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Abstract: The space of vacua of many four-dimensional, $\mathcal{N} = 2$ supersymmetric gauge theories can famously be identified with a family of complex curves. For gauge group $SU(2)$, this gives a fully explicit description of the low-energy effective theory in terms of an elliptic curve and associated modular fundamental domain. The two-dimensional space of vacua for gauge group $SU(3)$ parametrizes an intricate family of genus two curves. We analyze this family using the so-called Rosenhain form for these curves. We demonstrate that two natural one-dimensional subloci of the space of $SU(3)$ vacua, $\mathcal{E}_u$ and $\mathcal{E}_v$, each parametrize a family of elliptic curves. For these *elliptic loci*, we describe the order parameters and fundamental domains explicitly. The locus $\mathcal{E}_u$ contains the points where mutually local dyons become massless, and is a fundamental domain for a classical congruence subgroup. Moreover, the locus $\mathcal{E}_v$ contains the superconformal Argyres-Douglas points, and is a fundamental domain for a Fricke group.

Contents

1 Introduction

Supersymmetric field theories provide a rich ground for qualitative and quantitative analyses in quantum field theory [1–4]. Many observables have been determined non-perturbatively in terms of hypergeometric, modular or other special functions. The best understood example is $\mathcal{N} = 2$ supersymmetric Yang-Mills theory with gauge group $SU(2)$ [1, 2]. Its space of vacua is parametrized by the vacuum expectation value (vev) $u = \frac{1}{2}\langle \mathrm{Tr}\, \phi^2 \rangle$, where ϕ is the complex scalar in the $\mathcal{N} = 2$ vector multiplet. The renormalisation group flow generates a quantum scale Λ, at which the gauge coupling becomes strong. In the weak-coupling region $|u| \gg \Lambda^2$, the semi-classical BPS spectrum consists of massive monopoles and dyons. The theory can be solved non-perturbatively in terms of the Seiberg-Witten (SW) curve [1]. This solution demonstrates that the effective abelian gauge theory breaks down at two special points, $u = \pm\Lambda^2$. The electric-magnetic duality group is generated by the monodromies around these singular points. It is a subgroup of $SL(2, \mathbb{Z})$, which acts by linear fractional transformations on the effective coupling constant τ. With the SW solution, various physical quantities can be exactly determined as functions of τ using modular functions [5–9].

Similar non-perturbative solutions have been developed for gauge theories with matter multiplets [2] and theories with other gauge groups [11–17]. In pure Yang-Mills theory with compact gauge group G, the Coulomb branch has complex dimension $r = \mathrm{rank}(G)$. Classically, the moduli space is parametrized by the vevs $u_{I+1} \sim \langle \mathrm{Tr}\, \phi^{I+1} \rangle$, $I = 1, \dots, r$. The $r(r+1)/2$ couplings τ_{IJ} are determined by the r order parameters u_I. The electric-magnetic duality group is a subgroup of $Sp(2r, \mathbb{Z})$, generated by monodromies around singular loci. While this also demonstrates a link to modularity, the connection has remained more elusive, and the connection is best established for the superconformal theories [18–22].

One complication for asymptotically free theories is that the structure of the singular loci is in general quite intricate. This article focuses on the asymptotically free $SU(3)$ theory without hypermultiplets, whose singular loci have a rich structure [23–27]. There are six singular (complex) lines which intersect in five points. A particularly interesting phenomenon occurs at two of these five vacua, namely those where three mutually non-local dyons become massless, such that there is no duality frame in which all of these states only carry electric charge. This indicates that the system is in a critical phase, which led to the discovery of new superconformal theories [23–25].

Another complication for $SU(N > 2)$ is that the number of couplings exceeds the dimension of the Coulomb branch. The observables are therefore defined on a subspace of the genus $N-1$ Siegel upper half-space $\mathbb{H}_{N-1}$. For the $SU(3)$ theory, the Coulomb branch is parametrized by two order parameters which determine three coupling constants, τ_{11}, τ_{12} and τ_{22}. The curve and the SW differential for pure $SU(N)$ gauge theory have first been proposed in [11]. As a first step to explore the modularity of the $SU(3)$ theory, we relate the hyperelliptic Seiberg-Witten curve to the Rosenhain form, which is an algebraic expression in terms of Siegel theta series. To exactly match the Rosenhain curve and Seiberg-Witten curve, we use the fact that the complexified masses a_I and $a_{D,I} = \frac{\partial \mathcal{F}}{\partial a_I}$ are solutions of second order partial differential equations of Picard-Fuchs (PF) type. The solutions to such equations can be expressed in terms of the generalized hypergeometric function F_4 of Appell [13]. The Siegel theta series and their modular transformations can provide insights for the analytic continuation and monodromies of the solution in terms of F_4.

The Rosenhain curve allows us to characterize the $SU(3)$ Coulomb branch, parametrized by the two Casimirs $u = u_2$ and $v = u_3$, as the zero-locus of three equations inside a five-dimensional space. The structure of these equations simplifies on one-dimensional loci of the Coulomb branch. We study two of these loci in detail, namely $\mathcal{E}_u$ where $v = 0$ and $\mathcal{E}_v$ where $u = 0$. On each of these loci, the equations reduce to two algebraic relations of Siegel theta functions, relating the couplings τ_{IJ} to a single independent one. Interestingly, each of these loci in the space of genus two curves also parametrizes a family of (genus 1) elliptic curves. Both loci interpolate between a weak-coupling regime with large order parameters and a strong-coupling regime where u/Λ^2 and v/Λ^3 are $\mathcal{O}(1)$. Locus $\mathcal{E}_u$ contains three cusps where mutually local dyons become massless, while locus $\mathcal{E}_v$ contains two special points where mutually non-local dyons becomes massless. The latter are the superconformal Argyres-Douglas points.

Since an elliptic locus parametrizes a family of elliptic curves, there must be a coupling τ valued in a fundamental domain (or modular curve) for a discrete group in the upper half-plane $\mathbb{H}$. We derive the generators of the discrete subgroup from the monodromies of the $SU(3)$ theory. We provide two solutions for the locus $\mathcal{E}_u$. The coupling for the first solution is $\tau_- = \tau_{11} - \tau_{12}$, while $\tau_{22} = \tau_{11}$. The order parameter u equals a modular form u_- for the congruence subgroup $\Gamma^0(9) \subset SL(2, \mathbb{Z})$ (4.13),

$$u = u_-(\tau_-). \tag{1.1}$$

The cusps of the fundamental domain of $\Gamma^0(9)$ map exactly to the singular points on this locus. The coupling for the second solution is $\tau_+ = \tau_{11} + \tau_{12}$. In terms of this coupling, Equation (4.24) expresses u as

$$u = u_+(\tau_+), \tag{1.2}$$

where u_+ is expressed in terms of roots of modular forms, while it is not a modular function for a congruence subgroup of $SL(2,\mathbb{Z})$. We call it a *sextic modular function* since it is a solution to a sextic equation. The inverses of the identities (1.1) and (1.2) provide all order u-expansions for $\tau_{11} = \tau_{22}$ and τ_{12} on this locus. The function u_+ appeared earlier as the solutions for the order parameter on the Coulomb branch of the $\mathcal{N} = 2$, $SU(2)$ theory with one massless hypermultiplet [5]. While this Coulomb branch and $\mathcal{E}_u$ are isomorphic as four punctured spheres, it is striking that the solutions of the order parameters are identical.

We find another intriguing structure for the second locus $\mathcal{E}_v$ where $u = 0$. We are able to demonstrate for this locus that v is left invariant by the action of the principal congruence subgroup $\Gamma(6) \subset SL(2,\mathbb{Z})$. The fundamental domain $\Gamma(6)\backslash\mathbb{H}$ has 12 cusps, where v diverges. Surprisingly, this appears to imply the existence of strongly coupled vacua in the region where v is large, which is unexpected since large v is known to correspond to weak coupling. The paradox is resolved by realizing that v is invariant under a transformation which is not contained in $SL(2,\mathbb{Z})$, namely a *Fricke involution* $\tau \mapsto -1/n\tau$ for integer $n \geq 2$. This transformation maps the putative cusps to $i\infty$. The result is that v is a modular function for a discrete subgroup $\Gamma_v \subset SL(2,\mathbb{R})$ of Atkin-Lehner type, and we show that the non-trivial monodromies on this locus do generate this group.

We demonstrate furthermore that the elliptic curves underlying the two loci $\mathcal{E}_u$ and $\mathcal{E}_v$ are related to the genus two curve in a precise way. For a genus two curve Σ_2, a holomorphic map $\varphi : \Sigma_2 \to \Sigma_1$ to an elliptic curve Σ_1 may exist. Such maps were studied in the classic works by Legendre and Jacobi, and more recently in [28, 29]. The existence of the map φ depends on the complex structure moduli τ_{IJ}. The family of such curves spans a complex co-dimension one locus $\mathcal{L}_2$ in the complex three-dimensional space of genus two curves. At the elliptic loci of the Coulomb branch of the $SU(3)$ theory mentioned above, $\mathcal{L}_2$ intersects the $SU(3)$ Coulomb branch, such that for any point on the elliptic loci, there is a degree two map from the genus two curve to an elliptic curve, or in other words the genus two curve is a double cover of the elliptic curve. Besides $\mathcal{E}_u$ and $\mathcal{E}_v$, $\mathcal{L}_2$ also includes a third elliptic locus, $\mathcal{E}_3$ (6.1), which does not contain any of the singular points of the Coulomb branch.

Our work motivates a similar analysis for $SU(N)$ gauge theories, whose Coulomb branch parametrizes a curve of genus $N-1$. The order parameters u_I, $I = 2, \dots, N$, are expected to be given by higher genus modular functions of the coupling matrix τ_{IJ}. They should furthermore be invariant under a subgroup of $Sp(2r,\mathbb{Z})$ generated by the monodromies. The existence of maps to elliptic or lower genus curves is however more subtle for such theories [30, 31].

The outline of the paper is as follows. In Section 2 we give an overview of the $SU(2)$ theory. In Section 3 we review the geometry of the $SU(3)$ theory and write down asymptotic expansions of the periods which we later use. In Section

4 we discuss the Seiberg-Witten curve in Rosenhain form to match cross-ratios of the hyperelliptic curve with the theta constants. Sections 4 and 5 are devoted to studying these equations on the loci $v = 0$ and $u = 0$ respectively, which allows us to express u and v on these loci in terms of modular functions. In Section 7, we study the global symmetries of the moduli space and calculate the corresponding monodromies, along with the BPS spectrum at strong coupling. In Section 8, we comment on further directions.

We have included three appendices. Appendix A gives an overview of classical and Siegel modular forms, Appendix B discusses the Picard-Fuchs solutions, and Appendix C concludes with proofs of modular identities.

2 Review of the $SU(2)$ theory

We will begin our discussion by reviewing some of the features of pure $\mathcal{N} = 2$ Yang-Mills theory with gauge group $SU(2)$, in order to familiarize the reader with the concepts that are going to be expanded to higher rank in the following. For a more extensive review see [32–36]. The $\mathcal{N} = 2$ vector multiplet consists of a gauge field A, complex scalar field ϕ and Weyl fermions λ and ψ. They are all in the adjoint representation of the gauge group.

One of the important insights of Seiberg and Witten was that the quantum moduli space of pure $SU(2)$ super-Yang-Mills (SYM) coincides with that of an elliptic curve, parametrized by the quadratic Casimir $u = \frac{1}{2}\langle \mathrm{Tr}\phi^2 \rangle$ [1]. The complex structure of the curve is then identified with the complexified effective gauge coupling $\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$, where θ is the vacuum angle. The elliptic curve can be written in a few different ways depending on conventions. For example, in [2, 13] we find two different descriptions in terms of a cubic and a quartic polynomial, respectively. They do, however, correspond to isomorphic curves. Let us denote the two curves by $\mathcal{C}_1$ and $\mathcal{C}_2$,

$$\begin{aligned} \mathcal{C}_1 :\ \ y^2 &= x^3 - u\,x^2 + \frac{1}{4}\Lambda_1^4 x, \\ \mathcal{C}_2 :\ \ y^2 &= (x^2 - u)^2 - \Lambda_2^4. \end{aligned} \tag{2.1}$$

We will henceforth work in units where the dynamical scales are set to 1, $\Lambda_1 = \Lambda_2 = 1$. We can better understand the relation between these curves by studying the cross-ratios of the roots of the polynomials on the right hand side. The rank three polynomial can be considered as a rank four polynomial with one root at infinity, and with a certain choice of numbering the roots are[1]

$$\begin{aligned} &\mathcal{C}_1 :\ \ r_1 = \infty,\ \ r_2 = \frac{1}{2}(u + \sqrt{u^2 - 1}),\ \ r_3 = 0,\ \ r_4 = \frac{1}{2}(u - \sqrt{u^2 - 1}), \\ &\mathcal{C}_2 :\ \ r_1 = \sqrt{u + 1},\ \ r_2 = \sqrt{u - 1},\ \ r_3 = -\sqrt{u + 1},\ \ r_4 = -\sqrt{u - 1}. \end{aligned} \tag{2.2}$$

[1]the numbering is of course arbitrary, in the sense that the relations between the curves will continue to hold for other choices

The cross-ratios $C = \frac{(r_3-r_1)(r_4-r_2)}{(r_3-r_2)(r_4-r_1)}$ for both curves then become

$$\mathcal{C}_1,\, \mathcal{C}_2 \;:\; C = \frac{2\sqrt{u^2-1}}{u+\sqrt{u^2-1}}, \tag{2.3}$$

which demonstrates that the two curves are isomorphic for fixed u, $\tau_1 = \tau_2$.

A striking feature of Seiberg-Witten theory is modularity. To see this, we first note that every genus one curve can be written in Weierstraß form

$$y^2 = 4x^3 - g_2 x - g_3, \tag{2.4}$$

where the coefficients g_i depend on the complex structure. These curves have been studied extensively and the coefficients have been shown to be modular forms (see [37] for a pedagogical review). Moreover, the roots of the polynomial are functions of modular forms. Using this, one can relate the roots of the SW curve to elliptic theta functions and then further use the cross-ratios to derive an expression for u in terms of these theta functions,

$$u = \frac{\vartheta_2^4(\tau)+\vartheta_3^4(\tau)}{2\vartheta_2^2(\tau)\vartheta_3^2(\tau)} = \frac{1}{8}\left(q^{-\frac{1}{4}} + 20\,q^{\frac{1}{4}} - 62\,q^{\frac{3}{4}} + 216\,q^{\frac{5}{4}} + \mathcal{O}(q^{\frac{7}{4}})\right), \tag{2.5}$$

with $q = e^{2\pi i\tau}$. The q-expansion of $8\,u$ is known in the mathematics literature as the McKay-Thompson series of class 4C for the Monster group [38–41]. The order parameter u is a modular function of the effective coupling τ for the congruence subgroup $\Gamma^0(4)$ of $SL(2,\mathbb{Z})$ (see Appendix A.1 for the definition of the Jacobi theta functions and congruence subgroups). The fundamental domain of $\Gamma^0(4)$ is given by the image of $\mathcal{F} = SL(2,\mathbb{Z})\backslash\mathbb{H}$ under six elements in $SL(2,\mathbb{Z})$, see Figure 1. An equivalent way to derive (2.5) is to calculate the j-invariant of the SW curve and equate it with the j-invariant of the Weierstraß curve, which has a known expression in terms of Jacobi theta functions.

Singularities appear in the quantum moduli space when two branch points coincide, or equivalently when the discriminant

$$\Delta = \prod_{i<j}(r_i - r_j)^2 \tag{2.6}$$

of the curve vanishes. It is proportional to u^2-1 and we thus find three singularities, $u = \pm 1$ and $u \to \infty$. Using (2.5) it is easily shown that this corresponds to $\tau \to 0$, 2 and $i\infty$ respectively. The strong coupling points $\tau \to 0$ and $\tau \to 2$ correspond to the rational cusps of the fundamental domain of $\Gamma^0(4)$ (see Figure 1). These are the points where the monopole and the dyon become massless, respectively.

The complex masses a_D and a are given by period integrals of a meromorphic 1-form over the elliptic curve. In order to determine these, one can use the fact that

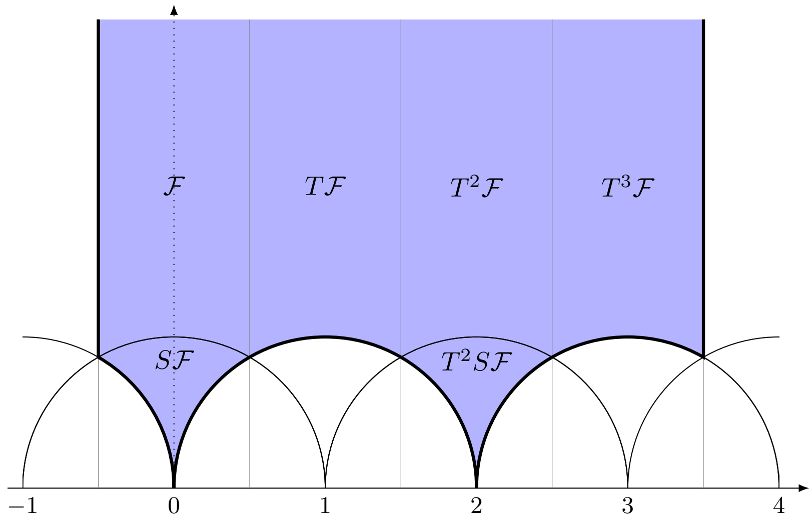

Figure 1. Fundamental domain $\Gamma^0(4)\backslash\mathbb{H}$ of the congruence subgroup $\Gamma^0(4)$. It consists of six images of the key-hole fundamental domain $\mathcal{F}$.

they form a system of solutions to a set of Picard-Fuchs equations. This allows to express the periods in terms of hypergeometric functions [42],

$$\begin{aligned} a_D(u) &= \tfrac{i}{2}(u-1)\,{}_2F_1\big(\tfrac12,\tfrac12,2;\tfrac{1-u}{2}\big),\\ a(u) &= \sqrt{\frac{(u+1)}{2}}\,{}_2F_1\big(-\tfrac12,\tfrac12,1;\tfrac{2}{1+u}\big)\,. \end{aligned} \tag{2.7}$$

At the strong coupling points, the periods $\pi(u) = (a_D(u), a(u))$ become $\pi(1) = (0, \frac{2}{\pi})$ and $\pi(-1) = (-\frac{4i}{\pi}, -\frac{2i}{\pi})$. According to the central charge formula $Z_\gamma = \gamma \cdot \pi$, these values confirm that for $u = 1$, the monopole $\gamma = (1, 0)$ becomes massless while for $u = -1$ the dyon $\gamma = (-1, 2)$ becomes massless. The limits $\lim_{u\to\pm1} a(u)$ depend on the direction from which ±1 are approached, which is due to the branch cut in the hypergeometric function [42]. We choose to take the limit from the lower half u-plane.

To demonstrate that the periods (2.7) are indeed correct, define the effective coupling as $\tau = \frac{\partial a_D}{\partial a}$. Using the chain rule, one can then compute τ as a function of u. The quantity $q = e^{2\pi i\tau}$ can be expanded for large u and the resulting relation can be inverted order by order, and one finds the same q-expansion as in (2.5).

The u-plane has a spontaneously broken global $\mathbb{Z}_2$ symmetry, acting by $u \mapsto e^{\pi i}u$. It acts on the periods π as $\rho = i\left(\begin{smallmatrix}1 & -2\\ 0 & 1\end{smallmatrix}\right)$ and sends $\tau \mapsto \tau - 2$, thus mapping the dyonic cusp, $u = -1$, to the monopole cusp, $u = 1$. Its square is the monodromy at infinity,

$$\rho^2 = M_\infty = \begin{pmatrix}-1 & 4\\ 0 & -1\end{pmatrix}. \tag{2.8}$$

It can be directly checked that (2.5) picks up a minus sign under T^2, $u(\tau-2) = -u(\tau)$, and therefore is invariant under (2.8). In fact, u also picks up a minus sign under $TST^{-1} = \left(\begin{smallmatrix} 1 & -2 \\ 1 & -1 \end{smallmatrix}\right)$, which together with T^2 generates $\Gamma^0(2)$. One could therefore consider u to be a modular function for the congruence group $\Gamma^0(2)$ with multipliers ± 1.

The zeros of (2.5) are given by the $\Gamma^0(4)$ orbit of $\tau_0 = 1+i$, lying on the boundary of the fundamental domain (proof in Appendix C.1). This can be understood as follows. The origin $u = 0$ is invariant under the global $\mathbb{Z}_2$ symmetry ρ, which acts as $\tau \mapsto \tau - 2$. The boundary arcs near the cusps 0 and 2 are identified, and the origin is the symmetric point τ_0 where the arcs from both cusps meet. The two points τ_0 and $\tau_0 + 2$ in Fig. 1 with this property are identified under $\left(\begin{smallmatrix} -3 & 4 \\ -1 & 1 \end{smallmatrix}\right) \in \Gamma^0(4)$.

3 The Coulomb branch of the $SU(3)$ theory

We study in this Section the $SU(3)$ Coulomb branch. We first recall the Seiberg-Witten geometry in Section 3.1 following [11, 13, 43]. Section 3.2 reviews the Picard-Fuchs solution for the complexified masses and couplings. Section 3.3 uses those results to write the curve in Rosenhain form.

3.1 Seiberg-Witten geometry

The vector multiplet scalar ϕ can be gauge rotated into the Cartan subalgebra of $SU(3)$. Then, ϕ can be expanded in terms of the two Cartan generators H_I, $I = 1, 2$, as

$$\phi = a_1 H_1 + a_2 H_2. \tag{3.1}$$

Non-vanishing vevs of ϕ break the gauge group in general to $U(1)^2$. The central charges of the gauge bosons are then given by

$$\begin{aligned} Z_1 &= 2a_1 - a_2, \\ Z_2 &= 2a_2 - a_1, \\ Z_3 &= a_1 + a_2. \end{aligned} \tag{3.2}$$

We denote electric-magnetic charges under $U(1)^2$ as $\gamma = (m_1, m_2, n_1, n_2)$, where m_i are the magnetic and n_i the electric charges respectively, and the period vector as $\pi = (a_{D,1}, a_{D,2}, a_1, a_2)^T$. The central charge for a generic γ is then given by $Z_\gamma = \gamma \cdot \pi$, where $\cdot$ is the standard scalar product.

The Coulomb branch is parametrized by vevs of Casimirs of ϕ, $u_I \sim \langle \mathrm{Tr}\phi^I \rangle$, $I = 2, 3$. Gauge invariant combinations for $SU(3)$ are

$$\begin{aligned} u = u_2 &= \frac{1}{2}\langle \mathrm{Tr}(\phi^2)\rangle_{\mathbb{R}^4} = a_1^2 + a_2^2 - a_1 a_2, \\ v = u_3 &= \frac{1}{3}\langle \mathrm{Tr}(\phi^3)\rangle_{\mathbb{R}^4} = a_1 a_2 (a_1 - a_2). \end{aligned} \tag{3.3}$$

These relations can be rewritten in terms of two cubic equations for a_1 and a_2 as

$$\begin{aligned} a_1^3 - u a_1 - v &= 0, \\ a_2^3 - u a_2 + v &= 0. \end{aligned} \tag{3.4}$$

There is a spontaneously broken global $\mathbb{Z}_6$ symmetry acting on u and v by $u \mapsto \alpha\, u$ and $v \mapsto -v$, with $\alpha = e^{2\pi i/3}$. Classically, the discriminant is the determinant $\Delta_{\text{classical}}$ of the matrix $B_{IJ} = \frac{\partial u_{I+1}}{\partial a_J}$. It reads

$$\Delta_{\text{classical}} = \det B_{IJ} = (a_1 - 2a_2)(2a_1 - a_2)(a_1 + a_2), \tag{3.5}$$

and vanishes when one of the gauge bosons (3.2) becomes massless.

Let us denote the space parametrized by u and v by $\mathcal{U}$. We parametrize points on this space by $(\underline{u}, v) \in \mathcal{U}$, where $\underline{u}$ is the normalized parameter, $\underline{u} = \sqrt[3]{\frac{4}{27}}\, u$. The moduli space $\mathcal{U}$ parametrizes a complex two-dimensional family of hyperelliptic curves of genus two [11, 43],

$$y^2 = (x^3 - u\,x - v)^2 - \Lambda^6, \tag{3.6}$$

which has discriminant

$$\Delta_\Lambda = \Lambda^{18}(4u^3 - 27(v + \Lambda^3)^2)(4u^3 - 27(v - \Lambda^3)^2). \tag{3.7}$$

This can be viewed as a product of the discriminants of two elliptic curves whose v parameters are separated by $2\Lambda^3$. Note that the $\mathbb{Z}_6$ global symmetry leaves the discriminant invariant. It vanishes if and only if $\underline{u}^3 = (v \pm \Lambda^3)^2$. We will frequently use units where the dynamical scale $\Lambda = 1$ and we note that it can always be restored from dimensional analysis.

If we restrict to $\operatorname{Im} v = 0$, the zero locus of the discriminant describes six singular curves which intersect in the following points. On the $v = 0$ plane, there are four singularities, namely $\underline{u} \in \{\infty, 1, \alpha, \alpha^2\}$. On the other hand for $\underline{u} = 0$, there are two singularities at $v = \pm 1$. These are the Argyres-Douglas points, where mutually non-local BPS states become massless and the theory becomes superconformal [23]. Figure 2 sketches the singular lines on the subset of $\mathcal{U}$ where $\operatorname{Im} v = 0$. The singular lines represent regions in $\mathcal{U}$ where the effective action of the pure $\mathcal{N} = 2$ theory becomes singular, and they are associated with vacua where hypermultiplets become massless.

Similarly to the $SU(2)$ case, the periods transform under monodromies which generate the duality group of the theory. The classical part of the monodromy group is given by the Weyl group of the $SU(3)$ root lattice, which acts as reflections on lines perpendicular to the positive roots. The perturbative quantum correction comes from the one-loop effective action. It contributes to the prepotential as

$$\mathcal{F}_{1-\text{loop}} = \frac{i}{2\pi} \sum_\alpha Z_\alpha^2 \log Z_\alpha, \tag{3.8}$$

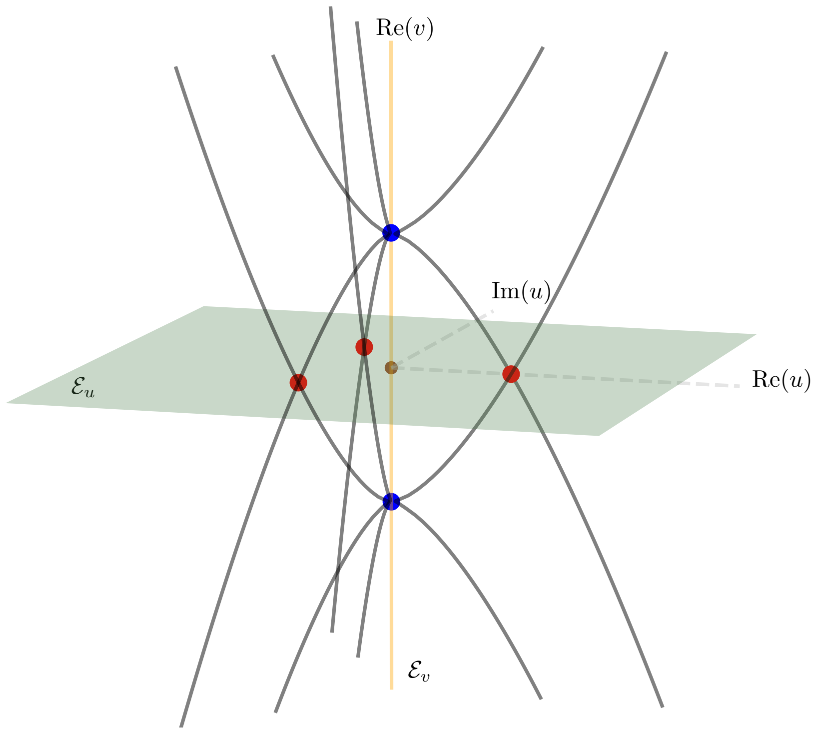

Figure 2. Singular lines $\Delta(u,v) = 0$ in the $SU(3)$ moduli space with Im $v = 0$, associated to massless dyons [13]. The red dots represent the strong coupling points $(\underline{u}, v) = (1, 0)$, $(\alpha, 0)$ and $(\alpha^2, 0)$ on the $v = 0$ plane $\mathcal{E}_u$, where two singular lines intersect. The blue dots represent the AD points $(\underline{u}, v) = (0, 1)$ and $(0, -1)$ respectively, where three singular lines intersect. They lie on $\mathcal{E}_v$, which is represented by the Re v axis here. The two loci $\mathcal{E}_u$ and $\mathcal{E}_v$ intersect in the origin $(u, v) = (0, 0)$ (brown).

where the sum runs over all positive roots α_1, α_2 and $\alpha_3 = \alpha_1 + \alpha_2$. Here, Z_α are the central charges (3.2) of the gauge bosons.

The semi-classical monodromies can be derived in the following way. The Weyl group of the root lattice A_2 is generated by two reflections, r_1 and r_2. The element r_k reflects the root lattice on the line perpendicular to α_k . For instance, r_2 induces the map $\alpha_2 \mapsto -\alpha_2$, $\alpha_1 \mapsto \alpha_1 + \alpha_2$. Using (3.2), we find that $a_1 \mapsto a_1$ and $a_2 \mapsto a_1 - a_2$. The semi-classical transformation of the dual variables can be obtained using (3.8) and the fact that, semi-classically, $a_{D,I} = \frac{\partial \mathcal{F}_{1-\text{loop}}}{\partial a_I}$ holds. The crucial insight is that $Z_2 \mapsto -Z_2$ induces a shift of πi due to the logarithm, and the result can be written as an integer linear combination of the periods. The other two Weyl elements transform a_1 and a_2 in the following way,

$$\begin{aligned} r_1 &: (a_1, a_2) \mapsto (a_2 - a_1, a_2), \\ r_2 &: (a_1, a_2) \mapsto (a_1, a_1 - a_2), \\ r_3 &: (a_1, a_2) \mapsto (-a_2, -a_1). \end{aligned} \tag{3.9}$$

The corresponding monodromies can be obtained in a similar fashion, the result is

$$\mathcal{M}^{(r_1)} = \begin{pmatrix} -1 & 0 & 4 & -2 \\ 1 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{M}^{(r_2)} = \begin{pmatrix} 1 & 1 & 1 & -2 \\ 0 & -1 & -2 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \mathcal{M}^{(r_3)} = \begin{pmatrix} 0 & -1 & 1 & -2 \\ -1 & 0 & 4 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{3.10}$$

which satisfy $\mathcal{M}^{(r_3)} = \mathcal{M}^{(r_2)}\mathcal{M}^{(r_1)}(\mathcal{M}^{(r_2)})^{-1}$.

3.2 Picard-Fuchs solution

One way to find the non-perturbative solution is to notice that the periods satisfy second order partial differential equations of Picard-Fuchs (PF) type, whose solution space is spanned by the generalized hypergeometric function F_4 of Appell [13]. We review some aspects of the PF solution in the following, and left further details for Appendix B. We study two interesting regions, one where u is large and v small, and the other one where v is large and u is small.

The non-perturbative effective action is characterized by the holomorphic prepotential $\mathcal{F}$, which allows to define the dual periods $a_{D,I} = \frac{\partial \mathcal{F}}{\partial a_I}$. Both periods a_I and $a_{D,I}$ are given by linear combinations of Appell functions. The large u expansion reads [13]

$$\begin{aligned} a_{D,1}(u,v) &= -\frac{i}{2\pi}\left(\sqrt{u} + \frac{3}{2}\frac{v}{u}\right)\log\left(\frac{27}{4u^3}\right) - \frac{1}{\pi}\left(\frac{i}{2} + 2\alpha_1\right)\sqrt{u} + \dots, \\ a_1(u,v) &= \sqrt{u} + \frac{1}{2}\frac{v}{u} + \dots, \end{aligned} \tag{3.11}$$

with $a_{D,2}(u,v) = a_{D,1}(u,-v)$, $a_2(u,v) = a_1(u,-v)$ and $\alpha_1 \in \mathbb{C}$ a constant (see Appendix B.2). The coupling constants $\tau_{IJ} = \frac{\partial a_{D,I}}{\partial a_J}$ are determined using the chain rule,

$$\tau_{11}(u,v) = \tau_{22}(u,-v) = \frac{i}{\pi}\log(8u^3) + \frac{9iv}{2\pi}u^{-3/2} - \left(\frac{129i}{32\pi} + \frac{63iv^2}{8\pi}\right)u^{-3} + \dots, \tag{3.12}$$

The off-diagonal τ_{12} is given by the series

$$\tau_{12}(u,v) = -\frac{\tau_{11}(u,v) + \tau_{22}(u,v)}{4} - \frac{1}{2\pi i}\log(8) + \frac{1}{2\pi i}\frac{27}{4}f(u,v), \tag{3.13}$$

where

$$f(u,v) = \frac{(1-4v^2)}{8}u^{-3} + \left(\frac{453}{1024} - 3v^2 - \frac{31}{16}v^4\right)u^{-6} + \dots. \tag{3.14}$$

Similarly, we find that the large v expansion of the coupling matrix reads (see Appendix B.3 for details, $\omega = e^{\pi i/6}$)

$$\tau_{11} \sim \frac{i}{\pi}\log(108v^2) - 1 + \frac{\omega}{\pi}uv^{-2/3} + \frac{\omega^5}{6\pi}u^2v^{-4/3} - \left(\frac{11i}{18\pi} + \frac{4i}{27\pi}u^3\right)v^{-2} + \dots, \tag{3.15}$$

and τ_{12} and τ_{22} are given by similar series. At $u = 0$ we have $\tau_{11} = \tau_{22} + 1$ and $\tau_{12} = -\frac{\tau_{11}}{2} + 1$.

For pure $SU(N)$ supersymmetric gauge theory, the periods satisfy the following interesting relation [44, 45],

$$\sum_{I=1}^{N-1} a_I a_{D,I} - 2\mathcal{F} = \frac{Ni}{\pi} u, \tag{3.16}$$

with $u = u_2 = \frac{1}{2}\langle \mathrm{Tr}(\phi^2)\rangle$. Here, $\mathcal{F}$ is the prepotential of the pure $SU(N)$ theory. For $N = 3$, we can differentiate (3.16) with respect to u and v to get

$$\begin{aligned} \frac{3i}{\pi} &= a_1 a'_{D,1} - a'_1 a_{D,1} + a_2 a'_{D,2} - a'_2 a_{D,2}, \\ 0 &= a_1 \dot{a}_{D,1} - \dot{a}_1 a_{D,1} + a_2 \dot{a}_{D,2} - \dot{a}_2 a_{D,2}, \end{aligned} \tag{3.17}$$

where $'$ ($\dot{}$) denotes $\frac{\partial}{\partial u}$ ($\frac{\partial}{\partial v}$). Both relations serve as useful checks of our solutions.

3.3 Seiberg-Witten curve in Rosenhain form

In this section, we will relate the $SU(3)$ Seiberg-Witten curve to the curve in Rosenhain form, which is a degree 5 equation. Every genus two hyperelliptic curve can be brought to the Rosenhain form [46]

$$y^2 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3). \tag{3.18}$$

The three roots λ_i of the polynomial are also referred to as *Rosenhain invariants*. These invariants are complementary to the Igusa invariants [47, 48].

By a lemma of Picard, the Rosenhain invariants can be expressed in terms of even theta constants as

$$\lambda_1 = \frac{\Theta_1^2\,\Theta_3^2}{\Theta_2^2\,\Theta_4^2}, \quad \lambda_2 = \frac{\Theta_3^2\,\Theta_8^2}{\Theta_4^2\,\Theta_{10}^2}, \quad \lambda_3 = \frac{\Theta_1^2\,\Theta_8^2}{\Theta_2^2\,\Theta_{10}^2}. \tag{3.19}$$

The functions Θ_j are instances of genus two Siegel modular forms,

$$\Theta\begin{bmatrix} a \\ b \end{bmatrix}(\Omega) = \sum_{k\in\mathbb{Z}^2} \exp\left(\pi i (k+a)^T \Omega (k+a) + 2\pi i (k+a)^T b\right), \tag{3.20}$$

where the entries of the column vectors a and b take values in the set $\{0, \frac{1}{2}\}$. The argument Ω is a 2×2-matrix

$$\Omega = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}, \tag{3.21}$$

valued in the Siegel upper half-plane $\mathbb{H}_2$. We refer to Appendix A.2 for a precise definition and references. The moduli space of genus two curves $\mathcal{M}_2$ is complex three-dimensional. Since the SW order parameters u and v are two complex parameters, the $SU(3)$ Coulomb branch maps out a complex two-dimensional space $\mathcal{U} \subset \mathcal{M}_2$ in the moduli space of genus two curves. In other words, $\mathcal{U}$ is a divisor of $\mathcal{M}_2$.

To relate the Rosenhain curve (3.18) to the Seiberg-Witten curve (3.6), note that a degree 5 polynomial as in (3.18) can be obtained by a linear fractional transformation of a degree 6 hyperelliptic equation $y^2 = \prod_{j=1}^{6}(x - r_j)$, which maps three of the roots to ∞, 0 and 1. Linear fractional maps leave cross-ratios invariant, which is a convenient way to relate the λ_j to u and v. Let us define the cross-ratio of four points $z_i \in \mathbb{CP}^1$ as

$$C(z_1, z_2, z_3, z_j) = \frac{(z_1 - z_3)(z_2 - z_j)}{(z_1 - z_j)(z_2 - z_3)}, \tag{3.22}$$

such that $C(\{\infty, 0, 1, \lambda_j\}) = \lambda_j$.

Note that we have 120 different possibilities to map three roots among the $\{r_j\}$ to $0, 1, \infty$, and another 3! possibilities to identify the three cross-ratios in the hyperelliptic setting with the λ_j. By studying the large u expansions of these for non-zero v, one can easily identify which cross-ratios, in terms of the r_i, correspond to which λ_j. To this end, let $\alpha = e^{2\pi i/3}$ as before. The roots of the rhs of (3.6) are then given by (with $\Lambda = 1$)

$$\begin{aligned}
r_1 &= s_+(u, v+1) + s_-(u, v+1), & r_4 &= s_+(u, v-1) + s_-(u, v-1),\\
r_2 &= \alpha\, s_+(u, v+1) + \alpha^2\, s_-(u, v+1), & r_5 &= \alpha\, s_+(u, v-1) + \alpha^2\, s_-(u, v-1),\\
r_3 &= \alpha^2\, s_+(u, v+1) + \alpha\, s_-(u, v+1), & r_6 &= \alpha^2\, s_+(u, v-1) + \alpha\, s_-(u, v-1),
\end{aligned} \tag{3.23}$$

where

$$s_\pm(u, v) = \sqrt[3]{\frac{v}{2} \pm \sqrt{\frac{v^2}{4} - \frac{u^3}{27}}}. \tag{3.24}$$

To simplify notation, let us set $s_{\pm\pm} \coloneqq s_\pm(u, v \pm 1)$. The large u, small v expansions for the roots are

$$\begin{aligned}
r_1 &= \sqrt{u} + \frac{1+v}{2u} + \ldots, & r_4 &= \sqrt{u} - \frac{1-v}{2u} + \ldots,\\
r_2 &= -\sqrt{u} + \frac{1+v}{2u} + \ldots, & r_5 &= -\sqrt{u} - \frac{1-v}{2u} + \ldots,\\
r_3 &= -\frac{1+v}{u} + \ldots, & r_6 &= \frac{1-v}{u} + \ldots.
\end{aligned} \tag{3.25}$$

Plugging the weak-coupling expansions (3.12) into the Rosenhain invariants gives the leading behaviour for the λ_j. From this we can see that each invariant λ_j approaches 1 in the large u limit.

We continue by determining which of the 720 possible sets of cross-ratios matches with the theta constants. We have to determine which roots correspond to the first three points z_i, $i = 1, 2, 3$, in the cross-ratio (3.22). Since the three theta constants approach 1 in the large u limit, we should take for $\{z_1, z_2\}$ in (3.22) the roots which vanish in this limit, thus $\{r_3, r_6\}$. Together with the choice of z_2, this reduces to 8 possible triplets. From a further comparison between the Rosenhain invariants and the cross-ratios, we determine that $z_1 = r_6$, $z_2 = r_3$ and $z_3 = r_2$. With $C_j := C(r_6, r_3, r_2, r_j)$ for $j = 1$, 4 and 5, we arrive at

$$\lambda_1 = C_5, \qquad \lambda_2 = C_1, \qquad \lambda_3 = C_4. \tag{3.26}$$

These are three equations for five unknowns, namely τ_{11}, τ_{12}, τ_{22}, u and v. To make it more manifest that the right hand side depends on only two variables, let us express the cross-ratios C_j in terms of $s_{\pm\pm}$,

$$\begin{aligned} C_1 &= \alpha^2 \frac{[\alpha\, s_{+-} + s_{--} - s_{++} - \alpha\, s_{-+}]\,[s_{++} - \alpha\, s_{-+}]}{[\alpha^2 s_{+-} + \alpha\, s_{--} - s_{++} - s_{-+}]\,[s_{-+} - s_{++}]}, \\ C_4 &= -\frac{[\alpha\, s_{+-} + s_{--} - s_{++} - \alpha\, s_{-+}]\,[\alpha^2\, s_{++} + \alpha\, s_{-+} - s_{+-} - s_{--}]}{3[s_{+-} - \alpha\, s_{--}]\,[s_{-+} - s_{++}]}, \\ C_5 &= -\alpha^2 \frac{[\alpha\, s_{+-} + s_{--} - s_{++} - \alpha\, s_{-+}]\,[\alpha\, s_{++} + s_{-+} - s_{+-} - \alpha\, s_{--}]}{3[s_{--} - s_{+-}]\,[s_{-+} - s_{++}]}. \end{aligned} \tag{3.27}$$

Note that these expressions are true on the full moduli space. For $u \neq 0$, we can define

$$X = \frac{s_{++}}{\sqrt{u/3}}, \qquad Y = \frac{s_{+-}}{\sqrt{u/3}}, \tag{3.28}$$

such that $X^{-1} = s_{-+}/\sqrt{u/3}$ and $Y^{-1} = s_{--}/\sqrt{u/3}$, since $s_{+\pm}\, s_{-\pm} = u/3$. The cross-ratios can then be expressed as

$$\begin{aligned} C_1 &= -\,\alpha^2 \frac{X(X-\alpha Y)(X-Y^{-1})(X-\alpha X^{-1})}{(X^2-1)(X-\alpha^2 Y)(X-\alpha Y^{-1})}, \\ C_4 &= -\,\frac{1}{3}\alpha^2 \frac{(X-\alpha Y)^2(X-Y^{-1})(X-\alpha Y^{-1})}{X(X^2-1)(Y-\alpha Y^{-1})}, \\ C_5 &= \quad \frac{1}{3} \frac{(X-\alpha Y)(X-Y^{-1})^2(X-\alpha^2 Y)}{X(X^2-1)(Y-Y^{-1})}. \end{aligned} \tag{3.29}$$

We thus see that the Coulomb branch can be identified with the zero-locus of the three equations (3.29) inside the space $(\lambda_1, \lambda_2, \lambda_3, X, Y)$. One may in principle eliminate X and Y to arrive at a single equation in terms of the λ_j. In the following two sections, we will restrict to the two one-dimensional sub-loci $\mathcal{E}_u$ and $\mathcal{E}_v$ of the solution space of (3.26), where $v = 0$ and $u = 0$ respectively.

4 Locus $\mathcal{E}_u$: $v = 0$

In this section we analyse the locus $v = 0$. We will demonstrate that the order parameter u can be expressed in terms of classical modular forms on this locus. In fact, we will arrive at two distinct expressions depending on a choice of effective coupling. In Section 6, we will discuss these aspects from the geometric point of view.

4.1 Algebraic relations

On the locus $v = 0$ we have that $\tau_{11}(u,0) = \tau_{22}(u,0)$ and $\tau_{12}(u,0)$ is given by (3.13). Let us analyse these coupling constants, now from the perspective of Section 3.3. For u large and positive, $s_{+\pm}$ has a large magnitude and phase $e^{\pi i/6}$. Similarly, the phase of $s_{-\pm}$ is approximately given by $e^{-\pi i/6}$ (see Appendix B.1 for a discussion on the subtlety of the cubic root). This means that

$$s_{--} = -\alpha\, s_{++}, \qquad s_{+-} = -\alpha^2\, s_{-+}, \qquad X = -\alpha^2\, Y^{-1}. \tag{4.1}$$

Using this and (3.28), we find that (3.29) now turns into

$$\begin{aligned} C_1 &= -\,\frac{(X+X^{-1})\,(X-\alpha X^{-1})}{(X-X^{-1})\,(X+\alpha X^{-1})}, \\ C_4 &= -\,\frac{1}{3}\frac{(X+X^{-1})^2}{(X-X^{-1})^2}, \\ C_5 &= +\,\frac{1}{3}\frac{(X+X^{-1})\,(X+\alpha X^{-1})}{(X-\alpha X^{-1})\,(X-X^{-1})}. \end{aligned} \tag{4.2}$$

Since the rhs of (4.2) depends only on one variable X, the cross-ratios C_j satisfy two algebraic equations, which can be determined by solving the equations for X^2. One finds

$$\begin{aligned} &C_1\,C_5 - C_4 = 0, \\ &(3C_4 - C_1)^2 - C_4(C_1+1)^2 = 0. \end{aligned} \tag{4.3}$$

Using (3.26) and (3.19), the cross-ratios are identified with quotients of Siegel theta functions (see Appendix A.2), and the above equations take the form

$$\begin{aligned} 0 &= \Theta_3^4 - \Theta_4^4, \\ 0 &= \Theta_1^2\Theta_2^2\Theta_8^4\Theta_3^4 - \Theta_2^4\Theta_8^2\Theta_{10}^2\Theta_3^4 + 8\,\Theta_1^2\Theta_2^2\Theta_4^2\Theta_8^2\Theta_{10}^2\Theta_3^2 + \Theta_1^2\Theta_2^2\Theta_4^4\Theta_{10}^4 - 9\,\Theta_1^4\Theta_4^4\Theta_8^2\Theta_{10}^2. \end{aligned} \tag{4.4}$$

The two systems of equations above are equivalent given that none of the λ_j vanish or are infinite, which is an assumption of Picard's lemma (3.19). We can use the second relation of (4.2) to solve for u,

$$u^3 = \frac{\sqrt{27}}{2}\frac{(3C_4+1)^3}{\sqrt{C_4}(C_4-1)}, \tag{4.5}$$

and in terms of theta constants this gives

$$u^3 = \frac{\sqrt{27}}{2}\frac{(3\Theta_1^2\Theta_8^2 + \Theta_2^2\Theta_{10}^2)^3}{\Theta_1\Theta_2^3\Theta_8\Theta_{10}^3(\Theta_1^2\Theta_8^2 - \Theta_2^2\Theta_{10}^2)}. \tag{4.6}$$

This can be viewed as a generalisation of the rank 1 result (2.5), in the sense that we can write the parameter u as a rational function of theta series. It follows naively that u transforms as a weight 0 function under a subgroup of $Sp(4,\mathbb{Z})$.

4.2 A modular expression for u

The solutions to the algebraic relations (4.4) are not unique due to the periodicity in the τ_{IJ}. The first equation implies $\tau_{11}-\tau_{22}=2k$ with $k\in\mathbb{Z}$, but we know from (3.12) that $k=0$. From (3.13) we can make a power series expansion for τ_{12} in terms of $p=e^{2\pi i\tau_{11}}$. One finds

$$\tau_{12}=-\frac{1}{2}\tau_{11}-\frac{1}{2\pi i}\log(8)+\frac{1}{2\pi i}\frac{27}{4}h(p), \tag{4.7}$$

with

$$h(p)=p^{\frac{1}{2}}-\frac{63}{16}p+\frac{1447}{64}p^{\frac{3}{2}}-\frac{307679}{2048}p^{2}+\mathcal{O}(p^{\frac{5}{2}}), \tag{4.8}$$

by satisfying the second relation in (4.4) order by order. Substitution of (4.7) in (4.5) gives the following p-expansion for u,

$$u=\frac{1}{2}p^{-\frac{1}{6}}+\frac{43}{8}p^{\frac{1}{3}}-\frac{2923}{128}p^{\frac{5}{6}}+\frac{1713}{16}p^{\frac{4}{3}}+\mathcal{O}(p^{\frac{11}{6}}). \tag{4.9}$$

One can verify agreement with the Picard-Fuchs approach by substituting this expansion in Eq. (3.12). As this series is only an expansion for small p, it is not very elucidating. To arrive at a closed expression, we aim to express u as a function of a "coupling constant" which transforms well under the duality transformations. This is not the case for τ_{11}.

However when $\tau_{11}=\tau_{22}$, the inversion $\mathcal{S}=\left(\begin{smallmatrix}0 & -\mathbb{1}\\ \mathbb{1} & 0\end{smallmatrix}\right)\in Sp(4,\mathbb{Z})$ acts naturally on the linear combinations $\tau_{\pm}=\tau_{11}\pm\tau_{12}$, which are in one-to-one correspondence with τ_{11} and τ_{12}. From (A.13), we deduce for the action of $\mathcal{S}$ on $\tau_{\pm}$

$$\mathcal{S}:\tau_{11}\pm\tau_{12}\mapsto-\frac{1}{\tau_{11}\pm\tau_{12}}. \tag{4.10}$$

That is to say, it reduces to the ordinary S-transformation $\tau_{\pm}\mapsto-1/\tau_{\pm}$. Moreover, $\tau_{\pm}\in\mathbb{H}$ for both $\pm$. To see this note that since $\mathrm{Im}(\Omega)$ is positive definite, we have that $y_{11}>0$ and $y_{11}y_{22}-y_{12}^2>0$, where $y_{IJ}=\mathrm{Im}(\tau_{IJ})$. Whenever $y_{11}=y_{22}$, the latter inequality implies that $y_{11}^2>y_{12}^2$. Since $y_{11}>0$, it implies $y_{11}>y_{12}$ and $y_{11}>-y_{12}$ simultaneously. From this we learn that $y_{11}-y_{12}$ and $y_{11}+y_{12}$ are both positive and therefore $\tau_{\pm}:=\tau_{11}\pm\tau_{12}\in\mathbb{H}$.

We will proceed by considering $\tau_{-}=:\tau$, leaving the discussion on τ_{+} for Section 4.3. To determine u as function of τ, one can first find the series expansion for τ in terms of p, invert and substitute $p(\tau)$ in (4.9). Alternatively, one can revert to the Picard-Fuchs solution, by inverting the series (3.12) for $v=0$,

$$q=e^{2\pi i(\tau_{11}(u)-\tau_{12}(u))}=U^3+45U^4+1512U^5+45672U^6+\dots,\qquad U=\frac{1}{4u^3}. \tag{4.11}$$

Either method gives us the following series for u,

$$\sqrt[3]{4}\,u=q^{-\frac{1}{9}}+5\,q^{\frac{2}{9}}-7\,q^{\frac{5}{9}}+3\,q^{\frac{8}{9}}+15\,q^{\frac{11}{9}}-32\,q^{\frac{14}{9}}+\mathcal{O}(q^{\frac{17}{9}}). \tag{4.12}$$

This expansion is also known as the McKay-Thompson series of class 9B for the Monster group [38–41]. Thus similarly to the u for rank 1 (2.5), we find a McKay-Thompson series. We then have

$$u = u_-(\tau) = \sqrt[3]{\frac{27}{4}}\,\frac{b_{3,0}\left(\frac{\tau}{3}\right)}{b_{3,1}\left(\frac{\tau}{3}\right)}, \tag{4.13}$$

where $b_{3,j}$ are theta series for the A_2 root lattice,

$$b_{3,j}(\tau) = \sum_{k_1,k_2\in\mathbb{Z}+\frac{j}{3}} q^{k_1^2+k_2^2+k_1k_2}, \quad j\in\{-1,0,1\}. \tag{4.14}$$

The theta series $b_{3,j}$ transform under the generators of $SL(2,\mathbb{Z})$ as ($\alpha = e^{2\pi i/3}$)

$$\begin{aligned} S: &\quad b_{3,j}\left(-\frac{1}{\tau}\right) = -\frac{i\tau}{\sqrt{3}}\sum_{l \bmod 3}\alpha^{2jl}\, b_{3,l}(\tau),\\ T: &\quad b_{3,j}(\tau+1) = \alpha^{j^2} b_{3,j}(\tau). \end{aligned} \tag{4.15}$$

The solution u_- can also be expressed in terms of the Dedekind η-function (A.8) as

$$u_-(\tau) = \sqrt[3]{\frac{27}{4}}\left(1+\frac{1}{3}\frac{\eta\left(\frac{\tau}{9}\right)^3}{\eta(\tau)^3}\right). \tag{4.16}$$

Using Theorem 1 in Appendix A.1, one finds that $u_-(9\tau)$ is a modular function for the congruence subgroup $\Gamma_0(9)$ (also defined in Appendix A.1). This implies that u is a modular function for $\Gamma^0(9)$, which is generated by the matrices T^9, STS and $(T^3S)T(T^3S)^{-1}$. In fact, it is easy to see from (4.15) that $u_-(\tau-3) = \alpha u_-(\tau)$ for all $\tau\in\mathbb{H}$. Furthermore, u rotates as well under TST^{-2}, $u_-\left(\frac{\tau-3}{\tau-2}\right) = \alpha u_-(\tau)$. The two elements T^3 and TST^{-2} generate $\Gamma^0(3)$ and u can therefore be interpreted as a modular function for $\Gamma^0(3)$ with multipliers α^k, analogous to the discussion on rank 1 in Section 2.

Let us analyse the strong coupling singularities $u^3 = \frac{27}{4}$ for $v=0$ in terms of the variable τ. We will demonstrate that these correspond to $\tau\to 0, 3$ and -3. Using (4.15), one finds that the expansion around 0 takes the form

$$\begin{aligned} \sqrt[3]{\frac{4}{27}}\,u_{-,D}(\tau_D) &= \frac{b_{3,0}(3\tau_D)+2b_{3,1}(3\tau_D)}{b_{3,0}(3\tau_D)-b_{3,1}(3\tau_D)}\\ &= 1+9\,q_D+27\,q_D^2+81\,q_D^3+198\,q_D^4+\mathcal{O}(q_D^5), \end{aligned} \tag{4.17}$$

with $\tau_D = -1/\tau$, $q_D = e^{2\pi i\tau_D}$ and $u_{-,D}(\tau_D) := u_-(-1/\tau_D)$. In the same notation we can invert the series to find

$$q_D = \chi - 3\chi^2 + 9\chi^3 - 22\chi^4 + 21\chi^5 + 207\chi^6 + \mathcal{O}(\chi^7), \tag{4.18}$$

where $\chi := (\sqrt[3]{4/27}\,u-1)/9$. It follows that $q_D \to 0$ for $\sqrt[3]{4/27}\,u \to 1$ or $\chi \to 0$. This can be directly confirmed by analytically continuing the Picard-Fuchs expansion around $u = \sqrt[3]{27/4}$.

The expansion around ± 3 can then be obtained from the one around 0 by shifting the argument $\tau_{D,\pm} = -\frac{1}{\tau} \pm 3$, and one finds using the T-transformation (4.15) that

$$u_{-,D}(\tau_{D,\pm}) = \alpha^{\mp 1}\sqrt[3]{\tfrac{27}{4}}\,\frac{b_{3,0}(3\tau_D)+2b_{3,1}(3\tau_D)}{b_{3,0}(3\tau_D)-b_{3,1}(3\tau_D)} \tag{4.19}$$

The expansions around the points 3 and -3 differ from the one around 0 only by the phases $\alpha^{-1} = \alpha^2$ and α. Together with (4.17), this proves that indeed $\tau \to \{0,-3,3\}$ corresponds to the three singularities $\underline{u} \to \{1,\alpha,\alpha^2\}$. Due to the T^9-invariance of the solution (4.13), there is an ambiguity in identifying the τ-parameter with $\tau + 9\,\mathbb{Z}$. These $\mathbb{Z}_2$ points are studied in detail in [43, 49]. They correspond to the 3 vacua of the $\mathcal{N}=1$ theory after deforming the $\mathcal{N}=2$ theory by relevant or marginal terms.

The modular analysis is completely analogous to the $SU(2)$ theory, as reviewed in Section 2: The cusps of $\Gamma^0(9)$ are $\{0,-3,3,i\infty\}$, which is exactly where u assumes the $\mathbb{Z}_2$ vacua and the semi-classical limit. The fundamental domain of $\Gamma^0(9)$ is given in Figure 3 and is the union of 12 images of the $SL(2,\mathbb{Z})$ fundamental domain $\mathcal{F} = SL(2,\mathbb{Z})\backslash\mathbb{H}$,

$$\Gamma^0(9)\backslash\mathbb{H} = \bigcup_{\ell=-4}^{4} T^\ell\mathcal{F} \cup S\mathcal{F} \cup T^3 S\mathcal{F} \cup T^{-3}S\mathcal{F}. \tag{4.20}$$

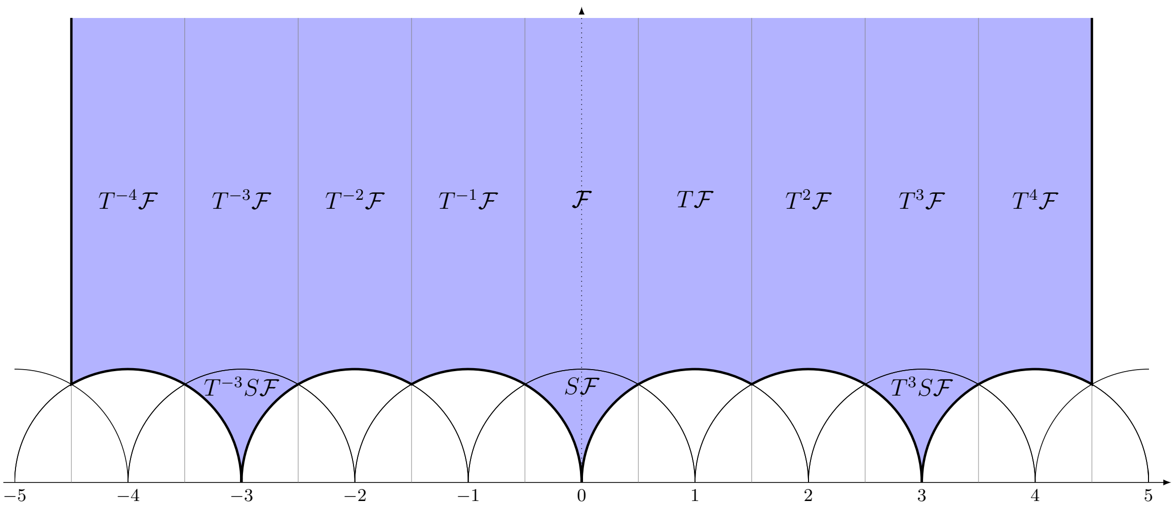

Figure 3. Fundamental domain $\Gamma^0(9)\backslash\mathbb{H}$ of the congruence subgroup $\Gamma^0(9)$. It consists of 12 images of the key-hole fundamental domain $\mathcal{F}$.

Using (4.16), we can find the exact coupling at the origin of the moduli space. We have that $u(\tau_0)=0$ for the $\Gamma^0(9)$ orbit of

$$\tau_0=\sqrt{3}\,\omega=\frac{3}{2}+\frac{\sqrt{3}}{2}i, \tag{4.21}$$

with $\omega=e^{\pi i/6}$. This is rigorously proven in Appendix C.1. The point τ_0 lies on the boundary of the fundamental domain, on the point where the boundary arcs from different cusps meet. The elements $(STS)^k\in\Gamma^0(9)$ map $\tau_0\mapsto\tau_0-3k$ for integer k, which identifies the "corners" in Figure 3. This is compatible with the global $\mathbb{Z}_3$ symmetry, which also acts by T^{-3} and leaves the origin invariant. It is in complete analogy to the $SU(2)$ picture, see Section 2: We find the nice picture that the cusps of $\Gamma^0(9)\backslash\mathbb{H}$ are in one-to-one correspondence with the singularities $u^3=\frac{27}{4}$ and $u=\infty$ and the origin is the symmetric point where the boundary arcs meet.

We will derive the modular expression for u from the SW geometry in Section 6. Section 7.2 will discuss how the action of the $SU(3)$ monodromies reduce to the generators of $\Gamma^0(9)$ for the action on τ_-.

The connection between elliptic curves and theta constants furthermore allows to express the periods $\frac{\partial a_I}{\partial u_J}$ as modular forms. Indeed, the period matrix $\frac{\partial a_I}{\partial u_J}$ can be written as a combination of even, odd and differentiated theta constants [50]. By substituting the solution for u and v into the asymptotic expansion of the periods, we can confirm this for some cases. Recall that in the $SU(2)$ theory, a is a quasi-modular form and $\frac{da}{du}$ is a modular form of $\Gamma^0(4)$ with non-trivial multipliers, both of weight 1 [35]. For rank 2, one finds that on $v=0$,

$$\frac{\partial a_1}{\partial v}(\tau)=-\frac{\partial a_2}{\partial v}(\tau)=\frac{1}{3\sqrt[3]{2}}b_{3,1}(\tfrac{\tau}{3})=\frac{1}{\sqrt[3]{2}}\frac{\eta(\tau)^3}{\eta(\frac{\tau}{3})}. \tag{4.22}$$

Theorem 1 in Appendix A.1 confirms that it is a modular form of weight 1 on $\Gamma^0(9)$, which is the same modular group as for u.

4.3 u as a sextic modular function

While we chose in the above the modular parameter $\tau_-=\tau_{11}-\tau_{12}$, Equation (4.10) shows that we could equally well consider $\tau_+=\tau_{11}+\tau_{12}$. We will consider the variable $\tau:=\tau_+$ in this subsection. We can determine the first terms in the q-expansion of u, which results in

$$u=u_+(\tau)=\frac{1}{4}\left(q^{-1/3}+104\,q^{2/3}-7396\,q^{5/3}+\mathcal{O}(q^{8/3})\right). \tag{4.23}$$

This series can be recognized as the q-expansion of

$$u_+(\tau)=\sqrt[3]{\tfrac{27}{2}}\,\frac{E_4(\tau)^{1/2}}{(E_4(\tau)^{3/2}-E_6(\tau))^{1/3}}, \tag{4.24}$$

where E_4 and E_6 are the Eisenstein series (A.3). We will derive this explicitly in Section 6. The function u_+ is a root of the sextic equation

$$X^6 - \frac{j(\tau)}{64}\,X^3 + \frac{27\,j(\tau)}{256} = 0, \tag{4.25}$$

where j is the $j-$invariant (A.4). Since the coefficients of this sextic equation are modular functions for $SL(2,\mathbb{Z})$, we call u_+ a *sextic modular function*. Due to the fractional powers in (4.24), u_+ is not a modular function for $SL(2,\mathbb{Z})$. In fact, $E_4^{1/2}$ and u_+ are not invariant under *any* subgroup of $SL(2,\mathbb{Z})$. One way to see this is that E_4 has a simple zero for $\tau = \alpha$, such that the square root introduces a branch cut. While the family of sextic modular functions thus includes functions which are not modular for $SL(2,\mathbb{Z})$, this family also includes functions which are modular for an index 6 congruence subgroup of $SL(2,\mathbb{Z})$. The order parameter for $SU(2)$ (2.5) is an example of the latter. One can thus view the family of sextic modular functions as an extension of the family of modular functions for index 6 congruence subgroups. We will discuss the modular properties of (4.24) in more detail in a future work [51].

Interestingly, u_+ is up to an overall factor the same function as the order parameter of the massless $N_f = 1$ theory with gauge group $SU(2)$ [5, 9, 52]. This aspect distinguishes massless $N_f = 1$ from $N_f = 0, 2, 3$, since for the latter theories the order parameters are modular functions for congruence subgroups isomorphic to $\Gamma^0(4)$ [5]. On the other hand, it is known since the time of Fricke and Klein that similar fractional powers of modular forms as in u_+ do appear in the context of Picard-Fuchs equations and hypergeometric functions [53, 54].

As mentioned before, the fractional powers in (4.24) are incompatable with any subgroup of $SL(2,\mathbb{Z})$. Nevertheless, if we choose a basepoint, we can show that u_+ is invariant under transformations of τ, which combine to a closed trajectory with starting and endpoint equal to the base point. We choose the base point τ_b with $\mathrm{Re}(\tau_b) = 0$ and $\mathrm{Im}(\tau_b) \gg 1$. First, using the modular transformation of E_4 and E_6, we find for the expansion of τ near 0,

$$\tau \to 0: \qquad u_+(\tau) = u_{+,D}(-1/\tau), \tag{4.26}$$

with

$$\begin{aligned} u_{+,D}(\tau_D) &= \sqrt[3]{\tfrac{27}{2}}\,\frac{E_4(\tau_D)^{1/2}}{(E_4(\tau_D)^{3/2} + E_6(\tau_D))^{1/3}} \\ &= \sqrt[3]{\tfrac{27}{4}}\,\left(1 + 144\,q_D - 3456\,q_D^2 + 596160\,q_D^3 + \dots\right). \end{aligned} \tag{4.27}$$

The S-transform $u_{+,D}$ is also a solution to (4.25) and thus also a sextic modular function. From Eq. (4.23) we see that u_+ is invariant under $\tau \mapsto \tau + 3$ at weak coupling, $\mathrm{Im}(\tau) \gg 1$. Let us introduce T_w for the translation at weak coupling. Moreover at strong coupling, $0 < \mathrm{Im}(\tau) \ll 1$, u_+ is invariant under $\tau_D = -1/\tau \mapsto$

$\tau_D + 1$. Let us introduce T_s for the translation at strong coupling. We can get the monodromies around the other cusps, $\tau = \pm 1$ from conjugation with T_w. We then find that u_+ is left invariant by

$$T_w^{3n}, \qquad (T_w^\ell S) T_s (T_w^\ell S)^{-1}, \qquad \ell, n \in \mathbb{Z}, \tag{4.28}$$

where S is the usual inversion $\tau \mapsto -1/\tau$, mapping τ from weak to strong coupling. These transformations are sketched in Figure 4 for $n = 1$ and $\ell = 0, \pm 1$.

We denote the invariance group of u_+ by Γ_{u_+}. It is generated by the elements in (4.28) with $n = 1$, and $\ell = 0, 1$. From the invariance under (4.28), one derives that a fundamental domain is given by

$$\Gamma_{u_+} \backslash \mathbb{H} = \bigcup_{\ell=-1}^{1} T^\ell \mathcal{F} \cup T^\ell S \mathcal{F}. \tag{4.29}$$

It consists of six copies of $\mathcal{F}$, which is directly related to u_+ being a sextic modular function. This fundamental domain is the grey area in Figure 4. The domain is clearly topologically equivalent to the fundamental domain in Figure 3. The expansions of u_+ and $u_{+,D}$ demonstrate that $u_+(i\infty) = \infty$, $u_+(0) = \sqrt[3]{\frac{27}{4}}$ and $u_+(\pm 1) = \alpha^{\mp} \sqrt[3]{\frac{27}{4}}$. We will derive u_+ from the SW geometry in Section 6, and the transformations (4.28) in Section 7.2 from the $SU(3)$ monodromies around the strong coupling cusps.

Because u_+ is not a weakly holomorphic modular form, but involves fractional powers of modular forms, it is problematic to identify the transformations (4.28) with elements of $SL(2,\mathbb{Z})$. One way to see that this identification is problematic is that the composition of S, T_w and T_s does not satisfy the relation $(ST)^3 = -\mathbb{1}$, if we identify $T_w = T_s = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. To further study this aspect, let us list the $SL(2,\mathbb{Z})$ matrices corresponding to (4.28),

$$\begin{aligned} T^3 &= \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \\ STS^{-1} &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \\ (TS)T(TS)^{-1} &= \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \\ (T^{-1}S)T(T^{-1}S)^{-1} &= \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{4.30}$$

These matrices fix each of the cusps $\{\infty, 0, 1, -1\}$. On the other hand, u_+ is not invariant under the modular action of the matrices on τ, $\tau \mapsto (a\tau + b)/(c\tau + d)$

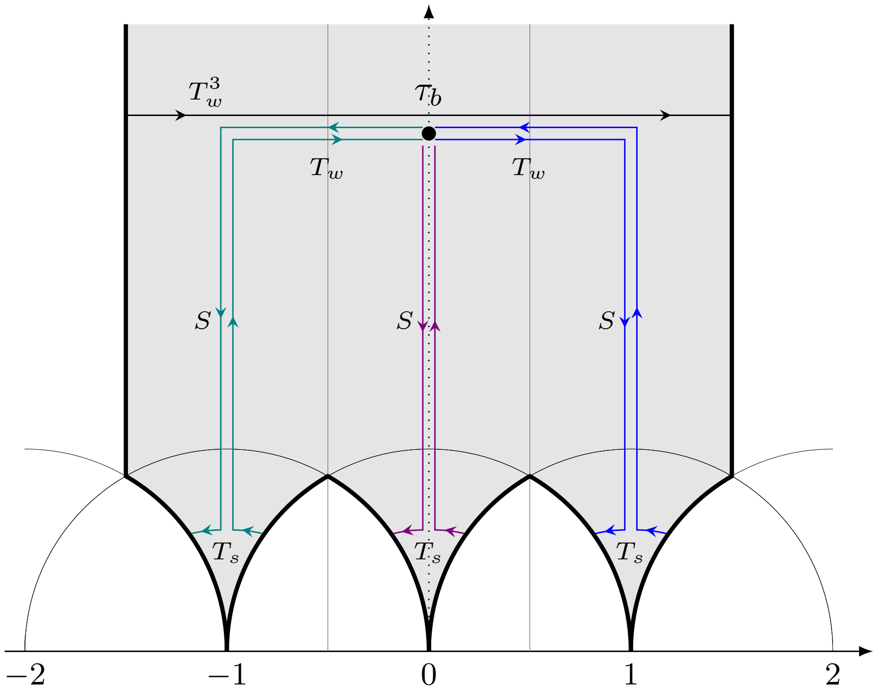

Figure 4. Fundamental domain for u_+. The vertical lines at $\tau = \pm 3/2$ are identified, as well as each pair of the two arcs meeting at a cusp $-1, 0$ or 1. The point τ_b is the base point for the monodromies, which are compositions of T_w, T_s and S. T_w is a shift $\tau \mapsto \tau + 1$ at weak coupling, T_s circles around a strong coupling cusp, and S maps τ from weak to strong coupling.

except for T^{3n}. For example, STS^{-1} would map $\tau = i\infty$ to -1. The values of u_+ are however different for these two arguments: $u_+(i\infty) = \infty$ and $u_+(-1) = \alpha\sqrt[3]{\frac{27}{4}}$. Furthermore, the matrices (4.30) generate the full modular group $SL(2, \mathbb{Z})$.

The origin $u_+(\tau_0) = 0$ of the moduli space is again given by the points where the boundary arcs meet: At $\tau_0 = \alpha$ we have that E_4 vanishes but E_6 does not. From (4.24) it is then clear that $\tau_0 + \mathbb{Z}$ are indeed the zeros of u_+. This is also compatible with the $\mathbb{Z}_3$ global symmetry, which according to (4.23) acts as T^{-1} and leaves the origin invariant.

5 Locus $\mathcal{E}_v$: $u = 0$

We will now consider the second elliptic locus, namely where $u = 0$. By doing a similar analysis as in Section 4 but now for large v, we find that the correct matching between the cross-ratios and the Rosenhain invariants for this limit is

$$\lambda_1 = C_5, \qquad \lambda_2 = C_4, \qquad \lambda_3 = C_1. \tag{5.1}$$

Note that the only difference from before is that the rôles of λ_2 and λ_3 have been interchanged. One could perform a change of symplectic basis to have the same matching as (3.26). This can be be done by acting on the periods with the matrix $\mathcal{T}_\theta = \left(\begin{smallmatrix} \mathbb{1} & \theta \\ 0 & \mathbb{1} \end{smallmatrix}\right) \in Sp(4,\mathbb{Z})$ with $\theta = \left(\begin{smallmatrix} -1 & 2 \\ 2 & -4 \end{smallmatrix}\right)$.[2] This changes the ω_1, ω_2 prefactors of $a_{D,1}$ in (B.16). This would however also change the Rosenhain form, and we therefore prefer to continue with the identification in (5.1).

We will proceed by deriving the relations satisfied by the couplings τ_{IJ} on the locus $u = 0$.

5.1 Algebraic relations

To determine the algebraic relations among the theta constants, we assume that v is real, large and positive. In this limit we find that $s_{+\pm} = \sqrt[3]{v \pm 1}$ and $s_{-\pm} = 0$. The cross-ratios (3.27) simplify to

$$\begin{aligned} C_1 &= -\,\alpha^2 \frac{s_{++} - \alpha s_{+-}}{s_{++} - \alpha^2 s_{+-}}, \\ C_4 &= -\,\frac{\alpha^2}{3} \frac{(s_{++} - \alpha s_{+-})^2}{s_{++} s_{+-}}, \\ C_5 &= +\,\frac{1}{3} \frac{(s_{++} - \alpha s_{+-})(s_{++} - \alpha^2 s_{+-})}{s_{++} s_{+-}}. \end{aligned} \tag{5.2}$$

From this we find two algebraic relations between the cross-ratios, namely

$$\begin{aligned} &C_1 C_5 - C_4 = 0, \\ &C_5^2 + C_4^2 - C_5 C_4 - C_4 = 0. \end{aligned} \tag{5.3}$$

Writing these in terms of the theta constants, we have

$$\begin{aligned} 0 &= \Theta_1^4 - \Theta_2^4, \\ 0 &= \Theta_2^4 \Theta_3^2 \Theta_8^4 + \Theta_1^4 \Theta_3^2 \Theta_{10}^4 - \Theta_1^2 \Theta_2^2 \Theta_3^2 \Theta_8^2 \Theta_{10}^2 - \Theta_2^4 \Theta_4^2 \Theta_8^2 \Theta_{10}^2. \end{aligned} \tag{5.4}$$

5.2 Modular expression for v

Our next aim is to determine a modular expression for v on this elliptic locus. The first relation in (5.4) implies $\tau_{11} = \tau_{22} + 2\mathbb{Z} + 1$, while the second one implies $\tau_{12} = \pm\frac{1}{2}\tau_{11} + \mathbb{Z}$. We claim that these are all the solutions. As in the case $v = 0$, the PF solution (3.15) fixes these relations,

$$\tau_{11} = \tau_{22} + 1, \qquad \tau_{12} = -\frac{\tau_{11}}{2} + 1. \tag{5.5}$$

In contrast to the locus $\mathcal{E}_u$, these linear relations between the τ_{11}, τ_{22} and τ_{12} are exact on $\mathcal{E}_v$. Using the first equation in (5.2), we can solve for v,

$$v = -\frac{i}{\sqrt{27}} \frac{(C_1 - 2)(C_1 + 1)(2C_1 - 1)}{C_1(C_1 - 1)}. \tag{5.6}$$

[2] Note that there is an ambiguity in the choice of $\mathcal{T}_\theta$. The λ_j are invariant under a subgroup of $Sp(4,\mathbb{Z})$. Multiplying $\mathcal{T}_\theta$ with an element of this group thus gives the same result.

This can again be written as a rational function of Siegel theta functions,

$$v = -\frac{i}{\sqrt{27}}\frac{(\Theta_8^2 - 2\Theta_{10}^2)(\Theta_8^2 + \Theta_{10}^2)(2\Theta_8^2 - \Theta_{10}^2)}{\Theta_8^2\Theta_{10}^2(\Theta_8^2 - \Theta_{10}^2)}. \tag{5.7}$$

As a function of $\tau_- = \tau_{11} - \tau_{12}$, one finds ($q_- = e^{2\pi i \tau_-}$)

$$v = \frac{i}{2\sqrt{27}}\left(\alpha\, q_-^{-\frac{1}{6}} - 33\,\alpha^2\, q_-^{\frac{1}{6}} - 153\, q_-^{\frac{1}{2}} - 713\,\alpha\, q_-^{\frac{5}{6}} + \mathcal{O}(q_-^{\frac{7}{6}})\right). \tag{5.8}$$

The expansion in terms of $\tau_+ = \tau_{11} + \tau_{12}$ is very similar. One can recognize these series as

$$\begin{aligned} v &= \tfrac{i}{2\sqrt{27}}\, m(\tfrac{\tau_+}{2}), \\ v &= \tfrac{i}{2\sqrt{27}}\, m(\tfrac{\tau_-}{6} + \tfrac{2}{3}), \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} m(\tau) &= \left(\frac{\eta\,(2\tau)}{\eta\,(6\tau)}\right)^6 - 27\left(\frac{\eta\,(6\tau)}{\eta\,(2\tau)}\right)^6 \\ &= q^{-1} - 33\,q - 153\,q^3 - 713\,q^5 - 2550\,q^7 - 7479\,q^9 + \mathcal{O}(q^{11}). \end{aligned} \tag{5.10}$$

The function m is known in the literature as the completely replicable function of class 6a [39–41]. Since the relations between the τ_{IJ} are linear in this case, one can prove the step from (5.7) to (5.9). The details of the proof are given in Appendix C.2. The perturbative expansion (5.8) can also be verified from the Picard-Fuchs solution by starting from Eq. (3.15) and setting $u = 0$. Then, expand $q = e^{2\pi i(\tau_{11}(v) - \tau_{12}(v))}$ as a series in v and invert it to find (5.8).

5.3 The $\mathbb{Z}_3$ vacua

Let us study the solution (5.9) near the strong coupling vacua. To this end, we eliminate the phases in (5.8) by substitution of $\tau := \tau_- + 1$ in (5.9). In the new variable τ, the solution reads

$$v = -\tfrac{i}{2\sqrt{27}}\, m\big(\tfrac{\tau}{6}\big)\,. \tag{5.11}$$

It can be shown that the values of τ at the Argyres-Douglas (AD) vacua $v_{\text{AD},1} = 1$ and $v_{\text{AD},2} = -1$ are ($\omega = e^{\pi i/6}$)

$$\begin{aligned} \tau_{\text{AD},1} &= -\frac{3}{2} + \frac{\sqrt{3}i}{2} = \sqrt{3}\,\omega^5, \\ \tau_{\text{AD},2} &= +\frac{3}{2} + \frac{\sqrt{3}i}{2} = \sqrt{3}\,\omega, \end{aligned} \tag{5.12}$$

and the origin $(u, v) = (0, 0)$ is located at $\tau_0 = \sqrt{3}i$. This is rigorously proven in Appendix C.2 using the properties of m. Note that these values lie in the interior of the upper half-plane, rather than at the boundary. Section 7.1 will demonstrate that these values of τ also match perfectly with the PF solutions.

The modular group of v is closely related to the duality group of the $SU(3)$ theory on this locus. It can be shown that v is a modular form for the principal congruence subgroup $\Gamma(6)$, as defined in Appendix A.1. However, the fundamental domain of this group has twelve cusps, and v diverges at all of them. This suggests that we found strongly coupled vacua in the region of the moduli space where v is large. But from the discriminant $\Delta_\Lambda|_{\mathcal{E}_v} = v^2 - 1$ we expect the only singularities to be at $v \in \{1, -1, \infty\}$.

To resolve this problem, let us study the function m in more detail. It is a linear combination of eta quotients, whose modular properties have been studied extensively [55, 56]. Applying Theorem 1 in Appendix A.1, one finds that m is a modular function for the Hecke congruence subgroup $\Gamma_0(12)$. In addition, it satisfies the following non-$SL(2,\mathbb{Z})$ transformations

$$m\left(\tau - \tfrac{1}{2}\right) = -m(\tau), \tag{5.13a}$$

$$m\left(-\tfrac{1}{12\tau}\right) = -m(\tau), \tag{5.13b}$$

and further properties are given in Appendix C.2. The transformation (5.13b) is also known as a *Fricke involution*. Translating both equations to the argument of v, we find that v picks up a minus sign under both T^{-3} and $F = \left(\begin{smallmatrix} 0 & -3 \\ 1 & 0 \end{smallmatrix}\right)$. Taking products, we find that v is properly invariant under $FT^{-3} = \left(\begin{smallmatrix} 0 & -3 \\ 1 & -3 \end{smallmatrix}\right)$ and T^{-6}. Let us normalize the former to $X = \frac{1}{\sqrt{3}}\left(\begin{smallmatrix} 0 & -3 \\ 1 & -3 \end{smallmatrix}\right)$, and denote the subgroup of $PSL(2,\mathbb{R})$ generated by these two elements as

$$\Gamma_v = \left\langle X, T^{-6} \right\rangle. \tag{5.14}$$

This group is a proper subgroup of the modular group $\Gamma^0(6|2) + 3$ of Atkin-Lehner type, in the notation of [41]. This Atkin-Lehner group extends the ordinary congruence subgroup $\Gamma^0(\frac{6}{2})$ by elements in $PSL(2,\mathbb{R})$. See Appendix A.1 for the precise definition. If we allow for a non-trivial multiplier system, the modular group associated with m is $\Gamma^0(6|2) + 3$ [41] . The latter contains for example T^{-3}, under which we have shown that v is anti-invariant. We can write a similar set of matrices as (4.30),

$$M_1 = \begin{pmatrix} -3 & -3 \\ 1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 3 \\ -1 & 3 \end{pmatrix}, \quad M_\infty = \begin{pmatrix} 1 & -6 \\ 0 & 1 \end{pmatrix} = T^{-6}, \tag{5.15}$$

under which $v \sim m(\tau/6)$ is invariant. If we consider their normalisation to unit determinant, $\Pi(M_j) \coloneqq |\det(M_j)|^{-1/2}\, M_j$, they lie in the group Γ_v (5.14), and furthermore satisfy

$$\Pi(M_1)\Pi(M_2) = M_\infty. \tag{5.16}$$

We will show in Section 7.2 that these generators match with the monodromies.

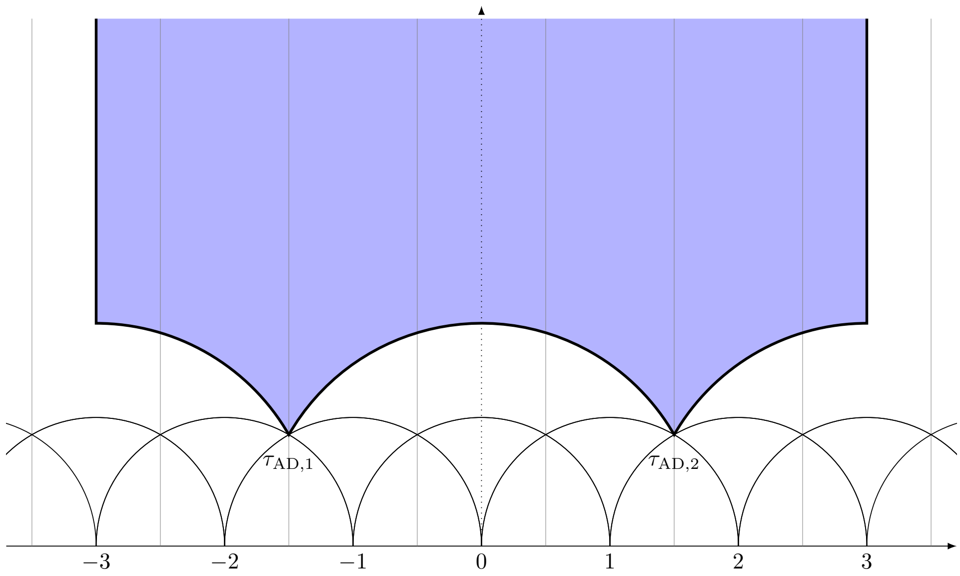

Figure 5. Fundamental domain $\Gamma_v\backslash\mathbb{H}$ for the group Γ_v. The values of the special points are: $\tau_{\mathrm{AD},1} = \sqrt{3}\,\omega^5$ and $\tau_{\mathrm{AD},2} = \sqrt{3}\,\omega$.

A fundamental domain for Γ_v can be drawn using the algorithm given in [41], and it is shown in Figure 5. The element T^6 contains the domain to $|\mathrm{Re}\,\tau| < 3$. X identifies the interior of the circle with radius $\sqrt{3}$ centered at 0, with a region inside the blue domain in Figure 5. Similarly, the interior of the circles centered at ± 3 is identified with a region of the blue domain. We conclude,

$$\Gamma_v\backslash\mathbb{H} = \{z \in \mathbb{H} \mid |\mathrm{Re}\, z| < 3\} \setminus \bigcup_{\ell=-1}^{1} \overline{D}_{\sqrt{3}}(3\ell). \tag{5.17}$$

where $\overline{D}_r(c)$ is the closed disc of radius r and center c.

The Argyres-Douglas vacua $v = 1$ and $v = -1$ correspond to the special points $\tau_{\mathrm{AD},j}$ (5.12). They are stabilized by M_1 and M_2, respectively. This makes the AD vacua elliptic points of Γ_v. They are in fact expected to *not* get mapped to cusps of v, since their coupling matrix (7.11) lies inside the Siegel upper half-space $\mathbb{H}_2$. This is a familiar property of superconformal points [23, 57]. It is different from the $\mathbb{Z}_2$ points where the coupling matrices (7.10) are located on the boundary $\partial\mathbb{H}_2$ and therefore mapped to the real line $\partial\mathbb{H}_1$. The origin $\tau_0 = \sqrt{3}i$ is mapped under FT^{-3} to $\tau_0 - 3$, which is identified with τ_0 since $v = 0$ is a fixed point under $T^{-3} : v \mapsto -v$. The anti-invariance under T^{-3} is in fact directly derived from the $\mathbb{Z}_2$ symmetry $\rho : v \mapsto e^{\pi i}v$ computed in (7.9). The large v monodromy ρ^2 acts on τ as T^{-6}, under which v is invariant. The origin of the Fricke involution can therefore be understood from the global structure on the $u = 0$ plane.

Similarly to Section 4.2, we can express periods in terms of modular forms. We have in terms of $\tau = \tau_{11} - \tau_{12} + 1$,

$$\frac{\partial a_1}{\partial u}(\tau) = \frac{\partial a_2}{\partial u}(\tau) = \frac{\sqrt[3]{2}\omega}{\sqrt{3}}\eta(\tfrac{\tau}{3})\eta(\tau). \tag{5.18}$$

The discussion is similar for the parameter $\tau_+ = \tau_{11} + \tau_{12}$. If we introduce here $\tau = \tau_+ - 1$, v equals $-\frac{i}{2\sqrt{27}}m(\tau/2)$, which is again invariant under $\Gamma(6)$. It is multiplied by a sign under T as well as under the Fricke involution $\tilde{F} = \left(\begin{smallmatrix} 0 & -1 \\ 3 & 0 \end{smallmatrix}\right)$. This means that it is invariant under T^2 together with the involution $\tilde{X} := \tilde{F}T^{-1} = \left(\begin{smallmatrix} 0 & -1 \\ 3 & -3 \end{smallmatrix}\right)$, which again generate an Atkin-Lehner type group. The fundamental domain of this group equals that in Figure 5, but with all points divided by 3.

6 Elliptic curves for the two loci

It is natural to expect that the complexified couplings $\tau_\pm$ for both loci $\mathcal{E}_u$ and $\mathcal{E}_v$ have an interpretation as complex structures of elliptic curves. Moreover, these elliptic curves are expected to be related to the geometry of the genus two Seiberg-Witten curve (3.6). We will make these expectations precise in this section.

Recall that the moduli space $\mathcal{M}_2$ of genus two curves is complex three-dimensional. The moduli space $\mathcal{M}_2$ contains two-dimensional loci $\mathcal{L}_2 \subset \mathcal{M}_2$, for which the genus two curves can be mapped to genus one with a map of degree 2 [58]. The map can be lifted to a map of the Jacobians of the curves. The Jacobian of the genus two curve is a four-torus, while the Jacobian of a genus one curve is a two-torus. For the curves contained in $\mathcal{L}_2$, there is a degree two map from the genus two Jacobian to the genus one Jacobian. The Jacobian of a curve in $\mathcal{L}_2$ factors, $T^4 \equiv T^2 \times T^2$, which demonstrates that for a generic curve in $\mathcal{L}_2$, there are two distinct maps $\varphi_j : \Sigma_2 \to \Sigma_{1,j}$, $j = 1, 2$ to two elliptic curves $\Sigma_{1,j}$. We will see in this section that these elliptic curves $\Sigma_{1,j}$ have precisely the complex structures $\tau_\pm$ introduced above.

The locus $\mathcal{L}_2$ can be characterized as the zero locus of a weight 30 polynomial in the genus two Igusa invariants J_2, J_4, J_6, J_{10} [28, Theorem 3].[3] See [59] for a definition of the Igusa invariants. Additionally, the $SU(3)$ vacuum moduli space also corresponds to a two-dimensional locus $\mathcal{U}$ in $\mathcal{M}_2$. On $\mathcal{U}$ the weight 30 polynomial factors in three terms, such that $\mathcal{U}$ and $\mathcal{L}_2$ intersect in three one-dimensional loci:

$$\begin{aligned} \mathcal{E}_1 = \mathcal{E}_u : &\quad v = 0, \\ \mathcal{E}_2 = \mathcal{E}_v : &\quad u = 0, \\ \mathcal{E}_3 \quad : &\quad 784u^9 - 24u^6\left(297v^2 + 553\right) - 15u^3\left(729v^4 + 5454v^2 - 4775\right) \\ &\quad + 8\left(27v^2 - 25\right)^3 = 0. \end{aligned} \tag{6.1}$$

[3] We found two small typos in Theorem 3 of [28]: For the coefficient of $J_{10}^2 J_4^2 J_2$, we find $+507384000$; and -6912 for the coefficient of $J_4^3 J_6^3$.

Not suprisingly, we have seen the first two of these loci before. The latter is a cubic equation in v^2 as well as in u^3, which does not reduce further. It does not include special points of the $SU(3)$ theory. For $v = 0$, the equation reduces to the points $u^3 = 8$ and $u^3 = \frac{125}{28}$ in the u-plane, and for $u = 0$ it intersects in $v^2 = \frac{25}{27}$ on the v-plane.

The locus $\mathcal{L}_2$ can also be characterized in terms of Rosenhain invariants of the curve [28, Equation (18)]. By plugging in the cross-ratios we can check that the $SU(3)$ Seiberg-Witten curve is not in $\mathcal{L}_2$ for generic u, v. For $v = 0$ we rediscover the first algebraic relation (4.3), while for $u = 0$ we find both relations (5.3). This arises from an additional symmetry of the $u = 0$ curve, which we will comment on below.

6.1 Elliptic curves for locus $\mathcal{E}_u$

In this subsection we will establish two elliptic curves corresponding to the two modular parameters $\tau_\pm$ in Section 4. The curves described by the locus $\mathcal{L}_2$ can be written in the form [28]

$$Y^2 = X^6 - s_1 X^4 + s_2 X^2 - 1, \tag{6.2}$$

with s_1 and s_2 complex coordinates for $\mathcal{L}_2$. This family of curves is left invariant by a non-trivial automorphism group, which contains the Klein four-group V_4 [60]. Namely, the curve (6.2) is left invariant by $(X, Y) \mapsto (-X, Y)$ and $(X, Y) \mapsto (X, -Y)$, which generate the dihedral group $D_4 \cong V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. We interpret this group as the symmetry group of BPS/anti-BPS spectrum, and more precisely the central charges of the W-bosons Z_j (3.2) and their charge conjugates. For $v = 0$, Eq. (3.11) shows that $a_1 = a_2 = a$, such that $Z_1 = Z_2 = a$, and $Z_3 = 2a$. One $\mathbb{Z}_2 \subset D_4$ corresponds to the charge conjugation symmetry, while the other $\mathbb{Z}_2$ corresponds to the $a_1 \leftrightarrow a_2$ symmetry on $\mathcal{E}_u$. Note that the automorphism group of a generic genus two curve is $\mathbb{Z}_2$, which is consistent with the charge conjugation symmetry for arbitrary (u, v).

For $v = 0$, the Seiberg-Witten curve $Y^2 = (X^3 - uX)^2 - 1$ is of the form (6.2), with $s_1 = 2u$ and $s_2 = u^2$. The degree two map to an elliptic curve is

$$(x, y) = (X^2, Y), \tag{6.3}$$

which maps the algebraic equation (6.2) to

$$y^2 = x(x - u)^2 - 1. \tag{6.4}$$

We can determine u in terms of the complex structure τ of the curve from the j-invariant, $j = 256u^6/(4u^3 - 27)$. This gives

$$4u(\tau) = q^{-1/3} + 104\, q^{2/3} - 7396\, q^{5/3} + \mathcal{O}(q^{8/3}). \tag{6.5}$$

We immediately recognize this function as the function u_+ (4.23), which was obtained from the Picard-Fuchs solution for the modular parameter $\tau_+ = \tau_{11} + \tau_{12}$. The curve (6.4) is exactly the Seiberg-Witten curve for the $SU(2)$ theory with one massless hypermultiplet in the fundamental representation and scales related by $\Lambda_{SU(2)} = 2\Lambda_{SU(3)}$ [2], which clarifies the observation in Section 4.3.

The curve that corresponds to $\tau_- = \tau_{11} - \tau_{12}$ can be constructed as follows. On the curve (6.2), the transformation $(X, Y) \mapsto (\frac{1}{X}, \frac{iY}{X^3})$ interchanges s_1 and s_2. Interchanging those coefficients, $s_1 = u^2$ and $s_2 = 2u$, and setting again $(x, y) = (X^2, Y)$, we obtain

$$y^2 = x(x^2 - u^2 x + 2u) - 1. \tag{6.6}$$

One finds $j = 256u^3(u^3 - 6)^3/(4u^3 - 27)$, which reproduces the solution u_- for the $\Gamma^0(9)$ curve (4.13). Note that the equation for j shows that u_- is the root of a degree 12 polynomial, which matches with the number of copies of $\mathcal{F}$ in Figure 3. Another way to obtain this curve is to set $x = X^2$ and $y = XY$, from which one gets a quartic curve with the same j-invariant.

We have thus demonstrated that the two natural choices $\tau_\pm$ of the modular parameter indeed correspond to the complex structures of two elliptic curves covering the hyperelliptic curve. The physical u is given in terms of two different functions $u_\pm : \mathbb{H} \to \mathbb{C}$ with arguments $\tau_\pm$.

6.2 Elliptic curves for locus $\mathcal{E}_v$

The Seiberg-Witten curve $Y^2 = (X^3 - v)^2 - 1$ for $u = 0$ is not in form (6.2) for a curve of $\mathcal{L}_2$. However, the discussion around (6.1) suggests that it can be written in this form. We can achieve this by comparing the invariants of the $u = 0$ hyperelliptic curve and (6.2), and solving for s_1, s_2. Just as two elliptic curves are isomorphic if and only if their j-functions are equal, higher genus curves are isomorphic if and only if their absolute invariants are equal [29, 47, 48]. On $\mathcal{L}_2$, there are only two independent invariants. Comparing the absolute invariants of (6.2) and the $SU(3)$ curve for $u = 0$, we arrive at

$$s_1 s_2 = 9\left(25 - 24v^2\right), \qquad s_1^3 + s_2^3 = 54\left(216v^4 - 340v^2 + 125\right). \tag{6.7}$$

These combinations of s_1 and s_2 are known as the "dihedral" constants, since they are left invariant by the action of the dihedral group D_6 on (6.2). To solve the two equations in (6.7), let us denote

$$\mathcal{Q}^\pm(v) = 27\left(216v^4 - 340v^2 + 125 \pm 8v\left(27v^2 - 25\right)\sqrt{v^2 - 1}\right). \tag{6.8}$$

Then, one of the six solutions is given by

$$s_1^\pm = \sqrt[3]{\mathcal{Q}^\mp(v)}, \quad s_2^\pm = 9\frac{25 - 24v^2}{\sqrt[3]{\mathcal{Q}^\mp(v)}}. \tag{6.9}$$

In order to get an elliptic curve, we again take the map $(x, y) = (X^2, Y)$. This gives us the two curves

$$y^2 = x^3 - s_1^{\pm} x^2 + s_2^{\pm} x - 1 \tag{6.10}$$

with j-function

$$j^{\pm} = -432 \left(1458v^6 - 2673v^4 + 1340v^2 - 125 \mp 2v \left(729v^4 - 972v^2 + 275\right) \sqrt{v^2 - 1}\right) \tag{6.11}$$

and discriminant $\Delta = v^2 - 1$. By inverting (6.11), the resulting function v matches precisely with (5.9) in Section 5.2. Note that $j^{\pm}$ vanish at the AD points $v = \pm 1$ and the curve (6.10) becomes a cusp $y^2 = x^3$. This implies that the AD points are elliptic fixed points and are in the $SL(2, \mathbb{Z})$ orbit of α, which is easy to check from (5.12): We have that $\tau_{\text{AD},1} = \alpha - 1$ and $\tau_{\text{AD},2} = \alpha + 2$. See also Figure 5. They do however not fall into the (classical) Kodaira classification of singular fibers, since the Weierstraß invariants of (6.10) are not polynomials in v and their order of vanishing is half-integer rather than integer.

The curve $Y^2 = X^6 - 2vX^3 + v^2 - 1$ for $u = 0$ has enhanced symmetry compared to the Klein four-group for (6.2). Since $v^2 - 1$ is the discriminant, we can divide and rescale X to find

$$Y^2 = X^6 - \frac{2v}{\sqrt{v^2 - 1}} X^3 + 1. \tag{6.12}$$

It is easy to show that any curve of the form $Y^2 = X^6 - aX^3 + 1$ is invariant under $(X, Y) \mapsto (\frac{1}{X}, \frac{Y}{X^3})$ and $(X, Y) \mapsto (\alpha X, -Y)$, where again $\alpha = e^{2\pi i/3}$. These order 2 and 6 elements generate the dihedral group D_{12}. Similarly to the enhanced automorphism group for $\mathcal{E}_u$, we interpret this group as a symmetry group of the BPS/anti-BPS spectrum. From Appendix B.3, we know that $a_2 = -\alpha a_1$ on $\mathcal{E}_v$. The central charges Z_j (3.2) of the W-bosons, together with their charge conjugates, span therefore a regular 6-gon, whose symmetry group is D_{12}.

Hyperelliptic curves $C \in \mathcal{L}_2$ with $\text{Aut}(C) \cong D_{12}$ satisfy an additional constraint, it is given by the zero locus of a weight 20 polynomial in the Igusa invariants [61, Eq. (24)]. Moreover, the elliptic subcovers of hyperelliptic curves with $\text{Aut}(C) \cong D_{12}$ are isogenous [28]. We can check explicitly that the $u = 0$ curve is of this form. This explains why the elliptic curves for the two complex structures produce a single modular function (5.9), rather than the two independent functions $u_{\pm}$ for $\mathcal{E}_u$. On $\mathcal{E}_u$ the first algebraic relation in (4.3) holds and places the curve in $\mathcal{L}_2$. On $\mathcal{E}_v$ both relations (5.3) hold, where the first one projects into $\mathcal{L}_2$ and the second one gives the augmented D_{12} symmetry. This is consistent with the argument of Section 4.2 that the maps φ_j should exist as long as $\text{Im}(\tau_{11}) = \text{Im}(\tau_{22})$, such that it is possible to define $\tau_{\pm} = \tau_{11} \pm \tau_{12} \in \mathbb{H}$. The first relations in both (4.3) and (5.3) are equivalent to this condition.

7 Monodromies

We study the weak and strong coupling monodromies in this section. In this way, we are able to derive the modular groups of the order parameters in Section 4, which parametrize the elliptic loci. As before, we are interested in studying the two patches of the moduli space where one of the parameters u and v is large compared to the other.

7.1 Weak coupling monodromies

The spontaneously broken global $\mathbb{Z}_3$ and $\mathbb{Z}_2$ symmetries are generated by $\sigma : u \mapsto \alpha u$ and $\rho : v \mapsto e^{\pi i} v$, respectively. Using the explicit Picard-Fuchs solutions (B.10) and (B.16), we can determine how these symmetries act on the periods in the weak coupling region of the Coulomb branch.

Weak coupling in locus $\mathcal{E}_u$

In the large u regime we are interested in the action of σ on the PF solutions in (B.10). We can readily determine that it acts on the periods as the matrix

$$\sigma_u = \alpha^2 \mathcal{P} \begin{pmatrix} 0 & 1 & 1 & -2 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{7.1}$$

where the subscript u indicates that the base point is at large u, and $\mathcal{P} = \left(\begin{smallmatrix} -\mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{smallmatrix}\right)$ is the central element of $Sp(4, \mathbb{Z})$. The matrix σ_u conjugates the semi-classical monodromies (3.10) to each other,

$$\begin{aligned} \sigma_u^{-1} \mathcal{M}^{(r_1)} \sigma_u &= \mathcal{M}^{(r_2)}, \\ \sigma_u^{-1} \mathcal{M}^{(r_2)} \sigma_u &= \mathcal{M}^{(r_1)}, \\ \sigma_u^{-1} \mathcal{M}^{(r_3)} \sigma_u &= \mathcal{M}^{(r_1)} \mathcal{M}^{(r_2)} (\mathcal{M}^{(r_1)})^{-1}. \end{aligned} \tag{7.2}$$

It holds that $\bar{\sigma}_u = \alpha \sigma_u \in Sp(4, \mathbb{Z})$. We introduce moreover the translation of τ_{IJ} at weak coupling,

$$\mathcal{T}_{w,u} = \begin{pmatrix} 0 & 1 & -1 & 2 \\ 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \alpha^2 \mathcal{P} \sigma_u^{-1} \in Sp(4, \mathbb{Z}), \tag{7.3}$$

which maps

$$\mathcal{T}_{w,u} : \quad \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} \mapsto \begin{pmatrix} \tau_{22} + 2 & \tau_{12} - 1 \\ \tau_{12} - 1 & \tau_{11} + 2 \end{pmatrix}. \tag{7.4}$$

Using (3.3), one checks that σ_u maps $u \mapsto \alpha u$, while $v \mapsto v$ is left invariant. Moreover, $\sigma_u^3 : u \mapsto e^{2\pi i}u$ leaves u invariant, but acts as a monodromy on the periods,

$$\sigma_u^3 = \mathcal{P}\mathcal{T}_{w,u}^{-3} = \mathcal{M}^{(r_2)}\mathcal{M}^{(r_1)}\mathcal{M}^{(r_2)} = \begin{pmatrix} 0 & -1 & -3 & 6 \\ -1 & 0 & 6 & -3 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{7.5}$$

This corresponds to the monodromy around $u = \infty$ by construction. In a similar way, we can determine the action of the $\mathbb{Z}_2$ symmetry generated by $\rho : v \mapsto e^{\pi i}v$. Here, one finds the matrix representation

$$\rho_u = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in Sp(4,\mathbb{Z}). \tag{7.6}$$

This matrix conjugates the semi-classical monodromies analogous to (7.2), with σ_u replaced by ρ_u. The large u monodromy for v is trivial, $\rho_u^2 = \mathbb{1}$. We will see later that σ_u and ρ_u have a natural action on the charge vectors of the dyons that become massless at the various strongly coupled singular vacua. The full $\mathbb{Z}_6$ symmetry can now be represented as

$$\rho_u^{-1}\mathcal{T}_{w,u} = \mathcal{T}_{\mathrm{q}}, \tag{7.7}$$

with $\mathcal{T}_{\mathrm{q}} = \left(\begin{smallmatrix} \mathbb{1} & C \\ 0 & \mathbb{1} \end{smallmatrix}\right)$, where $C = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$ is the Cartan matrix of $SU(3)$. This represents the quantum monodromy corresponding to a rotation of the scale $\Lambda^6 \to e^{2\pi i}\Lambda^6$ [12].

Weak coupling in locus $\mathcal{E}_v$

We now turn to the patch with v large and perform the analogous analysis as in the above. The action of $\sigma : u \mapsto \alpha u$ on the solution (B.12–B.16) can be represented by the matrix

$$\sigma_v = \alpha^2 \begin{pmatrix} -1 & -1 & 2 & -1 \\ 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \tag{7.8}$$

where now the subscript v indicates that we are in the large v regime. It satisfies $\sigma_v^3 = \mathbb{1}$ and the large v rotation is therefore a trivial monodromy. On this patch, the generator of the $\mathbb{Z}_2$ symmetry $\rho_v : v \mapsto e^{\pi i}v$ is more interesting. Here, instead of (7.6), we now find

$$\rho_v = \begin{pmatrix} 0 & -1 & 1 & 1 \\ 1 & 1 & -2 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{7.9}$$

Since $\rho_v^2 \neq \mathbb{1}$, $v \mapsto e^{2\pi i}v$ acts on the periods as a monodromy, while leaving v invariant. The full $\mathbb{Z}_6$ symmetry is again given by $\mathcal{P}\alpha^2\rho_v^{-1}\sigma_v^{-1} = \mathcal{T}_{\mathrm{q}}$, as in (7.7).

7.2 Strong coupling monodromies

Analytically continuing the PF solution (B.10) to strong coupling, we can compute the periods near the singularities. At the $\mathbb{Z}_2$ point $(\underline{u}, v) = (1, 0)$, the coupling matrix can be computed explicitly and we can then use σ_u to rotate to the other $\mathbb{Z}_2$ points $\underline{u} = \alpha, \alpha^2$ by means of the action (A.13). The coupling matrices at these points evaluate to

$$\Omega(1,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Omega(\alpha,0) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \quad \Omega(\alpha^2,0) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{7.10}$$

The above matrices lie on the boundary $\partial\mathbb{H}_2$ of the Siegel upper half-plane. The relations among the entries are consistent with the results from Section 4.

The coupling matrices at the $\mathbb{Z}_3$ (AD) points $(\underline{u}, v) = (0, \pm 1)$ are

$$\Omega(0,1) = \begin{pmatrix} -1 + \frac{i}{\sqrt{3}} & \frac{9-\sqrt{3}i}{6} \\ \frac{9-\sqrt{3}i}{6} & -2 + \frac{i}{\sqrt{3}} \end{pmatrix}, \quad \Omega(0,-1) = \begin{pmatrix} 1 + \frac{i}{\sqrt{3}} & \frac{3-\sqrt{3}i}{6} \\ \frac{3-\sqrt{3}i}{6} & \frac{i}{\sqrt{3}} \end{pmatrix}. \tag{7.11}$$

They lie in the interior of the Siegel upper half-space $\mathbb{H}_2$.

To determine the monodromies around these singularities, we recall the formula from [12, 13]. It gives the monodromy matrix in terms of the charge vector γ of the BPS state with vanishing mass. The charge vector is a left eigenvector with unit eigenvalue. The monodromy $\mathcal{M}_\gamma$ reads

$$\mathcal{M}_\gamma = \begin{pmatrix} \mathbb{1} + n \otimes m & n \otimes n \\ -m \otimes m & \mathbb{1} - m \otimes n \end{pmatrix} \tag{7.12}$$

for $\gamma = (m, n)$ with $m = (m_1, m_2)$ and $n = (n_1, n_2)$ the magnetic and electric charge vectors. In locus $\mathcal{E}_u$ we have three singular points where two mutually local dyons become massless, respectively, while in locus $\mathcal{E}_v$ three mutually non-local dyons become massless at each of the two singular points.

Strong coupling in locus $\mathcal{E}_u$

To calculate the monodromies using (7.12), we need to first choose a symplectic basis for the homology cycles. In locus $\mathcal{E}_u$ we choose it such that two monopoles $\gamma_1 = (1, 0, 0, 0)$ and $\gamma_2 = (0, 1, 0, 0)$ become massless at $(\underline{u}, v) = (1, 0)$. For gauge group $SU(N)$ this choice is always possible [13]. In this subsection, we will consider monodromies in locus $\mathcal{E}_u$, keeping $v = 0$ fixed. Restricting to this locus, a monodromy circles a point rather than a line. We denote the monodromy around the point $(\underline{u}, 0)$ in $\mathcal{E}_u$ by $\mathcal{M}_{(\underline{u},0)}$. The charges of the dyons that become massless at the singular points $(\underline{u}, v) = (\alpha, 0)$ and $(\underline{u}, v) = (\alpha^2, 0)$ are obtained by acting on the periods with

σ_u and σ_u^{-1} from the left, it turns out that this corresponds to acting on the charges $\gamma_{1,2}$ from the right with $-\mathcal{T}_{w,u}$ and its inverse. We find

$$\begin{aligned} \gamma_1 &= (1,0,0,0), & \gamma_2 &= (0,1,0,0), \\ \gamma_3 &= -\gamma_1 \mathcal{T}_{w,u} = (0,-1,1,-2), & \gamma_4 &= -\gamma_2 \mathcal{T}_{w,u} = (-1,0,-2,1), \\ \gamma_5 &= -\gamma_1 \mathcal{T}_{w,u}^{-1} = (0,-1,-1,2), & \gamma_6 &= -\gamma_2 \mathcal{T}_{w,u}^{-1} = (-1,0,2,-1), \end{aligned} \tag{7.13}$$

where each row corresponds to the charges of the mutually local states becoming massless at the respective points.

We will first derive the four-dimensional monodromy matrices, and then determine their action on the effective couplings constants $\tau_\pm$. The monodromy around $(\underline{u}, v) = (1,0)$ can be computed from the PF solution, it is

$$\mathcal{M}_{(1,0)} = \mathcal{M}_{\gamma_1}\mathcal{M}_{\gamma_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \tag{7.14}$$

and agrees with the product of the monodromies (7.12) of the singular lines associated with the massless states of charges γ_1 and γ_2. This monodromy can be written as a "trajectory" in the space of coupling constants as

$$\mathcal{M}_{(1,0)} = \mathcal{S}\mathcal{T}_{s,u}\mathcal{S}^{-1}, \tag{7.15}$$

where $\mathcal{S}$ is the symplectic inversion and $\mathcal{T}_{s,u}$ is the translation at strong-coupling,

$$\mathcal{S} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \mathcal{T}_{s,u} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.16}$$

The monodromies around $\underline{u} = \alpha$ and $\underline{u} = \alpha^2$ can be obtained from the charges of the corresponding states that become massless at the different points. Alternatively, we can write them as conjugations of $\mathcal{T}_{s,u}$. We find

$$\begin{aligned} \mathcal{M}_{(\alpha,0)} =& \mathcal{M}_{\gamma_3}\mathcal{M}_{\gamma_4} = (\mathcal{T}_{w,u}^{-1}\mathcal{S})\mathcal{T}_{s,u}(\mathcal{T}_{w,u}^{-1}\mathcal{S})^{-1} = \begin{pmatrix} 3 & -1 & 5 & -4 \\ -1 & 3 & -4 & 5 \\ -1 & 0 & -1 & 1 \\ 0 & -1 & 1 & -1 \end{pmatrix}, \\ \mathcal{M}_{(\alpha^2,0)} =& \mathcal{M}_{\gamma_5}\mathcal{M}_{\gamma_6} = (\mathcal{T}_{w,u}\mathcal{S})\mathcal{T}_{s,u}(\mathcal{T}_{w,u}\mathcal{S})^{-1} = \begin{pmatrix} -1 & 1 & 5 & -4 \\ 1 & -1 & -4 & 5 \\ -1 & 0 & 3 & -1 \\ 0 & -1 & -1 & 3 \end{pmatrix}. \end{aligned} \tag{7.17}$$

They satisfy the consistency condition

$$\mathcal{P}\mathcal{T}_{w,u}^{-3} = \mathcal{M}_\infty = \mathcal{M}_{(\alpha,0)}\mathcal{M}_{(1,0)}\mathcal{M}_{(\alpha^2,0)} = \begin{pmatrix} 0 & -1 & -3 & 6 \\ -1 & 0 & 6 & -3 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{7.18}$$

Due to the singularity structure, the matrices (7.14)-(7.18) are all the monodromies in the region where v is small. They all lie in $Sp(4,\mathbb{Z})$, since (7.12) do.

For the elliptic locus $v = 0$, we analyzed the couplings $\tau_\pm = \tau_{11} \pm \tau_{12}$ in Section 4. We will study here the action of $\mathcal{M}_\infty$ and $\mathcal{M}_{(\alpha^j,0)}$ on $\tau_\pm$. We will find for τ_- that the action of the monodromies generate a proper congruence subgroup $\Gamma^0(9) \subset SL(2,\mathbb{Z})$. Therefore, the action of $\mathcal{T}_{w,u}$ and $\mathcal{T}_{s,u}$ can be represented in terms of the same two-dimensional matrix $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The weak coupling shift $\mathcal{T}_{w,u}$ corresponds to the two-dimensional matrix T^3 for τ_-, while the strong coupling shift $\mathcal{T}_{s,u}$ corresponds to T. Moreover, the four-dimensional symplectic $\mathcal{S}$ reduces to the two-dimensional modular inversion S. Since $\tau_{11} = \tau_{22}$ on $\mathcal{E}_u$, it is easy to show that the four-dimensional monodromies reduce to the matrices

$$\begin{aligned} \mathcal{M}_{(\infty,0)} &\mapsto M^-_{(\infty,0)} = T^{-9} = \begin{pmatrix} 1 & -9 \\ 0 & 1 \end{pmatrix}, \\ \mathcal{M}_{(1,0)} &\mapsto M^-_{(1,0)} = STS^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \\ \mathcal{M}_{(\alpha,0)} &\mapsto M^-_{(\alpha,0)} = (T^{-3}S)T(T^{-3}S)^{-1} = \begin{pmatrix} 4 & 9 \\ -1 & -2 \end{pmatrix}, \\ \mathcal{M}_{(\alpha^2,0)} &\mapsto M^-_{(\alpha^2,0)} = (T^3S)T(T^3S)^{-1} = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}, \end{aligned} \tag{7.19}$$

for τ_-. They all lie in $\Gamma^0(9)$ and do in fact generate $\Gamma^0(9)$, and furthermore satisfy

$$M^-_{(\alpha,0)}M^-_{(1,0)}M^-_{(\alpha^2,0)} = M^-_{(\infty,0)}. \tag{7.20}$$

Note there is here no sign between $M^-_{(\infty,0)}$ and T^{-9}. Of course, this sign is irrelevant for the action on τ_-. A good consistency check is that these monodromies fix the τ_- at the cusps $\tau_- = \{-3, 0, 3\}$.

The weak coupling shift $\mathcal{T}_{w,u}$ corresponds to the two-dimensional matrix $T_{w,u}$ for τ_+, while the strong coupling shift is $T_{s,u}$. For the parameter τ_+, the monodromies reduce to

$$\begin{aligned} M^+_{(\infty,0)} &= PT_{w,u}^{-3}, \\ M^+_{(1,0)} &= ST_{s,u}S^{-1}, \\ M^+_{(\alpha,0)} &= (T_{w,u}^{-1}S)T_{s,u}(T_{w,u}^{-1}S)^{-1}, \\ M^+_{(\alpha^2,0)} &= (T_{w,u}S)T_{s,u}(T_{w,u}S)^{-1}, \end{aligned} \tag{7.21}$$

which satisfy

$$M^+_{(\alpha,0)} M^+_{(1,0)} M^+_{(\alpha^2,0)} = M^+_{(\infty,0)}. \tag{7.22}$$

This precisely reduces to the group Γ_{u_+} (4.29), which leaves the function u_+ invariant. As discussed in Section 4.3, these monodromies do not generate a congruence subgroup of $SL(2,\mathbb{Z})$ if we identify $T_{w,u}$ and $T_{s,u}$ with T.

Strong coupling in locus $\mathcal{E}_v$

We can perform a similar analysis in the region where v is large and u small. At each of the two singular points we find that three mutually non-local states become massless. The corresponding charges are

$$\begin{aligned}
\nu_1 &= (1,1,0,0), & \nu_2 &= (0,1,0,0),\\
\nu_3 &= \nu_1\bar{\sigma}_v^{-1} = (-1,0,-1,2), & \nu_4 &= \nu_2\bar{\sigma}_v^{-1} = (-1,-1,1,1),\\
\nu_5 &= \nu_1\bar{\sigma}_v = (0,-1,1,-2), & \nu_6 &= \nu_2\bar{\sigma}_v = (1,0,-1,-1),
\end{aligned} \tag{7.23}$$

where the left column represents the states that becomes massless at $(u,v)=(0,1)$ and the second column the ones for $(u,v)=(0,-1)$, and $\bar{\sigma}_v = \alpha\sigma_v \in Sp(4,\mathbb{Z})$.

The monodromy around $v=\infty$ is given by

$$\mathcal{M}_{(0,\infty)} = \rho_v^2 = \begin{pmatrix} -1 & -1 & 4 & 1 \\ 1 & 0 & -5 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}. \tag{7.24}$$

For $u=0$, the monodromy around the AD point $(u,v)=(0,1)$ can be calculated from the Picard-Fuchs solution,

$$\mathcal{M}_{(0,1)} = \begin{pmatrix} 2 & 0 & 1 & -2 \\ -2 & 1 & -2 & 4 \\ -1 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 \end{pmatrix} = \mathcal{M}_{\nu_1}\mathcal{M}_{\nu_3} = \mathcal{M}_{\nu_3}\mathcal{M}_{\nu_5}. \tag{7.25}$$

The remaining monodromy is fixed by the global consistency $\mathcal{M}_{(0,\infty)} = \mathcal{M}_{(0,1)}\mathcal{M}_{(0,-1)}$. This gives us

$$\mathcal{M}_{(0,-1)} = \begin{pmatrix} 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 2 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix} = \rho_v^{-1}\mathcal{M}_{(0,1)}\rho_v = \mathcal{M}_{\nu_2}\mathcal{M}_{\nu_4} = \mathcal{M}_{\nu_4}\mathcal{M}_{\nu_6}, \tag{7.26}$$

All of the above matrices are in $Sp(4,\mathbb{Z})$. Due to the relations (5.5) among τ_{11}, τ_{12} and τ_{22}, they act on $\tau_- = \tau_{11} - \tau_{12}$ as

$$\begin{aligned} M^-_{(0,1)} &= \begin{pmatrix} -4 & -7 \\ 1 & 1 \end{pmatrix}, \\ M^-_{(0,-1)} &= \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}, \\ M^-_{(0,\infty)} &= \begin{pmatrix} 1 & -6 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{7.27}$$

We conjugate with $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$, to match with the coupling $\tau = \tau_- + 1$ for (5.11). This reproduces precisely the matrices (5.15), which leave v invariant.

Similarly to the above, we can consider the action of the matrices $\mathcal{M}_{(0,\infty)}$ and $\mathcal{M}_{(0,\pm 1)}$ on the parameter $\tau_+ = \tau_{11} + \tau_{12}$. This gives

$$\begin{aligned} M^+_{(0,1)} &= \begin{pmatrix} 0 & 1 \\ -3 & 3 \end{pmatrix}, \\ M^+_{(0,-1)} &= \begin{pmatrix} -3 & 7 \\ -3 & 6 \end{pmatrix}, \\ M^+_{(0,\infty)} &= T^{-2}, \end{aligned} \tag{7.28}$$

with again $M^+_{(0,1)} M^+_{(0,-1)} = M^+_{(0,\infty)}$ up to normalisation. These matrices agree with what we found in Section 5, below (5.18).

7.3 BPS quiver and origin of $\mathcal{U}$

A potential application of the previous sections is to interpolate between weak and strong coupling. One may follow the BPS spectrum along such a trajectory using the connection to BPS quivers [26, 62, 63]. We briefly address this connection in this subsection, and leave a more detailed analysis for future work.

Let us consider the origin of the moduli space, $(u,v) = (0,0)$. At this point, the two elliptic loci, $\mathcal{E}_u$ and $\mathcal{E}_v$, touch. It is a perfectly regular point, since $\Delta = 729\Lambda^{18}$ does not vanish. We can compute the coupling matrix at the origin of the moduli space $\mathcal{U}$ starting from large u, and find

$$\Omega^u(0,0) = \begin{pmatrix} 1 + \frac{\sqrt{3}}{2} i & -\frac{1}{2} \\ -\frac{1}{2} & 1 + \frac{\sqrt{3}}{2} i \end{pmatrix}. \tag{7.29}$$

The above matrix can be obtained by expanding the periods to first order in v but exact in u, computing the coupling matrix, setting $v = 0$ and taking the limit $u \to 0$ for $u < 0$. This is consistent with the argument given in [62] that the origin should be approached on the negative real u-line, as it avoids the singularity $\underline{u} = 1$ where the periods pick up a monodromy.

Analytically continuing the solutions for large v (B.16), we find that the coupling at the origin $(0,0) \in \mathcal{U}$ is given by

$$\Omega^v(0,0) = \begin{pmatrix} \frac{2i}{\sqrt{3}} & 1 - \frac{i}{\sqrt{3}} \\ 1 - \frac{i}{\sqrt{3}} & -1 + \frac{2i}{\sqrt{3}} \end{pmatrix} \tag{7.30}$$

The two different matrices (7.29) and (7.30) are related through the action (A.13) as

$$\mathcal{T}_\theta \, (\mathcal{M}^{(r_2)})^{-1} \mathcal{M}_{\nu_2} : \Omega^u(0,0) \mapsto \Omega^v(0,0), \tag{7.31}$$

with $\mathcal{T}_\theta$ as below (5.1). The two effective couplings at the origin $\Omega^{u,v}(0,0)$ are therefore related by a monodromy up to $\mathcal{T}_\theta$. This is expected, since $\mathcal{T}_\theta$ transforms (5.1) to (3.26).

As shown in [62], the central charge configuration at the origin can be obtained from the one for large u by following the negative real axis on the $v = 0$ plane from large u to 0. At this point, the full $\mathbb{Z}_6$-symmetry is restored and none of the central charges are zero. We find that, for example, $Z_{\nu_1} = Z_{\nu_2} = e^{\frac{9\pi i}{6}} = -i$, $Z_{\nu_3} = Z_{\nu_4} = e^{\frac{5\pi i}{6}}$ and $Z_{\nu_5} = Z_{\nu_6} = e^{\frac{\pi i}{6}}$ in the normalisation of Table 1. Together with their charge conjugates, they all map into each other by $\frac{2\pi}{6}$ rotations. In fact, the symmetry group is larger than $\mathbb{Z}_6$. Since the symmetry group for the central charges of $(\nu_j, \nu_{j+1}, -\nu_j, -\nu_{j+1})$ for $j = 1, 3, 5$ is D_4, and the symmetry group of the equilateral triangle is D_6, the total symmetry group becomes $D_4 \rtimes D_6$. This group is known to be isomorphic to the group $\mathbb{Z}_3 \rtimes D_8$, which is the automorphism group of this genus 2 curve [28]. Moreover, this group is isomorphic to $D_{12} \rtimes \mathbb{Z}_2$, such that the automorphism group D_4 of $\mathcal{E}_u$, and D_{12} of $\mathcal{E}_v$ are both subgroups of the automorphism group at the origin.

The BPS quiver for strong coupling [62] is presented in Figure 6. Every charge vector in the basis is represented by a node. The number of arrows is determined by the symplectic inner product between a pair of charges. The global $\mathbb{Z}_2$ symmetry σ_v acts in the picture to the right as $\nu_k \mapsto \nu_{k+2 \mod 6}$.

8 Discussion

In this paper we discussed the modular properties of pure $\mathcal{N} = 2$ Yang-Mills theory in four dimensions with gauge group $SU(3)$. On the two loci $\mathcal{E}_u$ and $\mathcal{E}_v$, where $v = 0$ and $u = 0$ respectively, we express the parameters u and v of the moduli space as modular functions for discrete subgroups of $SL(2, \mathbb{R})$. See (4.13) and (5.11). To this end, we formulate the genus two $SU(3)$ SW curve in Rosenhain form in terms of Siegel theta series. The parameters of the theory are then found by relating the Rosenhain form to the PF solution of [13]. We provide an explicit fundamental domain for the effective coupling on the two elliptic loci $\mathcal{E}_u$ and $\mathcal{E}_v$. The relation between cross-ratios of the curve and theta constants suggests that the full moduli

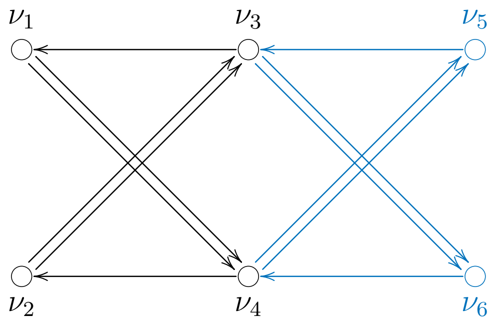

Figure 6. The mutation algorithm produces a finite spectrum consisting of 6 particles at strong coupling [62]. The generating matrix $\bar{\sigma}_v = \alpha\sigma_v$ maps the charges to the right. The coloured part does not belong to the $SU(3)$ quiver, it merely highlights how all the charges at strong coupling can be obtained from $\bar{\sigma}_v$.

space can be parametrized by higher genus modular forms. It would be interesting to find a general solution to (3.29) by expressing u and v as algebraic functions of theta constants.

On $\mathcal{E}_u$, we established a nice generalisation of the structure appearing in the $SU(2)$ case. In rank one, the parameter u is a weakly holomorphic modular function for the congruence subgroup $\Gamma^0(4)$. For $SU(3)$, we instead found that on $\mathcal{E}_u$ the parameter u is a weakly holomorphic modular function of τ_- for the group $\Gamma^0(9) \subset SL(2,\mathbb{Z})$. The structure of the moduli space near the special points of this locus also seems to generalize the rank one picture: We find that u maps the $\mathbb{Z}_2$ singularities to the cusps of its fundamental domain. Furthermore, the duality group is generated by the nontrivial monodromies on $\mathcal{E}_u$. For the other choice of modular parameter $\tau_+ = \tau_{11} + \tau_{12}$, we find that u is not invariant under a congruence subgroup, but is rather a *sextic modular function*, which is the same function as appears for rank 1 $N_f = 1$ SQCD. Nevertheless, we are able to show that the monodromies can be viewed as paths in a new fundamental region, which we propose.

On the other locus $\mathcal{E}_v$ where $u = 0$, we find that v can be expressed as a modular function for a subgroup $\Gamma_v \subset SL(2,\mathbb{R})$ of Atkin-Lehner type. The AD points are mapped to the elliptic fixed points of the quotient $\Gamma_v\backslash\mathbb{H}$. The group Γ_v includes a Fricke involution, which can be viewed as a manifestation of S-duality [18, 19, 65]. We derive it from the monodromy group on $\mathcal{E}_v$. On the locus $\mathcal{E}_v$, the genus two hyperelliptic curve splits into two *elliptic* curves with complex structures $\tau_\pm = \tau_{11} \pm \tau_{12}$. The appearance of the Fricke involution is a consequence of the two families of elliptic curves being isogenous [54, 66]. Fricke dualities also appear in String theory, where they have been shown to play an important rôle in the web of dualities of CHL models, i.e. orbifolds of heterotic string theory on T^6 or type II on $K3 \times T^2$ [67, 68]. They are also the natural generalisation of S-duality in the context of Olive-Montonen duality in $\mathcal{N} = 4$ super-Yang-Mills theory for non-simply laced gauge groups [69, 70] and the geometric Langlands program [71]. Moreover, Fricke involutions are familiar in topological string theory where they act on higher genus amplitudes, which are described by quasi modular forms. They exchange the large complex structure of the Calabi-Yau threefold with the conifold loci, which gives an analogue of the action of electric-magnetic duality or $\mathcal{N}=2$ S-duality in topological string theory [54, 66].

It would be interesting to extend this work to other theories, such as those with gauge group $SU(N)$, including matter multiplets, theories of class S [72], or gravitational couplings to these theories [9, 10]. For theories with $SU(N>2)$, one can for example consider to turn on only the bottom Casimir u_2 and setting $u_3,\dots,u_N$ to zero. Our analysis naively suggests that it should be parametrized by a modular function for $\Gamma^0(N^2)$. The discriminant of the $SU(N)$ curve [11]

$$y^2=\left(x^N-\sum_{j=2}^{N}u_jx^{N-j}\right)^2-1 \tag{8.1}$$

intersects with this locus in $u_2^N=N^N(N-2)^{2-N}/4$, confirming that there are N singularities at strong coupling. However, it is easy to show that $\Gamma^0(N^2)$ has N cusps aside from $i\infty$ if and only if N is prime. Note that this worked for $N=2,3$. It is furthermore not obvious how the modular parameter would relate to the coupling matrix, and the map to elliptic subcovers is more subtle in the higher rank case [30].

We would like to finish by mentioning a few potential applications and directions for further research:

- We observe that the functions parametrising the $SU(2)$ and $SU(3)$ moduli spaces are all *replicable* [38–41] modular functions. The $SU(2)$ order parameter u is of class 4C, u_- of class 9B, and v of class 6a. It would be interesting to explore whether there is an underlying reason for the functions to have this property.

- This work motivates exploring subloci of Coulomb branches for theories with other gauge groups and including matter multiplets. This could provide a better understanding of the modularity of these theories. Moreover, it would be interesting to understand whether the solution of the theory on a sublocus is equivalent to the solution of another theory, such as we found for $\mathcal{E}_u$ and the massless $N_f=1$, $SU(2)$ theory for example.

- The elliptic loci we consider are somewhat analogous to the special Kähler strata of Coulomb branches being studied in the recent work [75–77]. The latter aims to classify higher rank $\mathcal{N}=2$ SCFTs by decomposing the singular locus into a nested series of one-dimensional building blocks. It would be interesting to see if our methods find applications in this programme.

- The last application which we would like to mention, is topological quantum field theory [78]. Evaluation of the path integral or correlation functions for a compact four-manifold X involves the integration over the Coulomb branch (the so-called u-plane integral) of the theory [6, 64, 79]. For gauge group $SU(2)$, the integral becomes an integral over the modular fundamental domain $\Gamma^0(4)\backslash\mathbb{H}$ [6, 80–82]. A better understanding of the modularity of $SU(N>2)$ Seiberg-Witten theory could possibly allow further progress in this direction for theories with $N>2$.

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A Automorphic forms

In this appendix we collect examples of modular forms that are used in the text above and discuss some general structures related to these. For further reading see [46, 55, 56, 83–85].

A.1 Elliptic modular forms

Modular groups and fundamental domains

We first recall the notion of the congruence subgroups $\Gamma_0(n)$ and $\Gamma^0(n)$ of $SL(2,\mathbb{Z})$. They are defined as

$$\begin{aligned}\Gamma_0(n) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \middle|\, c \equiv 0 \mod n \right\}, \\ \Gamma^0(n) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \middle|\, b \equiv 0 \mod n \right\},\end{aligned} \tag{A.1}$$

and are related by conjugation with the matrix $\mathrm{diag}(n,1)$. We furthermore define $\Gamma(n)$ as the subgroup of $SL(2,\mathbb{Z}) \ni A$ with $A \equiv \mathbb{1} \mod n$.

The modular groups of $n|h$-type are defined in the following way [41]. Consider matrices of the form

$$\begin{pmatrix} ae & b/h \\ cn & de \end{pmatrix} \tag{A.2}$$

with determinant e, where $a,b,c,d,e,h,n \in \mathbb{Z}$, and h is the largest integer for which $h^2|N$ and $h|24$ with $n = N/h$. These matrices are also referred to as *Atkin-Lehner involutions*.

In the case that n is a positive integer and $h|n$, we define $\Gamma_0(n|h)$ as the set of above matrices with $e=1$. For any positive integer e which satisfies $e|n/h$ and $(e, n/eh) = 1$ (e is called an *exact divisor* of n/h), one can include also matrices of the above form with $e > 1$, forming a group denoted by $\Gamma_0(n|h) + e$. In fact, this construction works for any choice $\{e_1, e_2, \dots\}$ of exact divisors of n/h, resulting in the group $\Gamma_0(n|h) + e_1, e_2, \dots$. If $h = 1$, the $|h$ is omitted in the notation, and in case that all the possible e_i are included, the group is simply denoted by $\Gamma_0(n|h)+$.

In the Γ^0 convention the notation simplifies, since $\Gamma^0(n|h) = \Gamma^0(\frac{n}{h})$. This can be checked by conjugating (A.2) with $\mathrm{diag}(n, 1)$. The extension by non-unity determinant matrices follows by analogy.

A key concept of the theory of modular forms is the *fundamental domain*. A fundamental domain for a group $\Gamma \subset SL(2,\mathbb{R})$ is an open subset $\mathcal{F} \subset \mathbb{H}$ with the property that no two distinct points of $\mathcal{F}$ are equivalent under the action of Γ and every point in $\mathbb{H}$ is mapped to some point in the closure of $\mathcal{F}$ by the action of an element in Γ. The quotient $\Gamma\backslash\mathbb{H}$ can be compactified by adding finitely many points called *cusps*. Cusps are Γ-equivalence classes of $\mathbb{Q} \cup \{i\infty\}$. Special points in the fundamental domain are the *elliptic fixed points*, which are points in $\mathbb{H}$ that have a non-trivial Γ-stabiliser. There, the quotient $\Gamma\backslash\mathbb{H}$ becomes singular. Elliptic points can always be mapped to the boundary of the fundamental domain. They furthermore contribute non-trivially to the order of vanishing, which determines the dimension of the spaces of modular forms for fixed weight.

Examples of modular forms

The Eisenstein series $E_k : \mathbb{H} \to \mathbb{C}$ for even $k \geq 2$ are defined as the q-series

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n, \quad q = e^{2\pi i \tau}, \tag{A.3}$$

with B_k the Bernoulli numbers and $\sigma_k(n) = \sum_{d|n} d^k$ the divisor sum. For $k \geq 4$ even, E_k is a modular form of weight k for $\mathrm{SL}(2,\mathbb{Z})$. With this normalisation, the j-invariant can be written as

$$j = 1728 \frac{E_4^3}{E_4^3 - E_6^2}. \tag{A.4}$$

The Jacobi theta functions $\vartheta_j : \mathbb{H} \to \mathbb{C}$, $j = 2, 3, 4$, are defined as

$$\begin{aligned} \vartheta_2(\tau) &= \sum_{r \in \mathbb{Z}+\frac{1}{2}} q^{r^2/2}, \\ \vartheta_3(\tau) &= \sum_{n \in \mathbb{Z}} q^{n^2/2}, \\ \vartheta_4(\tau) &= \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2/2}, \end{aligned} \tag{A.5}$$

with $q = e^{2\pi i\tau}$. These functions transform under the generators T and S of $SL(2,\mathbb{Z})$ as

$$\begin{aligned} S:&\quad \vartheta_2(-1/\tau) = \sqrt{-i\tau}\vartheta_4(\tau), &\vartheta_3(-1/\tau) = \sqrt{-i\tau}\vartheta_3(\tau), &\quad \vartheta_4(-1/\tau) = \sqrt{-i\tau}\vartheta_2(\tau) \\ T:&\quad \vartheta_2(\tau+1) = e^{\frac{\pi i}{4}}\vartheta_2(\tau), &\vartheta_3(\tau+1) = \vartheta_4(\tau), &\quad \vartheta_4(\tau+1) = \vartheta_3(\tau). \end{aligned} \tag{A.6}$$

Some special values that we use are

$$\vartheta_2(i) = \vartheta_4(i) = \sqrt[4]{\tfrac{\pi}{2}}/\Gamma(\tfrac{3}{4}), \qquad \vartheta_3(i) = \sqrt[4]{\pi}/\Gamma(\tfrac{3}{4}). \tag{A.7}$$

The Dedekind eta function $\eta : \mathbb{H} \to \mathbb{C}$ is defined as the infinite product

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n), \quad q = e^{2\pi i\tau}. \tag{A.8}$$

It transforms under the generators of $SL(2,\mathbb{Z})$ as

$$\begin{aligned} S:&\quad \eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau), \\ T:&\quad \eta(\tau+1) = e^{\frac{\pi i}{12}}\,\eta(\tau). \end{aligned} \tag{A.9}$$

Quotients of η functions are frequently used to generate bases for the spaces of modular forms for congruence subgroups of $SL(2,\mathbb{Z})$. We use the following

Theorem 1 [55, 56]: Let $f(\tau) = \prod_{\delta|N} \eta(\delta\tau)^{r_\delta}$ be an eta-quotient with $k = \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z}$ and $\sum_{\delta|N} \delta r_\delta \equiv \sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \mod 24$. Then, f is a weakly holomorphic modular form for $\Gamma_0(N)$ with weight k. In particular, f transforms as $f(\tau|_\gamma) = \chi(d)(c\tau+d)^k f(\tau)$ under $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ with character $\chi(d) = \left(\frac{(-1)^k s}{d}\right)$, where $s = \prod_{\delta|N} \delta^{r_\delta}$.

A.2 Siegel modular forms

Ordinary modular forms are constructed by the action of an $SL(2,\mathbb{Z})$ Möbius transformation on the upper half-plane $\mathbb{H}$. Siegel modular forms [83, 85] generalize this notion by introducing an action of $Sp(2g,\mathbb{Z})$ on the so-called Siegel upper half-plane $\mathbb{H}_g$, which works for any *genus* $g \in \mathbb{N}$.

Define the Siegel modular group of genus g as

$$Sp(2g,\mathbb{Z}) = \{M \in \mathrm{Mat}(2g;\mathbb{Z}) \,|\, M^T J M = J\} \quad \text{with } J = \begin{pmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{pmatrix}. \tag{A.10}$$

The group $Sp(4,\mathbb{Z})$ can be generated [83] by the elements J and $T = \begin{pmatrix} \mathbb{1}_g & s \\ 0 & \mathbb{1}_g \end{pmatrix}$ with $s = s^T$. The Siegel upper half-plane

$$\mathbb{H}_g = \{\Omega \in \mathrm{Mat}(g;\mathbb{C}) \,|\, \Omega^T = \Omega,\ \mathrm{Im}\Omega > 0\} \tag{A.11}$$

consists of complex symmetric $g \times g$ matrices whose (componentwise) imaginary part is positive definite. This generalizes the ordinary upper half-plane $\mathbb{H} = \mathbb{H}_1$. For example, for $g = 2$ this means that

$$\Omega = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}, \quad \mathrm{Im}\tau_{11} > 0, \quad \mathrm{Im}\tau_{11}\mathrm{Im}\tau_{22} - (\mathrm{Im}\tau_{12})^2 > 0. \tag{A.12}$$

An element $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(2g, \mathbb{Z})$ acts on the Siegel upper half-plane by

$$\Omega \longmapsto \gamma(\Omega) = (A\Omega + B)(C\Omega + D)^{-1}. \tag{A.13}$$

A (classical) Siegel modular form of weight k and genus g is then a holomorphic function $f : \mathbb{H}_g \to \mathbb{C}$ satisfying

$$f(\gamma(\Omega)) = \det(C\Omega + D)^k f(\Omega) \qquad \forall \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{Z}), \tag{A.14}$$

where for $g = 1$ holomorphicity at $i\infty$ is required in addition.

Theta series provide an explicit class of classical Siegel modular forms. For a, $b \in \mathbb{Q}^2$ and $\Omega \in \mathbb{H}_2$, define

$$\Theta \begin{bmatrix} a \\ b \end{bmatrix} (\Omega) = \sum_{k \in \mathbb{Z}^2} \exp\left(\pi i (k+a)^T \Omega (k+a) + 2\pi i (k+a)^T b\right). \tag{A.15}$$

We are especially interested in the case where the entries of these column vectors take values in the set $\{0, \frac{1}{2}\}$. The corresponding theta functions are usually referred to as the theta characteristics. We call $\gamma = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ an even (odd) characteristic if $4a^T b$ is even (odd). In the case of genus two there are ten even theta constants [84],

$$\begin{aligned} &\Theta_1 = \Theta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; \Theta_2 = \Theta \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \; \Theta_3 = \Theta \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}, \; \Theta_4 = \Theta \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \; \Theta_5 = \Theta \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}, \\ &\Theta_6 = \Theta \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \; \Theta_7 = \Theta \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}, \; \Theta_8 = \Theta \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}, \; \Theta_9 = \Theta \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \; \Theta_{10} = \Theta \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \end{aligned} \tag{A.16}$$

All even theta constants can be related through algebraic identities to four *fundamental* ones, Θ_1, Θ_2, Θ_3, Θ_4 [84].

The above theta functions are weight $\frac{1}{2}$ Siegel modular forms for a subgroup of $Sp(4, \mathbb{Z})$. Their transformation properties under the Siegel modular group can be found in [85].

B Picard-Fuchs solution

In the limit of large u and small v, reference [13] determines the a_I and $a_{D,I}$ non-perturbatively in terms of the fourth Appell hypergeometric function $F_4(a, b, c, d; x, y)$.

For $\sqrt{|x|} + \sqrt{|y|} < 1$, this function is given by

$$F_4(a, b, c, d; x, y) = \sum_{m,n \geq 0} \frac{(a)_{m+n}\,(b)_{m+n}}{m!\,n!\,(c)_m (d)_n}\, x^m\, y^n, \tag{B.1}$$

where $(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)}$ is the Pochhammer symbol. We will also need expansions of F_4 for large y, which can be achieved by replacing the sum over n by the hypergeometric series ${}_2F_1$,

$$F_4(a, b, c, d; x, y) = \sum_{m \geq 0} \frac{(a)_m\,(b)_m}{m!\,(c)_m}\, {}_2F_1(a+m, b+m, d; y)\, x^m. \tag{B.2}$$

While analytic continuations are known for ${}_2F_1$, they are not well established for F_4.

B.1 Classical roots

In order to match the Picard-Fuchs solutions with the periods, we need to expand the periods around the classical solutions in (3.4). We therefore need to find the roots of these two cubics.

The general formula for the roots of a depressed cubic equation, $ax^3 + bx + c = 0$, is given by

$$\xi_k = -\frac{1}{3a}\left(\alpha^k C + \frac{\Delta_0}{\alpha^k C}\right), \quad k \in \{0, 1, 2\}, \tag{B.3}$$

where $\alpha = e^{2\pi i/3}$, $C^3 = \frac{\Delta_1 \pm \sqrt{\Delta_1^2 - 4\Delta_0}}{2}$, $\Delta_0 = -3ab$ and $\Delta_1 = 27a^2c$ [86]. The choice of sign in front of the square root in C is arbitrary, in the sense that it only corresponds to a permutation of the roots.

It is however important to fix the ambiguities in taking the square and cubic root. We fix the ambiguity in the square root by the following choice for the branch of the logarithm: For any complex number $z \in \mathbb{C}^*$, we set $\log(z) = \log|z| + i\mathrm{Arg}(z)$ with $-\pi < \mathrm{Arg(z)} \leq \pi$. The ambiquity in the cubic root of a complex number z is fixed by demanding that the real part of $\sqrt[3]{z}$ has the largest absolute value among the three solutions to $\rho^3 = z$. Thus $\sqrt[3]{1} = 1$ and $\sqrt[3]{-1} = -1$. Two of the cube roots of i and $-i$ have equal real parts. We fix the remaining ambiguity by setting $\sqrt[3]{i} = e^{\pi i/6} = \frac{\sqrt{3}}{2} + \frac{i}{2}$ and $\sqrt[3]{-i} = e^{-\pi i/6} = \frac{\sqrt{3}}{2} - \frac{i}{2}$.

To list the roots of our two equations, we define

$$s_\pm(a, b) = \sqrt[3]{\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}}. \tag{B.4}$$

Using Eq. (B.3), we then find that the roots of (3.4) for a_1 are given by

$$\begin{aligned} \xi_1(u, v) &= s_+(u, v) + s_-(u, v), \\ \xi_2(u, v) &= \alpha\, s_+(u, v) + \alpha^2\, s_-(u, v), \\ \xi_3(u, v) &= \alpha^2\, s_+(u, v) + \alpha\, s_-(u, v), \end{aligned} \tag{B.5}$$

and the roots for a_2 by $-\xi_j(u,v)$. This gives the $3\times3=9$ solutions to the equations in (3.4). However, (3.3) is supposed to have only $2\times3=6$ solutions. Let us determine the 6 solutions in one of the regimes of interest for $SU(3)$ Yang-Mills theory: we assume u is large and close to the positive axis: $u=\lambda-i\epsilon\lambda$ with λ real and very large and $0<\epsilon\ll1$. Note that in this regime

$$s_\pm(u,v)=\sqrt[3]{\frac{v}{2}\pm i\sqrt{\frac{u^3}{27}-\frac{v^2}{4}}}. \tag{B.6}$$

Furthermore, $s_+(u,v)\,s_-(u,v)=u/3$ and $s_-(u,-v)=e^{-\pi i/3}s_+(u,v)=-\alpha s_+(u,v)$ hold. For $v=0$, we have $s_+(u,0)=e^{\pi i/6}\sqrt{u/3}$ and $s_-(u,0)=e^{-\pi i/6}\sqrt{u/3}$, and thus

$$\begin{aligned}\xi_1(u,0)&=\sqrt{u},\\ \xi_2(u,0)&=-\sqrt{u},\\ \xi_3(u,0)&=0.\end{aligned} \tag{B.7}$$

This demonstrates that the solutions to (3.3) for (a_1,a_2) are given by

$$(\xi_1,-\xi_2),\ (\xi_1,-\xi_3),\ (\xi_2,-\xi_1),\ (\xi_2,-\xi_3),\ (\xi_3,-\xi_1),\ (\xi_3,-\xi_2). \tag{B.8}$$

B.2 Picard-Fuchs system for large u

To express a_I and $a_{D,I}$ in terms of u and v, we will start by working in the patch with large u and small v, and use the variables $x=\frac{27v^2}{4u^3}$ and $y=\frac{27\Lambda^6}{4u^3}$. In [13] the authors use the notation P_3 for this patch and, similarly, P_2 for the patch where v is large and u is small and we will adopt this notation in the following. We have four solutions [13, Eq. (6.1)] to the Picard-Fuchs system [13, Eq. (5.11)] for $SU(3)$,

$$\begin{aligned}\omega_1^{P_3}&=\sqrt{3}\,2^{\frac{2}{3}}\Lambda\,y^{-\frac{1}{6}}\,F_4\big(-\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{2},1;x,y\big)\,,\\ \omega_2^{P_3}&=\frac{2^{\frac{2}{3}}\Lambda}{3}\sqrt{x}\,y^{-\frac{1}{6}}\,F_4\big(\tfrac{1}{3},\tfrac{2}{3},\tfrac{3}{2},1;x,y\big)\,,\\ \Omega_1^{P_3}&=36\pi\,e^{-\pi i/6}\,2^{2/3}\Lambda\,\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{6})^2}\,F_4\Big(-\tfrac{1}{6},-\tfrac{1}{6},\tfrac{1}{2},\tfrac{2}{3};\tfrac{x}{y},\tfrac{1}{y}\Big)+\beta_1^{P_3}\,\omega_1^{P_3},\\ \Omega_2^{P_3}&=-e^{\frac{\pi i}{3}}\frac{2^{\frac{2}{3}}\Lambda}{\sqrt{3}\,2\pi}\,\Gamma(\tfrac{1}{3})^3\sqrt{\frac{x}{y}}\,F_4\Big(\tfrac{1}{3},\tfrac{1}{3},\tfrac{3}{2},\tfrac{2}{3};\tfrac{x}{y},\tfrac{1}{y}\Big)+\beta_2^{P_3}\,\omega_2^{P_3},\end{aligned} \tag{B.9}$$

where $\beta_1^{P_3}=(i-\sqrt{3})\pi+4\log(2)+3\log(3)-5$ and $\beta_2^{P_3}=1+(i+\frac{1}{\sqrt{3}})\pi+3\log(3)$.[4] The a_I and $a_{D,I}$ are linear combinations of these periods found by comparing the expansions of these solutions with the classical and semi-classical solutions in the

[4] We corrected the power of $\Gamma(\frac{1}{3})$ in the expression for $\Omega_2^{P_3}$ compared to [13], and removed the factor of $\sqrt{3}\Lambda$ from the second terms of $\Omega_i^{P_3}$ which have been placed incorrectly in [13] as they are already included in $\omega_1^{P_3}$ and $\omega_2^{P_3}$.

previous section for large u. Using the classical solutions $(a_1, a_2) = (\xi_1, -\xi_2)$ one finds [13, Eq. 6.4],

$$\begin{aligned}
a_{D,1}(u,v) &= -\frac{i}{4\pi}(\Omega_1^{P_3} + 3\Omega_2^{P_3}) - \frac{1}{\pi}(\alpha_1\omega_1^{P_3} - \alpha_2\omega_2^{P_3}) \\
&= -\frac{i}{2\pi}\left(\sqrt{u} + \frac{3}{2}\frac{v}{u}\right)\log\left(\frac{27\Lambda^6}{4u^3}\right) - \frac{1}{\pi}\left(\frac{i}{2} + 2\alpha_1\right)\sqrt{u} + O(u^{-1}), \\
a_{D,2}(u,v) &= -\frac{i}{4\pi}(\Omega_1^{P_3} - 3\Omega_2^{P_3}) - \frac{1}{\pi}(\alpha_1\omega_1^{P_3} + \alpha_2\omega_2^{P_3}) = a_{D,1}(u,-v) \\
a_1(u,v) &= \frac{1}{2}(\omega_1^{P_3} + \omega_2^{P_3}) \sim \sqrt{u} + \frac{1}{2}\frac{v}{u} + \ldots, \\
a_2(u,v) &= \frac{1}{2}(\omega_1^{P_3} - \omega_2^{P_3}) \sim \sqrt{u} - \frac{1}{2}\frac{v}{u} + \ldots,
\end{aligned} \tag{B.10}$$

with $\alpha_1 = \frac{5i}{4} - i\log(2) - \frac{3i}{4}\log(3)$ and $\alpha_2 = \frac{3i}{4} + \frac{9i}{4}\log(3)$. The chain rule then allows to compute the coupling matrix,

$$\Omega(u,v) = \begin{pmatrix} \partial_u a_1 & \partial_u a_2 \\ \partial_v a_1 & \partial_v a_2 \end{pmatrix}^{-1} \begin{pmatrix} \partial_u a_{D,1} & \partial_u a_{D,2} \\ \partial_v a_{D,1} & \partial_v a_{D,2} \end{pmatrix}. \tag{B.11}$$

B.3 Picard-Fuchs system for large v

We can run a similar analysis as in the previous section for the patch P_2, i.e., for large v and small u. This is not done explicitly in [13] but the authors hint at how it should be done. Here, we use the variables $x = \frac{4u^3}{27v^2}$ and $y = \frac{\Lambda^6}{v^2}$ to express the solutions of the Picard-Fuchs equations as

$$\begin{aligned}
\omega_1^{P_2} &= 2y^{-1/6}F_4\left(-\frac{1}{6}, \frac{1}{3}, \frac{2}{3}, 1; x, y\right), \\
\omega_2^{P_2} &= 2^{1/3}x^{1/3}y^{-1/6}F_4\left(\frac{1}{6}, \frac{2}{3}, \frac{4}{3}, 1; x, y\right), \\
\Omega_1^{P_2} &= -\frac{\alpha^2}{2}\pi^{-3/2}\Gamma\left(-\tfrac{1}{6}\right)\Gamma\left(\tfrac{2}{3}\right)F_4\left(-\tfrac{1}{6}, -\tfrac{1}{6}, \tfrac{2}{3}, \tfrac{1}{2}; \tfrac{x}{y}, \tfrac{1}{y}\right) + \beta_1^{P_2}\omega_1^{P_2}, \\
\Omega_2^{P_2} &= -\frac{\alpha}{3}\pi^{-3/2}\sqrt[3]{\tfrac{x}{y}}\,\Gamma\left(-\tfrac{2}{3}\right)\Gamma\left(\tfrac{1}{6}\right)F_4\left(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{4}{3}, \tfrac{1}{2}; \tfrac{x}{y}, \tfrac{1}{y}\right) + \beta_2^{P_2}\omega_2^{P_2},
\end{aligned} \tag{B.12}$$

with

$$\begin{aligned}
\beta_1^{P_2} &= -\frac{i}{4\pi}\left(2\log 2 + 3\log 3 - 6 + \pi(i - 2/\sqrt{3})\right), \\
\beta_2^{P_2} &= -\frac{i}{2^{4/3}\pi}\left(2\log 2 + 3\log 3 + \pi(i + 2/\sqrt{3})\right).
\end{aligned} \tag{B.13}$$

Comparing the expansions of these solutions with the asymptotic expansions of $a_{(D),I}$ for the semi-classical contributions fixes the coefficients. For this, one needs to match the F_4 expansions with the leading coefficients of the (differentiated) prepotential [11]

$$\mathcal{F} = \frac{\tau_0}{6}\sum_{i=1}^{3} Z_i^2 + \mathcal{F}_{1-\mathrm{loop}} + \mathcal{F}_{inst.}, \tag{B.14}$$

where[5]

$$\tau_0 = \frac{9 - \log 4}{2\pi i}. \tag{B.15}$$

From this, one finds

$$\begin{aligned}
a_{D,1} &= -i\sqrt{3}\alpha\left(\Omega_1^{P_2} - 2^{-2/3}\alpha\Omega_2^{P_2}\right) + \left(\alpha c_1 - \tfrac{i\sqrt{3}}{2}\right)\omega_1^{P_2} + \left(\alpha^2 c_2 + \tfrac{i\sqrt{3}}{2}\right)\omega_2^{P_2},\\
a_{D,2} &= -i\sqrt{3}\left(\Omega_1^{P_2} - 2^{-2/3}\Omega_2^{P_2}\right) + \left(c_1 + \tfrac{i\sqrt{3}}{2}\right)\omega_1^{P_2} + \left(c_2 - \tfrac{i\sqrt{3}}{2}\right)\omega_2^{P_2},\\
a_1 &= \frac{1}{2}\left(\omega_1^{P_2} + \omega_2^{P_2}\right),\\
a_2 &= -\frac{\alpha}{2}\left(\omega_1^{P_2} + \alpha\omega_2^{P_2}\right),
\end{aligned} \tag{B.16}$$

where $c_1 = \frac{\sqrt{3}}{4\pi}\left(2\log 2 + 3\log 3 + \frac{\pi}{\sqrt{3}} - 6\right)$ and $c_2 = -\frac{\sqrt{3}}{4\pi}\left(2\log 2 + 3\log 3 - \frac{\pi}{\sqrt{3}}\right)$. We note that for $u = 0$, we find $a_2 = -\alpha a_1$.

B.4 The $\mathbb{Z}_2$ vacua and massless states

In deriving the above results for the large v regime we have used a different symplectic basis than what is used in for example [13, 79]. In this subsection we briefly comment on how the two bases relate. The basis chosen in [13, 79] is more natural to use when comparing such quantities as the strong coupling periods for the two different loci, and in this basis we also compute the periods for all the points of interest. The change of basis is done by interchanging the roots $\xi_2 \leftrightarrow \xi_3$ as given in (B.5). Quantum mechanically, the singular branch of the classical theory splits into two branches separated by the scale Λ. Therefore, we must also interchange $r_2 \leftrightarrow r_3$ and $r_5 \leftrightarrow r_6$. One finds that this symplectic change of basis is given by the semi-classical version of the second Weyl reflection of the A_2 root lattice,

$$\mathcal{R}_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \in Sp(4, \mathbb{Z}). \tag{B.17}$$

This merely changes some prefactors of the solution (B.16). The change of roots modifies the cross-ratios in a trivial way, and they agree asymptotically with the theta quotients (3.19) computed from the new periods, as expected. One can show that the algebraic relations (5.3) for $u = 0$ take the same form. However, on this locus we now find

$$\tau_{12} = \frac{1 - \tau_{11}}{2}, \qquad \tau_{22} = \tau_{11} - 2, \tag{B.18}$$

from which it follows that

$$\begin{aligned}
2i\sqrt{27}\, v &= -\alpha^2 q^{-\frac{1}{6}} + 33\alpha q^{\frac{1}{6}} + 153 q^{\frac{1}{2}} + 713\alpha^2 q^{\frac{5}{6}} + \mathcal{O}(q^{\frac{7}{6}})\\
&= m\left(-\alpha q^{\frac{1}{6}}\right) = m\left(\tfrac{\tau}{6} - \tfrac{1}{6}\right),
\end{aligned} \tag{B.19}$$

[5] We correct a typo in [13, Eq. 6.8].

$(\underline{u}, v)$	$\pi(\underline{u}, v)$	normalisation
$(0,1)$	$(0,-\sqrt{3}i,1,-\alpha^2)$	$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{7}{6}\right)/2^{1/3}\sqrt{\pi}$
$(0,-1)$	$(-\sqrt{3}i,0,-\alpha,1)$	
$(1,0)$	$(0,0,1,1)$	
$(\alpha,0)$	$(\alpha^2,\alpha^2,0,-\alpha^2)$	$\sqrt[3]{2}\,\pi/3\sqrt{3}$
$(\alpha^2,0)$	$(-\alpha,-\alpha,-\alpha,0)$	
$(0,0)$	$(-i,-i,-\omega^5,\omega)$	$2\sqrt{\frac{\pi}{3}}\Gamma\left(\frac{7}{6}\right)/\Gamma\left(\frac{2}{3}\right)$

Table 1. Periods at the $\mathbb{Z}_3$, $\mathbb{Z}_2$ points and the origin, computed from the analytic continuation of the large v PF solution and appropriately normalized.

which is identical to (5.8) up to phases.

We can use the new solution to analyse the $\mathbb{Z}_3$ symmetry $u \mapsto \alpha u$. This leads to the matrix

$$\tilde{\sigma}_v = \alpha^2 \begin{pmatrix} 0 & 1 & -1 & 2 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{B.20}$$

It can also be obtained from the previous result (7.8) by conjugation with $\mathcal{R}_2$. It satisfies $\tilde{\sigma}_v^3 = \mathbb{1}$ and we can use it to generate the charges of the states that become massless at the $\mathbb{Z}_2$ points. To this end, we introduce the purely integral matrix $U = \alpha^2 \tilde{\sigma}_v^{-1} \in Sp(4,\mathbb{Z})$, which is the matrix used in [13, 79], and act with this on the monopole basis,

$$\begin{aligned} \tilde{\nu}_1 &= (1,0,0,0), & \tilde{\nu}_2 &= (0,1,0,0), \\ \tilde{\nu}_3 &= \tilde{\nu}_1 U = (-1,-1,1,-2), & \tilde{\nu}_4 &= \tilde{\nu}_2 U = (1,0,-2,1), \\ \tilde{\nu}_5 &= \tilde{\nu}_1 U^{-1} = (0,1,-1,2), & \tilde{\nu}_6 &= \tilde{\nu}_2 U^{-1} = (-1,-1,2,-1). \end{aligned} \tag{B.21}$$

Using the periods from Table 1 we can confirm that $\tilde{\nu}_{\{1,3,5\}}$ become massless at the AD point $(0,1)$ and $\tilde{\nu}_{\{2,3,6\}}$ at the AD point $(0,-1)$. Furthermore, the charges in row $k+1$ in (B.21) become massless at the $\mathbb{Z}_2$ point $(\underline{u}, v) = (\alpha^k, 0)$. It can be checked that the charges in each row are mutually local with respect to the symplectic inner product induced by J, given in (A.10). The charges in both columns however are mutually non-local. This is a crucial observation that lead to the discovery of new superconformal theories [23–25].

The matrix (B.20) conjugates the strong coupling matrices [13] as well as the semi-classical matrices according to

$$\tilde{\sigma}_v^{-1} M^{(r_1)} \tilde{\sigma}_v = M^{(r_2)}, \quad \tilde{\sigma}_v^{-1} M^{(r_2)} \tilde{\sigma}_v = M^{(r_3)}, \quad \tilde{\sigma}_v^{-1} M^{(r_3)} \tilde{\sigma}_v = M^{(r_1)}. \tag{B.22}$$

The same equations hold for the $\mathbb{Z}_2$ symmetry

$$\tilde{\rho}_v = \begin{pmatrix} 1 & 1 & -2 & 1 \\ -1 & 0 & 4 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \tag{B.23}$$

as is also the case for large u. As a consistency check, the pair $(\tilde{\sigma}_v, \tilde{\rho}_v)$ again satisfies the relation (7.7), and $\tilde{\rho}_v^2$ is a non-trivial monodromy. The matrix $\tilde{\rho}_v$ maps $\{\tilde{\nu}_2, \tilde{\nu}_4, \tilde{\nu}_6\}$ to $\{-\tilde{\nu}_1, -\tilde{\nu}_3, -\tilde{\nu}_5\}$ and therefore exchanges the AD points $v = \pm 1$.

The periods in Table 1 obtain different values depending on the direction from which the various points are approached.[6] On the locus $\mathcal{E}_u$, where $v = 0$, we have three singularities located at $\underline{u} = 1, \alpha, \alpha^2$. Reference [62] argues that one finds consistent values if the points are approached from the negative real axis. In this way we can go from weak to strong coupling without crossing walls of the second kind.[7] On $\mathcal{E}_v$, with $u = 0$, we instead have two singularities on the real line at $v = \pm 1$, analogous to the u-plane in the $SU(2)$ theory. There, we find a consistent picture by taking the limits from the lower half-plane in order to avoid the singular points (see discussion in [42]).

The two patches with large u and large v (from this subsection) respectively are connected by a simple change of basis. It is given by

$$\mathcal{M} = \mathcal{M}_{\tilde{\nu}_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}. \tag{B.24}$$

This matrix is the strong coupling monodromy (7.12) associated with the magnetic monopole $\tilde{\nu}_2 = (0, 1, 0, 0)$.

B.5 The $\mathbb{Z}_3$ vacua

With the explicit result (7.11) for the coupling matrix, the charges of the massless states (B.21) and the periods from Table 1 we can revisit the results of [23]. Starting from the three states $\tilde{\nu}_{\{1,3,5\}}$ which become massless at $(\underline{u}, v) = (0, 1)$, we aim to find a symplectic projection such that the massless states are charged only under the first $U(1)$ factor. Following the logic of [23, 79], in this basis the coupling matrix becomes

[6] This is not only a problem involving monodromies. By computing coupling matrices at the origin from different directions we find that they generally do not lie in the Siegel upper half-plane $\mathbb{H}_2$, even though it is a regular point of the curve. One cannot place them back in $\mathbb{H}_2$ by acting on them with monodromy matrices in $Sp(4, \mathbb{Z})$.

[7] Walls of the second kind are generally defined as hypersurfaces where a fixed quiver QM description of the BPS spectrum breaks down, and one needs to mutate the quiver to find the spectrum on the other side of the wall [62].

diagonal ($\tau_{12} = 0$) and the curve splits into a small and a large torus, parametrized by τ_{11} and τ_{22}, respectively. The modulus of the large torus is fixed by the $\mathbb{Z}_3$ symmetry to be $\tau_{22} = -\alpha^2$. The small torus $\tau_{11} = \tau(\rho)$ depends on the direction ρ from which the AD point is approached, where $\delta v = 2\varepsilon^3$, $\delta u = 3\varepsilon^3\rho$. The small torus near the $\mathbb{Z}_3$ point takes the form $w^2 = z^3 - 3\rho z - 2$. This curve degenerates if $\rho^3 = 1$, has a $\mathbb{Z}_2$ symmetry at $\rho^3 = \infty$ and a $\mathbb{Z}_3$ symmetry at $\rho^3 = 0$.

If we approach the AD point from the $\rho = 0$ plane we find that $\tau_{11} = \alpha$. By an $Sp(4,\mathbb{Z})$ transformation we can go to a basis where the mutually non-local states $\tilde{\nu}_1$, $\tilde{\nu}_3$ and $\tilde{\nu}_5$ are mapped to an electron, a monopole and a dyon, all charged with respect to the first $U(1)$ factor only. This is done, for example, by the transformation

$$\mathcal{A} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix} \in Sp(4,\mathbb{Z}). \tag{B.25}$$

This furthermore diagonalizes the coupling matrix

$$\mathcal{A} : \Omega(0,1) \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha^2 \end{pmatrix}, \tag{B.26}$$

as anticipated. The periods $\pi(0,1) = (0,*,0,*)$ depend on the exact transformation, but the relations $a_1 = \alpha^2 a_{D,1} \to 0$ and $a_2 = -\alpha^2 a_{D,2}$ are fixed.

C Proofs of modular identities

In this section we collect some rigorous proofs of exact statements made in the sections above.

C.1 The origin of the moduli space

The zeros of $u(\tau)$ for both the $SU(2)$ and $SU(3)$ theory can be derived from the properties of the Jacobi theta functions.

The $SU(2)$ theory

The moduli space of the pure $SU(2)$ theory is parametrized by the modular function

$$u(\tau) = \frac{\vartheta_2(\tau)^4 + \vartheta_3(\tau)^4}{2\vartheta_2(\tau)^2\vartheta_3(\tau)^2} = 1 + \frac{1}{8}\left(\frac{\eta(\frac{\tau}{4})}{\eta(\tau)}\right)^8. \tag{C.1}$$

The Jacobi theta functions ϑ_j and their transformation properties are given in Appendix A.1. The zeros of u are given by the $\Gamma^0(4)$-orbit of $1+i$. To prove this, it suffices to observe that

$$\vartheta_2(1+i)^4 + \vartheta_3(1+i)^4 = \left(e^{\frac{\pi i}{4}}\vartheta_2(i)\right)^4 + \vartheta_4(i)^4 = -\vartheta_2(i)^4 + \vartheta_2(i)^4 = 0, \tag{C.2}$$

where we have used the T-transformation in the first equation and the S-transformation of ϑ_4 in the second equation. Using the result (A.7), we know that the denominator is nonzero. Therefore we have proven that $u(1+i)=0$.

The $SU(3)$ theory

Let us prove that (4.21) is a root of (4.13). Notice that

$$b_{3,0}(\tau)=\vartheta_3(2\tau)\vartheta_3(6\tau)+\vartheta_2(2\tau)\vartheta_2(6\tau). \tag{C.3}$$

Without computing any of these sums, we can simplify the terms in $b_{3,0}\left(\frac{\tau_0}{3}\right)$ by making use of the transformation identities in Section A.1,

$$\begin{aligned}
\vartheta_3(1+\tfrac{i}{\sqrt{3}}) &= \vartheta_4(\tfrac{i}{\sqrt{3}}),\\
\vartheta_3(3+\sqrt{3}i) &= \vartheta_4(\sqrt{3}i),\\
\vartheta_2(1+\tfrac{i}{\sqrt{3}}) &= e^{\frac{\pi i}{4}}\vartheta_2(\tfrac{i}{\sqrt{3}}) = \sqrt[4]{3}\,e^{\frac{\pi i}{4}}\vartheta_4(\sqrt{3}i)\\
\vartheta_2(3+\sqrt{3}i) &= e^{\frac{3\pi i}{4}}\vartheta_2(\sqrt{3}i) = \tfrac{1}{\sqrt[4]{3}}\,e^{\frac{3\pi i}{4}}\vartheta_4(\tfrac{i}{\sqrt{3}}).
\end{aligned} \tag{C.4}$$

We thus find that $b_{3,0}(\frac{\tau_0}{3})=b_{3,0}(\frac{1}{2}+\frac{i}{2\sqrt{3}})=0$. The denominator

$$b_{3,1}(\tau)=3\,\frac{\eta(3\tau)^3}{\eta(\tau)} \tag{C.5}$$

vanishes nowhere on $\mathbb{H}$, as $\eta^{24}(\tau)=\Delta(\tau)$ is a holomorphic cusp form of weight 12 for $SL(2,\mathbb{Z})$. This proves that indeed $u(\tau_0)=0$.

C.2 The function v

Since on the locus $\mathcal{E}_v$ the relations (5.5) among the τ_{IJ} are exact, it is possible to prove the step from (5.6) to (5.11) by computing the theta constants analytically instead of perturbatively (as done on $\mathcal{E}_u$). First, note that $C_1=\lambda_3=\frac{\Theta_8^2}{\Theta_{10}^2}$, since $\Theta_1=\Theta_2$ due to (5.5). This lets us simplify,

$$v=-\frac{i}{\sqrt{27}}\frac{(\Theta_8^2-2\Theta_{10}^2)(\Theta_8^2+\Theta_{10}^2)(2\Theta_8^2-\Theta_{10}^2)}{\Theta_8^2\Theta_{10}^2(\Theta_8^2-\Theta_{10}^2)}. \tag{C.6}$$

Both sides of this equation are functions of $\Omega=\begin{pmatrix}\tau_{11} & -\tau_{11}/2+1\\ -\tau_{11}/2+1 & \tau_{11}-1\end{pmatrix}$. We have that $\tau_-=\frac{3}{2}\tau_{11}-1$ and therefore $\tau=\tau_-+1=\frac{3}{2}\tau_{11}$, as defined in Section 5.3. In view of the claim $v\propto m(\frac{\tau}{6})$, let us further define $\sigma=\frac{\tau}{6}=\frac{\tau_{11}}{4}$ to obtain integral powers of $\mathsf{q}:=e^{2\pi i\sigma}$. This allows to compute the theta constants,

$$\begin{aligned}
\Theta_8(\Omega) &= e^{\frac{\pi i}{4}}\sum_{k,l\in\mathbb{Z}+\frac{1}{2}}(-1)^{k+l}\mathsf{q}^{2(k^2+kl+l^2)},\\
\Theta_{10}(\Omega) &= e^{\frac{\pi i}{4}}\sum_{k,l\in\mathbb{Z}+\frac{1}{2}}(-1)^{2k}\mathsf{q}^{2(k^2+kl+l^2)}.
\end{aligned} \tag{C.7}$$

The q-series on the rhs should be interpreted as functions of $\sigma(\tau_{11}) = \frac{\tau_{11}}{4}$, however it is convenient to consider them as functions of the new variable $\sigma \in \mathbb{H}$. On $\mathcal{E}_v$, the theta constants Θ_8 and Θ_{10} collapse to shifted theta functions of the A_2 root lattice, as defined in (4.14). In order to see this, define

$$f(\tau) = \frac{1}{3}(b_{3,0}(\tau) - b_{3,0}(4\tau)) = \frac{2\eta(4\tau)^2\,\eta(12\tau)^2}{\eta(2\tau)\,\eta(6\tau)} = 2(q + q^3 + 2\,q^7 + \dots). \tag{C.8}$$

According to Theorem 1 in Appendix A.1, f is a modular form of weight 1 for $\Gamma_0(12)$. By splitting the $b_{3,0}$ theta functions into even and odd exponents, it can be easily shown that

$$\Theta_8(\sigma) = -ie^{\frac{\pi i}{4}} f(\tfrac{\sigma}{2} + \tfrac{1}{4}), \quad \Theta_{10}(\sigma) = -e^{\frac{\pi i}{4}} f(\tfrac{\sigma}{2}), \tag{C.9}$$

with the abuse of notation $\Theta_j(\Omega) = \Theta_j(\sigma)$ with Ω and σ as below (C.6). By modularity, it is enough to compare a finite number of coefficients between (C.6) and (5.10), which proves $v = -\frac{i}{2\sqrt{27}}\, m(\sigma)$.

Special points of v

The solutions to $v = 1$ and $v = -1$ are not straightforward to obtain. Let us start with the point $(\underline{u}, v) = (0, -1)$. In the following, all arguments are those of m. Due to the prefactor in (5.11), $v = -1$ is in fact a quadratic equation with zero discriminant and therefore satisfied if and only if

$$\left(\frac{\eta(2\tau)}{\eta(6\tau)}\right)^6 = -\sqrt{27}\, i. \tag{C.10}$$

A solution to this equation can be found to be

$$\tau_{-1} = \frac{\omega}{2\sqrt{3}} = \frac{1}{4} + \frac{i}{4\sqrt{3}} = \frac{\tau_{\text{AD},2}}{6} \tag{C.11}$$

with $\omega = e^{\pi i/6}$ as before and $\tau_{\text{AD},2}$ the argument of v in (5.12). The other AD point can be found using the symmetry of m, and it is given by

$$\tau_{+1} = \frac{\omega^5}{2\sqrt{3}} = -\frac{1}{4} + \frac{i}{4\sqrt{3}} = \frac{\tau_{\text{AD},1}}{6}. \tag{C.12}$$

The zero of m (and therefore of v) is given by

$$\tau_0 = \frac{i}{2\sqrt{3}}. \tag{C.13}$$

Note that all these numbers have the same absolute value $\frac{1}{2\sqrt{3}}$.

Let us prove (C.11) first: In order to compute both the numerator and the denominator, we can resort to the S- and T-transformations of η as given in A.1,

$$\begin{aligned} \eta(2\tau_{-1}) &\overset{S}{=} 3^{\frac{1}{4}}\omega e^{-\frac{\pi i}{12}}\eta(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i) \overset{T}{=} 3^{\frac{1}{4}} e^{\frac{\pi i}{12}}\eta(\alpha) \\ \eta(6\tau_{-1}) &= \eta(\tfrac{3}{2} + \tfrac{\sqrt{3}}{2}i) \overset{T}{=} e^{\frac{\pi i}{6}}\eta(\alpha). \end{aligned} \tag{C.14}$$

Equation (C.10) follows immediately.
In order to find the point where $v = +1$, we can make the observation that $m(-\frac{1}{\tau}) = -m(\frac{\tau}{12})$. This implies that under the Fricke involution $\left(\begin{smallmatrix} 0 & -1 \\ 12 & 0 \end{smallmatrix}\right)$, the solution receives a minus sign,

$$m\left(-\frac{1}{12\tau}\right) = -\,m(\tau). \tag{C.15}$$

Using the T-transformation of η, one also finds that $m\left(\tau \pm \frac{1}{2}\right) = -m(\tau)$. We can use either of those maps, $\tau_{+1} = \tau_{-1} - \frac{1}{2} = -\frac{1}{12\tau_{-1}}$ to obtain (C.12).

We can also study the zeros of v. Every root of $m(\tau)$ is given by the equation $\eta(2\tau)^{12} = 27\,\eta(6\tau)^{12}$. A solution to this equation is (C.13), which we can prove: Using the S-transformation, we find

$$\eta(2\tau_0) = \eta(\tfrac{i}{\sqrt{3}}) = 3^{\frac{1}{4}}\eta(\sqrt{3}i) = 3^{\frac{1}{4}}\eta(6\tau_0). \tag{C.16}$$

The result follows immediately. Another proof follows simply from the fact that τ_0 is the fixed point under (C.15).

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