\frac{q^{n/2}}{q^{n/2}-1} \right)}{5|H|^{1/2}q^{n/2}p} < 1.$$

If H is not a transitive linear group, then $|H|^{1/2}q^{n/2}p \geq 2$, so it suffices to have

$$\left(\frac{q}{q-1} \right) \left(\frac{q^{n/2}}{q^{n/2}-1} \right) \leq \frac{10}{3},$$

which is true if $(q, n) \neq (2, 2)$. On the other hand, we may exclude the case $(q, n) = (2, 2)$: indeed the only soluble irreducible subgroup of $GL(2, 2)$ with order ≥ 4 is $GL(2, 2)$, which is transitive on the nonzero vectors.

If H is a transitive linear group, then $|H| = (q^n - 1)\rho$, with ρ the order of the stabilizer in H of a nonzero vector and

$$\frac{3 \left(\frac{q}{q-1}\right) \left(\frac{q^{n/2}}{q^{n/2}-1}\right)}{5|H|^{1/2}q^{n/2}p} \leq \frac{3 \left(\frac{q}{q-1}\right) \left(\frac{q^{n/2}}{q^{n/2}-1}\right)}{5\sqrt{\rho}},$$

so it suffices to have

$$\left(\frac{q}{q-1}\right) \left(\frac{q^{n/2}}{q^{n/2}-1}\right) \leq \frac{5\sqrt{\rho}}{3},$$

which is true if $q \geq 3$ and if $(q, n, \rho) \notin \{(2, 4, 2), (2, 3, 2), (2, 3, 3), (2, 2, 2)\}$. We may exclude the case $(q, n, \rho) = (2, 3, 2)$ (there is no transitive linear subgroup of $GL(3, 2)$ of order 14). If $(q, n, \rho) = (2, 4, 2)$, then $H = GL(1, 16) \times C_2$, hence $p = 6/30$ so $|H|^{1/2}q^{n/2}p \geq 2$ and (5.4) is true. If $(q, n, \rho) = (2, 3, 3)$ then $H = \Gamma L(1, 8)$ and consequently $p = 15/21$ and

$$\frac{3 \left(\frac{q}{q-1}\right) \left(\frac{q^{n/2}}{q^{n/2}-1}\right)}{5|H|^{1/2}q^{n/2}p} = \frac{3 \cdot 2 \cdot \sqrt{8} \cdot 21}{5 \cdot (\sqrt{8} - 1) \cdot 15 \cdot \sqrt{21}\sqrt{8}} < 1.$$

If $(q, n, \rho) = (2, 2, 2)$ then $H = GL(2, 2)$ and consequently $p = 2/3$ and

$$\frac{3 \left(\frac{q}{q-1}\right) \left(\frac{q^{n/2}}{q^{n/2}-1}\right)}{5|H|^{1/2}q^{n/2}p} = \frac{3 \cdot 2 \cdot 2 \cdot 3}{5 \cdot 2 \cdot 2 \cdot \sqrt{6}} < 1. \quad \blacksquare$$

LEMMA 12: *If G is one of the exceptional cases in the statement of Lemma 11, then $C(G) < \frac{5}{3}\sqrt{|G|}$.*

Proof. This follows easily by direct computation. We use MAGMA, and the code from [9, Appendix, page 36] to compute $C(G)$ explicitly whenever G is a group satisfying the conditions of one of the exceptional cases of Lemma 11. \blacksquare

The next step is to deal with the case of a central chief factor.

LEMMA 13: *If $G \cong C_p^\delta$, then $C(G) \leq \frac{5}{3}\sqrt{|G|}$, with equality if and only if $G = C_2 \times C_2$.*

Proof. If $p \neq 3$ or $p = 2$ and $\delta > 3$, then

$$C(G) = \sum_{0 \leq i \leq \delta-1} \frac{p^\delta}{p^\delta - p^i} \leq \delta + \frac{p}{(p-1)^2} < \frac{5 \cdot p^{\delta/2}}{3} = \frac{5 \cdot \sqrt{|G|}}{3}.$$

If $(p, \delta) = (2, 1)$ then

$$\frac{C(G)}{\sqrt{G}} = \frac{2}{\sqrt{2}} = \sqrt{2};$$

if $(p, \delta) = (2, 2)$ then

$$\frac{C(G)}{\sqrt{G}} = \frac{\frac{4}{2} + \frac{4}{3}}{2} = \frac{5}{3};$$

if $(p, \delta) = (2, 3)$ then

$$\frac{C(G)}{\sqrt{G}} = \frac{\frac{8}{4} + \frac{8}{6} + \frac{8}{7}}{\sqrt{8}} \sim 1.5826. \quad \blacksquare$$

Proof of Part (ii) of Theorem 1. We prove the claim by induction on the order of $|G|$. If $\text{Frat}(G) \neq 1$, then the conclusion follows immediately since $C(G) = C(G/\text{Frat}(G))$. Otherwise G contains a normal subgroup U as in Lemma 4. If $G = U \cong C_p^\delta$, then the conclusion follows from Lemma 13. Otherwise, Lemma 11, together with the inductive hypothesis gives

$$C(G) \leq C(G/U) + \alpha_U < \frac{5\sqrt{|G|}}{3\sqrt{|U|}} + \frac{5(\sqrt{|U|} - 1)\sqrt{|G|}}{3\sqrt{|U|}} = \frac{5}{3}\sqrt{|G|}$$

as claimed. \blacksquare

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