

## A note on a question of Markman

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DOI:

[10.1016/j.jpaa.2021.106673](https://doi.org/10.1016/j.jpaa.2021.106673)

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*Document Version*

Peer reviewed version

*Citation for published version (Harvard):*

Huang, S 2021, 'A note on a question of Markman', *Journal of Pure and Applied Algebra*, vol. 225, no. 9, 106673. <https://doi.org/10.1016/j.jpaa.2021.106673>

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# A note on a question of Markman

SHENGYUAN HUANG

ABSTRACT: Let  $\mathcal{F}$  be a vector bundle on a complex projective algebraic variety  $X$ . If  $\mathcal{F}$  deforms along a first order deformation of  $X$ , its Mukai vector remains of Hodge type along this deformation. We prove an analogous statement for all polyvector fields, not only for those in  $H^1(X, T_X)$  corresponding to deformations of the complex structure. This answers a question of Markman. We also explore a Lie theoretic analogue of the statement above.

**Mathematics Subject Classification (2020).** 14F06.

**Key words.** Hochschild homology and cohomology, HKR isomorphisms, deformation.

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## 1. Introduction

**1.1.** Let  $X$  be a smooth complex algebraic variety. Consider the first order deformation  $\tilde{X}$  of  $X$  associated to a class  $\tilde{\alpha} \in H^1(X, T_X)$ .

In general, a vector bundle  $\mathcal{F}$  may not deform to a bundle  $\tilde{\mathcal{F}}$  on  $\tilde{X}$ . The obstruction  $\alpha_{\mathcal{F}} \in \text{Ext}^2(\mathcal{F}, \mathcal{F})$  to the existence of a vector bundle  $\tilde{\mathcal{F}}$  on  $\tilde{X}$  such that  $\tilde{\mathcal{F}}|_X \cong \mathcal{F}$  was described in [B94, T09] as the contraction

$$\alpha_{\mathcal{F}} = \tilde{\alpha} \lrcorner at_{\mathcal{F}} \in \text{Ext}^2(\mathcal{F}, \mathcal{F}).$$

Here  $at_{\mathcal{F}} \in \text{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \Omega_X)$  is the Atiyah class of  $\mathcal{F}$ .

Moreover, if  $\mathcal{F}$  does deform, then its Chern classes and hence its Mukai vector stay of Hodge type on the deformed space  $\tilde{X}$ . This implies that the class

$$\tilde{\alpha} \lrcorner v(\mathcal{F}) \in \mathbf{H}\Omega_*(X) \stackrel{\text{def}}{=} \bigoplus_{q-p=*} H^p(X, \wedge^q \Omega_X)$$

vanishes, where  $v(\mathcal{F})$  is the Mukai vector of  $\mathcal{F}$ .

Thus, in the simple case where  $\tilde{\alpha} \in H^1(X, T_X)$  we conclude that if  $\tilde{\alpha} \lrcorner at_{\mathcal{F}}$  is zero, then  $\tilde{\alpha} \lrcorner v(\mathcal{F})$  is zero.

**1.2.** In an email correspondence, Eyal Markman asked if the above statement can be generalized to the case where  $\tilde{\alpha}$  is an arbitrary polyvector field in  $\mathbf{HT}^*(X) = \bigoplus_{p+q=*} H^p(X, \wedge^q T_X)$ . According to Markman, this question is central to his study of the deformations of hyperkähler manifolds. We provide an answer to this question in this paper.

**1.3.** First, there are two ways to generalize the class that appears in (1.1) above. For a class  $\tilde{\alpha} \in \mathbf{HT}^*(X)$  we can define classes in  $\mathbf{Ext}^*(\mathcal{F}, \mathcal{F})$  in the following two ways.

The first is defined by using the HKR isomorphism

$$I^{HKR} : \mathbf{HT}^*(X) \rightarrow \mathbf{HH}^*(X),$$

where the latter is the Hochschild cohomology of  $X$ . Since  $\mathbf{HH}^*(X)$  can be interpreted as natural transformations of the identity functor at the dg level, this yields a natural map  $\mathbf{HH}^*(X) \rightarrow \mathbf{Ext}^*(\mathcal{F}, \mathcal{F})$ .

The second construction was defined by Toda [T09]. Consider the exponential Atiyah class

$$\exp(at_{\mathcal{F}}) = 1 + at_{\mathcal{F}} + \cdots + \frac{(at_{\mathcal{F}})^k}{k!} + \cdots,$$

where  $(at_{\mathcal{F}})^k \in \mathbf{Ext}^k(\mathcal{F}, \mathcal{F} \otimes \wedge^k \Omega_X)$ . Let  $\tilde{\alpha}^{p,k} \in H^p(X, \wedge^k T_X)$  be the homogenous degree  $(p, k)$  part of  $\tilde{\alpha}$ . We can contract  $\tilde{\alpha}^{p,k}$  with  $\frac{(at_{\mathcal{F}})^k}{k!}$  to get an element in  $\mathbf{Ext}^{p+k}(\mathcal{F}, \mathcal{F})$ . Taking the sum over all  $(p, k)$ , we get the desired class which will be denoted by  $\alpha \lrcorner \exp(at_{\mathcal{F}}) \in \mathbf{Ext}^*(\mathcal{F}, \mathcal{F})$ . When  $\tilde{\alpha}$  is a class in  $H^1(X, T_X)$ , we recover the previous contraction  $\tilde{\alpha} \lrcorner at_{\mathcal{F}}$ .

Our first result is below.

**1.4. Theorem A.** *The two classes defined above are the same. In other words the diagram*

$$\begin{array}{ccc}
 \mathrm{HH}^*(X) & \longrightarrow & \mathrm{Ext}^*(\mathcal{F}, \mathcal{F}) \\
 \uparrow I^{HKR} & \nearrow & \\
 \mathrm{HT}^*(X) & & (-) \lrcorner \exp(\mathrm{at}_{\mathcal{F}})
 \end{array}$$

*is commutative.*

There is an analogous result for Hopf algebras. See Theorem 2.7 in [CI17] and see [PW09] for more details.

**1.5.** The space  $\mathrm{H}\Omega_*(X)$  is naturally a module over  $\mathrm{HT}^*(X)$ , mimicking the module structure of Hochschild homology over cohomology. For an object  $\mathcal{F}$  in the derived category of  $X$ , its Mukai vector  $v(\mathcal{F})$  lies in  $\mathrm{H}\Omega_*(X)$ . Thus we can act with the class  $\tilde{\alpha}$  to obtain  $\tilde{\alpha} \lrcorner v(\mathcal{F}) \in \mathrm{H}\Omega_*(X)$ .

**Theorem B.** *If  $\tilde{\alpha} \lrcorner \exp(\mathrm{at}_{\mathcal{F}}) = 0$ , then we have*

$$D(\tilde{\alpha} \lrcorner v(\mathcal{F})) = 0.$$

*Here  $D$  is the Duflo operator,*

$$D(\tilde{\alpha}) = Td(X)^{\frac{1}{2}} \lrcorner \tilde{\alpha},$$

*where  $Td(X)$  is the Todd class of  $X$ .*

**Remark:** We are using the contraction symbol  $\lrcorner$  in three different ways in this paper.

- A polyvector field  $\tilde{\alpha} \in \mathrm{HT}^*(X)$  acts on a class  $v \in \mathrm{H}\Omega_*(X)$ . This action is denoted by  $\tilde{\alpha} \lrcorner v \in \mathrm{H}\Omega_*(X)$ .
- A class  $v \in \mathrm{H}\Omega_*(X)$  acts on a polyvector field  $\tilde{\alpha} \in \mathrm{HT}^*(X)$ . This action yields an element  $v \lrcorner \tilde{\alpha} \in \mathrm{HT}^*(X)$ . We only use the second contraction in the Duflo operator  $D(\tilde{\alpha}) = Td(X)^{\frac{1}{2}} \lrcorner \tilde{\alpha}$  in this paper. Note that  $D$  is an automorphism of  $\mathrm{HT}^*(X)$ . The inverse operator is  $D^{-1}(\tilde{\alpha}) = Td(X)^{-\frac{1}{2}} \lrcorner \tilde{\alpha}$ .
- The third contraction map is  $\beta \lrcorner \exp(\mathrm{at}_{\mathcal{F}}) \in \mathrm{Ext}^*(\mathcal{F}, \mathcal{F})$  for  $\beta \in \mathrm{HT}^*(X)$ . An element  $\beta$  in  $H^p(X, \wedge^k T_X)$  can only contract with the term  $\frac{(\mathrm{at}_{\mathcal{F}})^k}{k!}$  in the Taylor expansion of  $\exp(\mathrm{at}_{\mathcal{F}})$ . It is easy to distinguish this map from the previous two maps.

**1.6.** The inspiration for Theorem A comes from a similar statement in Lie theory. Let  $\mathfrak{g}$  be a finite dimensional Lie algebra and  $V$  be a finite dimensional representation. One can draw the diagram

$$\begin{array}{ccc} (U\mathfrak{g})^{\mathfrak{g}} & \longrightarrow & \text{Hom}(V, V) \\ \uparrow & \nearrow & \\ (S\mathfrak{g})^{\mathfrak{g}} & & \end{array}$$

which is similar to the one in Theorem A. We will provide more details and prove that the diagram above is commutative in section 2.

**1.7.** Note that our statement in Theorem B appears to be different from the original one, which did not have the Duflo operator  $D$ . We will prove that the original statement follows easily from ours.

**1.8. Plan of the paper.** Section 2 contains the proof of Theorem A and of its Lie theoretic analogue.

Section 3 is devoted to the proof of Theorem B. It is a consequence of Theorem A. At the end we prove that we can recover the result in (1.1) from Theorem B.

**1.9. Acknowledgments.** I would like to thank Andrei Căldăraru for discussing details with me during our weekly meeting. I have benefited from stimulating email correspondence with Dror Bar-Natan. I am also grateful to Eyal Markman for many valuable comments. The author is partially supported by the National Science Foundation under Grant No. DMS-1811925.

## 2. The proof of Theorem A

We prove Theorem A in this section. The diagram in Theorem A has a Lie theoretic background. We draw the corresponding diagram for Lie algebras and we explain the similarity between the Lie algebra diagram and the diagram in Theorem A. We provide a proof for the commutativity of the Lie algebra diagram and explain that the proof can be generalized to the diagram in Theorem A.

**2.1. A similar diagram for Lie algebras.** Let  $\mathfrak{g}$  be a finite dimensional Lie algebra over a field of characteristic zero and let  $V$  be a finite dimensional representation of  $\mathfrak{g}$ . There is a diagram

$$\begin{array}{ccc} (U\mathfrak{g})^{\mathfrak{g}} & \longrightarrow & \text{Hom}(V, V) \\ \text{PBW} \uparrow & \nearrow & \\ (S\mathfrak{g})^{\mathfrak{g}} & & \end{array}$$

The PBW map from the symmetric algebra  $S\mathfrak{g}$  to the universal enveloping algebra  $U\mathfrak{g}$  is defined on the degree  $n$ -th component of  $S\mathfrak{g}$  as follows

$$x_1 \cdots x_n \rightarrow \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

Here  $S_n$  is the symmetric group on a finite set of  $n$  symbols. The universal enveloping algebra  $U\mathfrak{g}$  acts naturally on  $V$ . This natural action defines the map  $(U\mathfrak{g})^{\mathfrak{g}} \rightarrow \text{Hom}(V, V)$  on the top of the diagram above. The map  $(S\mathfrak{g})^{\mathfrak{g}} \rightarrow \text{Hom}(V, V)$  is defined as follows. We can rewrite the representation map  $\mathfrak{g} \otimes V \rightarrow V$  as a map  $\Lambda : V \rightarrow V \otimes \mathfrak{g}^*$ . Take the exponent

$$\exp(\Lambda) = id_V + \Lambda + \cdots + \frac{\Lambda^k}{k!} + \cdots$$

of the map  $\Lambda$ . Then we can contract  $\exp(\Lambda)$  with  $S\mathfrak{g}$ .

In algebraic geometry, Kapranov and Kontsevich [K99] observed that the shifted tangent bundle  $T_X[-1]$  has a Lie algebra structure in the derived category of  $X$ . Roberts and Willerton [RW10] proved that the category of representations of  $T_X[-1]$  is the derived category of  $X$  and the universal enveloping algebra of  $T_X[-1]$  is the Hochschild cochain complex  $\mathcal{R}\mathcal{H}om(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X)$ , where  $\Delta$  is the diagonal embedding  $\Delta : X \hookrightarrow X \times X$ . The functor  $(-)^{\mathfrak{g}}$  is the 0-th Lie algebra cohomology which is similar to  $H^*(X, -)$ . Setting  $\mathfrak{g}$  to be equal to  $T_X[-1]$  in the Lie algebra diagram, we get the diagram in Theorem A for a smooth complex variety  $X$ . The Hochschild cohomology  $\text{HH}^*(X)$  plays the role of  $(U\mathfrak{g})^{\mathfrak{g}}$ ,  $\text{HT}^*(X)$  plays the role of  $(S\mathfrak{g})^{\mathfrak{g}}$ , and the HKR map is precisely the PBW map.

**Proof of the commutativity for the Lie algebra diagram.** We can prove that the diagram in (2.1) is commutative even before taking  $\mathfrak{g}$ -invariants, i.e., the diagram

$$\begin{array}{ccc} U\mathfrak{g} & \longrightarrow & \text{Hom}(V, V) \\ \text{PBW} \uparrow & \nearrow & \\ S\mathfrak{g} & & \end{array}$$

is commutative. The map PBW factors through the tensor algebra  $T\mathfrak{g}$

$$\text{PBW} : S\mathfrak{g} \xrightarrow{\psi} T\mathfrak{g} \longrightarrow U\mathfrak{g},$$

so we can replace  $U\mathfrak{g}$  at the top left corner of the diagram by  $T\mathfrak{g}$ . It is easy to check that the map  $S\mathfrak{g} \rightarrow \text{Hom}(V, V)$  is equal to the following map

$$S\mathfrak{g} \xrightarrow{\psi} T\mathfrak{g} \xrightarrow{\varphi} \text{Hom}(V, V),$$

where the map  $\varphi : T\mathfrak{g} \rightarrow \text{Hom}(V, V)$  is defined as follows. Rewrite the representation map  $\mathfrak{g} \otimes V \rightarrow V$  as a map  $\Lambda : V \rightarrow V \otimes \mathfrak{g}^*$ . Instead of taking the exponential of the map  $\Lambda$ , we compose the map  $\Lambda$  with itself  $k$  times. We get a map  $\Lambda^{\otimes k} : V \rightarrow V \otimes (\mathfrak{g}^*)^{\otimes k}$  in this way. Contract  $\Lambda^{\otimes k}$  with  $\mathfrak{g}^{\otimes k}$  and get a map  $\mathfrak{g}^{\otimes k} \rightarrow \text{Hom}(V, V)$ . Adding the  $k$ -th components for all  $k \in \mathbb{N}$ , we obtain the desired map  $\varphi : T\mathfrak{g} \rightarrow \text{Hom}(V, V)$ .

Now we have two maps  $T\mathfrak{g} \rightarrow \text{Hom}(V, V)$ . One of them is the map  $\varphi$ , and the other one is  $\Theta : T\mathfrak{g} \rightarrow U\mathfrak{g} \rightarrow \text{Hom}(V, V)$ . We want to show that they agree. This follows from Lemma 2.2 below by setting  $W_1$  to be  $V$  and  $W_2$  to be  $\mathfrak{g}^{\otimes k}$ .

□

**2.2. Lemma.** *Let  $W_1$  and  $W_2$  be finite dimensional vector spaces over a field  $k$  and  $f$  be a map  $W_2 \otimes W_1 \rightarrow W_1$ . Rewrite the map as  $g : W_1 \rightarrow W_2^* \otimes W_1$  by the adjunction formula  $\text{Hom}(W_2 \otimes_k W_1, W_1) = \text{Hom}(W_1, W_2^* \otimes_k W_1)$ . Fix an element  $x \in W_2$ . Then  $f(x \otimes -)$  is a map from  $W_1$  to  $W_1$ . This map is precisely  $g$  followed by the contraction with  $x$ .*

*Proof.* This is due to the adjunction property

$$\text{Hom}(W_2 \otimes_k W_1, W_1) = \text{Hom}(W_1, W_2^* \otimes_k W_1).$$

□

**Proof of Theorem A.** The proof above reduces the commutativity of the Lie algebra diagram to a statement about tensor algebras. The statement about tensor algebras remains valid in the case of derived categories.

One can define a map  $\text{Sym}(T_X[-1]) \rightarrow T(T_X[-1])$  given by the formula

$$x_1 \wedge \cdots \wedge x_n \rightarrow \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} x_{\sigma(1)} \cdots x_{\sigma(n)},$$

where  $T(T_X[-1])$  is the tensor algebra on  $T_X[-1]$ .

The map above is a differential graded version of the map  $\psi$  in (2.1). Let  $X^{(1)}$  be the first order neighborhood of  $X$  in  $X \times X$ . There are embeddings  $i : X \hookrightarrow X^{(1)}$  and  $j : X^{(1)} \hookrightarrow X \times X$ . Arinkin and Căldăraru [AC12] showed that  $T(T_X[-1])$  is isomorphic to  $(i^*i_*\mathcal{O}_X)^\vee$ , where  $(-)^\vee$  is the dual. The map

$$(i^*i_*\mathcal{O}_X)^\vee \rightarrow (i^*j_*i_*\mathcal{O}_X)^\vee = (\Delta^*\Delta_*\mathcal{O}_X)^\vee = \mathcal{R}\mathcal{H}om(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X)$$

is defined by the adjunction  $j^* \dashv j_*$ . The composite map

$$\begin{aligned} \text{Sym}(T_X[-1]) &\rightarrow T(T_X[-1]) \cong (i^*i_*\mathcal{O}_X)^\vee \rightarrow (i^*j_*i_*\mathcal{O}_X)^\vee \\ &= (\Delta^*\Delta_*\mathcal{O}_X)^\vee = \mathcal{R}\mathcal{H}om(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X) \end{aligned}$$

is the sheaf version HKR isomorphism as showed in [AC12]. Taking cohomology on both sides of the equality above, we get the HKR isomorphism

$$I^{HKR} : \text{HT}^*(X) = \bigoplus_{p+q=*} H^p(X, \wedge^q T_X) \rightarrow \text{HH}^*(X).$$

Now it is clear that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{R}\mathcal{H}om(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X) & \longrightarrow & \mathcal{R}\mathcal{H}om(\mathcal{F}, \mathcal{F}) \\ \uparrow & \nearrow & \uparrow \\ T(T_X[-1]) & & \\ \uparrow & \nearrow & \\ \text{Sym}(T_X[-1]), & & \end{array}$$

which is similar to the Lie algebra diagram in (2.1). Taking cohomology on the diagram above, we get the diagram

$$\begin{array}{ccc} \text{HH}^*(X) & \longrightarrow & \text{Ext}^*(\mathcal{F}, \mathcal{F}) \\ \uparrow I^{HKR} & \nearrow & \uparrow \\ \text{HT}^*(X) = \bigoplus_{p+q=*} H^p(X, \wedge^q T_X) & & \end{array} \quad \begin{array}{l} \\ \\ (-) \lrcorner \exp(at_{\mathcal{F}}) \end{array}$$

that we start with in Theorem A. □

### 3. The proof of Theorem B

We use Theorem A to prove Theorem B in this section.



**3.1.** Denote  $I^{hkr}(\tilde{\alpha})$  by  $\alpha \in \mathrm{HH}^*(X) = \mathrm{Ext}_{X \times X}^*(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ , where  $\mathcal{O}_\Delta = \Delta_* \mathcal{O}_X$ . Denote the image of  $\alpha$  in  $\mathrm{Ext}^*(\mathcal{F}, \mathcal{F})$  by  $\alpha_{\mathcal{F}}$ . For any vector bundle  $\mathcal{F}$  on  $X$ , Căldăraru and Willerton [CW10] defined an abstract Chern character  $\mathrm{ch}(\mathcal{F})$  which lies in the degree zero part of the Hochschild homology  $\mathrm{HH}_*(X) = \mathrm{Ext}_{X \times X}^*(S_\Delta^{-1}, \mathcal{O}_\Delta)$ , where  $S_\Delta^{-1} = \Delta_*(\omega_X^\vee[-\dim X])$ . There is an HKR isomorphism for Hochschild homology

$$I_{HKR} : \mathrm{HH}_*(X) \rightarrow \mathrm{H}\Omega_*(X) = \bigoplus_{q-p=*} H^p(X, \wedge^q \Omega_X).$$

The image of the abstract Chern character under the map  $I_{HKR}$  is the usual Chern character of  $\mathcal{F}$  [C05]. We need the lemma below.

**3.2. Lemma.** *If  $\alpha_{\mathcal{F}}$  is zero, then  $\alpha \circ \mathrm{ch}(\mathcal{F})$  is zero. Here  $\circ$  is the composition of morphisms in  $\mathbf{D}^b(X \times X)$  and  $\mathrm{ch}(\mathcal{F})$  is the abstract Chern character.*

*Proof.* The proof is known in an email correspondence with Eyal Markman. Let  $\beta$  be any class in  $\mathrm{Ext}_{X \times X}^*(\mathcal{O}_\Delta, S_\Delta)$ , where  $S_\Delta = \Delta_*(\omega_X[\dim X])$ . Similar to the definition of the class  $\alpha_{\mathcal{F}}$  associated to  $\alpha \in \mathrm{Ext}_{X \times X}^*(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ , we get a class  $\beta_{\mathcal{F}} \in \mathrm{Ext}_X^*(\mathcal{F}, S_X \mathcal{F})$ , where  $S_X(-) = \omega_X[\dim X] \otimes -$ . It is shown in [C05] that the class  $\mathrm{ch}(\mathcal{F})$  is characterized by the identity

$$\mathrm{Tr}_{X \times X}(\beta \circ \mathrm{ch}(\mathcal{F})) = \mathrm{Tr}_X(\beta_{\mathcal{F}}).$$

Due to the equality above, we have

$$\mathrm{Tr}_{X \times X}(\gamma \circ \alpha \circ \mathrm{ch}(\mathcal{F})) = \mathrm{Tr}_X((\gamma \circ \alpha)_{\mathcal{F}}) = \mathrm{Tr}_X(\gamma_{\mathcal{F}} \circ \alpha_{\mathcal{F}})$$

for any  $\gamma \in \mathrm{Ext}_{X \times X}^*(\mathcal{O}_\Delta, S_\Delta)$ . The right hand side is zero since we assume that  $\alpha_{\mathcal{F}}$  is zero. We can conclude that  $\alpha \circ \mathrm{ch}(\mathcal{F})$  is zero because the equality  $\mathrm{Tr}_{X \times X}(\gamma \circ \alpha \circ \mathrm{ch}(\mathcal{F})) = 0$  holds for any  $\gamma$  and  $\mathrm{Tr}(-)$  is non-degenerate.  $\square$

The two HKR isomorphisms  $I_{HKR}$  and  $I^{HKR}$  can be twisted by the Todd class. We denote the resulting twisted isomorphisms by  $I_K$  and  $I^K$

$$I_K : \mathrm{HH}_*(X) \rightarrow \mathrm{H}\Omega_*(X) = \bigoplus_{q-p=*} H^p(X, \wedge^q \Omega_X),$$

$$I^K : \mathrm{HT}^*(X) = \bigoplus_{p+q=*} H^p(X, \wedge^q T_X) \rightarrow \mathrm{HH}^*(X).$$

They are given by the formula  $I_K = (- \wedge \mathrm{Td}(X)^{\frac{1}{2}}) \circ I_{HKR}$  and  $I^K = I^{HKR} \circ D^{-1}$ , where  $D^{-1}$  is the inverse of the Duflo operator.

The Mukai vector  $v(\mathcal{F})$  of  $\mathcal{F}$  is  $I_K(\mathrm{ch}(\mathcal{F}))$  by definition. There are natural ring structures on  $\mathrm{HH}^*(X)$  and  $\mathrm{HT}^*(X)$ : the product on  $\mathrm{HH}^*(X)$  is the Yoneda product, and

the product on  $\mathrm{HT}^*(X)$  is the wedge product. Kontsevich [Kont03] claimed that the map  $I^K$  is a ring isomorphism. This statement was proved by Calaque and Van den Bergh [CV10]. The Hochschild homology is a module over the Hochschild cohomology and similarly  $\mathrm{H}\Omega_*(X)$  is a module over  $\mathrm{HT}^*(X)$ . Calaque, Rossi, and Van den Bergh [CRV12] proved that the maps  $I_K$  and  $I^K$  respect the module structures.

**Proof of Theorem B.** The commutative diagram in Theorem A shows that

$$\tilde{\alpha} \lrcorner \exp(at_{\mathcal{F}}) = \alpha_{\mathcal{F}},$$

which is zero under the assumption of Theorem B. We conclude that  $\alpha \circ \mathrm{ch}(\mathcal{F})$  is zero by Lemma 3.2. Since  $I_K$  and  $I^K$  respect the module structures, we have

$$0 = I_K(\alpha \circ \mathrm{ch}(\mathcal{F})) = (I^K)^{-1}(\alpha) \lrcorner I_K(\mathrm{ch}(\mathcal{F})) = (I^K)^{-1}(\alpha) \lrcorner v(\mathcal{F}).$$

The inverse map of  $I^K$  is the composite map

$$(I^K)^{-1} : \mathrm{HH}^*(X) \xrightarrow{(I^{HKR})^{-1}} \mathrm{HT}^*(X) \xrightarrow{D} \mathrm{HT}^*(X).$$

As a consequence

$$0 = I_K(\alpha \circ \mathrm{ch}(\mathcal{F})) = (I^K)^{-1}(\alpha) \lrcorner v(\mathcal{F}) = D(\tilde{\alpha}) \lrcorner v(\mathcal{F}).$$

□

**3.3. The special case when  $\tilde{\alpha} \in H^1(X, T_X)$ .** The result in (1.1) says that  $\tilde{\alpha} \lrcorner v(\mathcal{F})$  is zero if  $\tilde{\alpha} \lrcorner \exp(at_{\mathcal{F}})$  is zero for any  $\tilde{\alpha} \in H^1(X, T_X)$ . We end this paper by proving that Theorem B implies the result in (1.1).

From now on let  $\tilde{\alpha}$  be an element in  $H^1(X, T_X)$ . The only term in  $\exp(at_{\mathcal{F}}) = 1 + at_{\mathcal{F}} + \frac{(at_{\mathcal{F}})^2}{2!} + \dots$  that can contract with  $\tilde{\alpha}$  is  $at_{\mathcal{F}}$ , so  $\tilde{\alpha} \lrcorner \exp(at_{\mathcal{F}}) = \tilde{\alpha} \lrcorner at_{\mathcal{F}}$  in this case.

Choose  $\mathcal{F} = O_X$ . We have  $\tilde{\alpha} \lrcorner \exp(at_{O_X}) = 0$ . Therefore

$$D(\tilde{\alpha}) \lrcorner v(O_X) = (Td(X)^{\frac{1}{2}} \lrcorner \tilde{\alpha}) \lrcorner Td(X)^{\frac{1}{2}} = 0$$

according to Theorem B.

Expand the Todd class  $Td(X)$  as  $1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \dots$ , and note that the only term of  $(Td(X)^{\frac{1}{2}} \lrcorner \tilde{\alpha}) \lrcorner Td(X)^{\frac{1}{2}}$  in  $H^2(X, O_X)$  is  $\tilde{\alpha} \lrcorner \frac{c_1}{2}$ . Since  $(Td(X)^{\frac{1}{2}} \lrcorner \tilde{\alpha}) \lrcorner Td(X)^{\frac{1}{2}} = 0$ , we can conclude that  $\tilde{\alpha} \lrcorner c_1$  is zero for any  $\tilde{\alpha} \in H^1(X, T_X)$ . The fact that  $\tilde{\alpha} \lrcorner c_1 = 0$  for

$\tilde{\alpha} \in H^1(X, T_X)$  is also known due to Griffiths. Consider the first order deformation of  $X$  corresponding to  $\tilde{\alpha}$ . The vanishing of  $\tilde{\alpha} \lrcorner c_1$  is equivalent to the class  $c_1$  remaining of type  $(p, p)$ .

The term  $\tilde{\alpha} \lrcorner \frac{c_1}{4}$  is exactly the difference between  $D(\tilde{\alpha})$  and  $\tilde{\alpha}$  because

$$D(\tilde{\alpha}) = Td(X)^{\frac{1}{2}} \lrcorner \tilde{\alpha} = (1 + \frac{c_1}{4} + \cdots) \lrcorner \tilde{\alpha} = \tilde{\alpha} + \frac{c_1}{4} \lrcorner \tilde{\alpha} + 0.$$

We conclude that  $\tilde{\alpha} \lrcorner v(\mathcal{F})$  is zero if and only if  $D(\tilde{\alpha}) \lrcorner v(\mathcal{F})$  is zero for  $\tilde{\alpha} \in H^1(X, T_X)$ .

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