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# Equivalence and Strong Equivalence between Sparsest and Least $\ell_1$ -Norm Nonnegative Solutions of Linear Systems and Their Applications

Yun-Bin Zhao\*

**Abstract.** Many practical problems can be formulated as  $\ell_0$ -minimization problems with non-negativity constraints, which seek the sparsest nonnegative solutions to underdetermined linear systems. Recent study indicates that  $\ell_1$ -minimization is efficient for solving  $\ell_0$ -minimization problems. From a mathematical point of view, however, the understanding of the relationship between  $\ell_0$ - and  $\ell_1$ -minimization remains incomplete. In this paper, we further address several theoretical questions associated with these two problems. We prove that the fundamental strict complementarity theorem of linear programming can yield a necessary and sufficient condition for a linear system to admit a unique least  $\ell_1$ -norm nonnegative solution. This condition leads naturally to the so-called range space property (RSP) and the ‘full-column-rank’ property, which altogether provide a new and broad understanding of the equivalence and the strong equivalence between  $\ell_0$ - and  $\ell_1$ -minimization. Motivated by these results, we introduce the concept of ‘RSP of order  $K$ ’ that turns out to be a full characterization of uniform recovery of all  $K$ -sparse nonnegative vectors. This concept also enables us to develop a nonuniform recovery theory for sparse nonnegative vectors via the so-called weak range space property.

**Key words** Strict complementarity, linear programming, underdetermined linear system, sparsest nonnegative solution, range space property, uniform recovery, nonuniform recovery.

**Mathematics Subject Classification** 90C05, 90C25, 65K05, 94A08, 94A20.

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# 1 Introduction

In this paper, we use  $\|\cdot\|_0$  to denote the number of nonzero components of a vector. We investigate the following optimization problem with nonnegativity constraints:

$$\min\{\|x\|_0 : Ax = b, x \geq 0\}, \quad (1)$$

which is called an  $\ell_0$ -minimization problem or  $\ell_0$ -problem. It is well known that the nonnegativity constraints are quite common in mathematical optimization and numerical analysis (see, e.g., [12] and the references therein). Clearly, the aim of the problem (1) is to find a *sparsest nonnegative solution* to a system of linear equations. This problem has found so many applications in such areas as signal and image processing [16, 2, 6, 34, 42, 17, 44, 29], machine learning [32, 3, 4, 31, 25], pattern recognition and computer vision [42, 40], proteomics [39], to name but a few. This problem is a special case of the compressed nonnegative sparse coding [26, 43] and rank minimization with positive semi-definite constraints (see, e.g., [37, 44, 48]). It is closely related to the so-called nonnegative matrix factorization as well [30, 36, 35].

The  $\ell_0$ -minimization problem is NP-hard [33]. The current theory and algorithms for  $\ell_0$ -minimization are mainly developed through heuristic methods and continuous approximations. A large amount of recent attention is attracted to the  $\ell_1$ -problem

$$\min\{\|x\|_1 : Ax = b, x \geq 0\} \quad (2)$$

which has been shown efficient for solving (1) in many situations, so does the reweighted  $\ell_1$ -minimization (e.g., [11, 49]). In this paper, the optimal solution to the problem (2) is called the *least  $\ell_1$ -norm nonnegative solution* to the linear system  $Ax = b$ . Any linear programming solver can be used to solve the problem (2). Various specialized algorithms for this problem have also been proposed in the literature (e.g., [6, 42, 45, 25]).

Over the past few years,  $\ell_0$ -problems without nonnegativity constraints have been extensively studied in the field of sparse signal and image processing and compressed sensing. Both theory and numerical methods have been developed for this problem (e.g., [7, 10, 14, 11, 5, 18, 49]). However, the sparsest solution and sparsest nonnegative solution to a linear system are very different from a mathematical point of view. The analysis and many results developed for the sparsest solution to a linear system cannot apply to the sparsest nonnegative solution straightaway. So far, the understanding of the relationship between (1) and (2), and the  $\ell_1$ -method-based recovery theory for sparse nonnegative vectors remains very incomplete. For example, the following important questions have not well addressed at present:

- (a) How to completely characterize the uniqueness of least  $\ell_1$ -norm nonnegative solutions to a linear system?
- (b) How to deterministically explain the efficiency and the limitation of the  $\ell_1$ -method for solving  $\ell_0$ -problems?
- (c) Are there any other matrix properties that are different from the existing ones (such as restricted isometric property (RIP) [9, 7, 8] and null space property (NSP) [13, 29]) and can fully characterize the exact recovery of  $K$ -sparse nonnegative vectors?

- (d) Is it possible to develop some theory for the exact recovery of sparse nonnegative vectors that may go beyond the scope of uniform recovery?

In general, for a given pair  $(A, b)$ , the sparsest nonnegative solution to the system  $Ax = b$  is not unique. So it is important to distinguish the equivalence and the strong equivalence between (1) and (2). In this paper,  $\ell_0$ - and  $\ell_1$ -problems are said to be equivalent if the  $\ell_0$ -problem has an optimal solution that coincides with the unique optimal solution to the  $\ell_1$ -problem. We say that the  $\ell_0$ - and  $\ell_1$ -problems are strongly equivalent if the  $\ell_0$ -problem has a unique optimal solution that coincides with the unique optimal solution to the  $\ell_1$ -problem. Clearly, the ‘strong equivalence’ implies the ‘equivalence’, but the converse is not true in general. The ‘equivalence’ does not require an  $\ell_0$ -problem to have a unique optimal solution. Of course, the above-mentioned questions (a)-(d) can be partially addressed by applying the existing theory based on such concepts as the mutual coherence [19, 15, 20], ERC [21, 41], RIP [9, 7, 8], NSP [46, 13, 29, 47], outwardly  $k$ -neighborliness property [16], and the verifiable condition [27, 28]. However, these existing conditions are restrictive in the sense that they imply the strong equivalence (instead of the equivalence) between  $\ell_0$ - and  $\ell_1$ -problems. For instance, Donoho and Tanner [16] have given a geometric condition, i.e., the outwardly  $K$ -neighborliness property of the sensing matrix, which guarantees that a  $K$ -sparse nonnegative vector is unique to both problems (1) and (2). An equivalent form of this result was also discovered by Zhang (see Theorem 1 in [46]). From a null-space perspective, Zhang [47], and Khajehnejad et al [29] have shown that  $K$ -sparse nonnegative vectors can be exactly recovered by  $\ell_1$ -minimization if and only if the null space of  $A$  satisfies certain property. Thus the outwardly  $K$ -neighborliness property [16] and the null space property [29, 47] imply the strong equivalence between problems (1) and (2). In addition, the mutual coherence condition (e.g. [19, 20, 24]) and RIP [9, 7, 8] can be extended to guarantee the uniqueness of least  $\ell_1$ -norm nonnegative solutions to a linear system. However, these sufficient conditions are not necessary conditions for the uniqueness of optimal solutions to the  $\ell_1$ -problem and for the equivalence between  $\ell_0$ - and  $\ell_1$ -problems. As shown by our later analysis, the ‘equivalence’ concept enables us to deeply and broadly understand the relationship between  $\ell_0$ - and  $\ell_1$ -minimization, making it possible to address the aforementioned questions (a)-(d).

The first purpose of this paper is to completely address the question (a) by developing a necessary and sufficient condition for the uniqueness of least  $\ell_1$ -norm nonnegative solutions to a linear system. We establish this condition through the strict complementarity theory of linear programming, which leads naturally to the new concept of range space property (RSP) of  $A^T$ . Based on this result, we first show that the equivalence between  $\ell_0$ - and  $\ell_1$ -problems can be interpreted by the RSP of  $A^T$ , which is remarkably different from existing analyses in [16, 29]. We prove that the  $\ell_1$ -method can guarantee to solve an  $\ell_0$ -problem if and only if the RSP holds at an optimal solution to the  $\ell_0$ -problem. The RSP-based analysis can yield a broad understanding of the efficiency and the restriction of the  $\ell_1$ -method for solving  $\ell_0$ -problems, and can efficiently explain the theoretical and actual numerical performance of the  $\ell_1$ -method, leading to an answer to the question (b).

Furthermore, we introduce a matrix property, called the RSP of order  $K$ , through which we provide a characterization of uniform recovery of sparse nonnegative vectors. Interestingly, the variants of this new concept make it possible to extend uniform recovery to non-uniform

recovery of some sparse nonnegative vectors, to which the uniform recovery does not apply. Such an extension is important not only from a mathematical point of view, but from the viewpoint of many practical applications as well. For instance, when many columns of  $A$  are important, the sparsest solution to the linear system  $Ax = b$  may not be sparse enough to satisfy the uniform recovery conditions. The RSP of order  $K$  and its variants make it possible to address the aforementioned questions (c) and (d).

This paper is organized as follows. In Sect. 2, we develop a necessary and sufficient condition for a linear system to have a unique least  $\ell_1$ -norm nonnegative solution. In Sect. 3, we provide an efficiency analysis for the  $\ell_1$ -problem in solving  $\ell_0$ -problems through the RSP of  $A^T$ . In Sect. 4, we develop a guaranteed recovery of  $K$ -sparse nonnegative vectors via the so-called RSP of order  $K$ , and conclusions are given in the last section.

## 2 Uniqueness of least $\ell_1$ -norm nonnegative solutions

Throughout this paper, we use the following notation: Let  $R_+^n$  be the first orthant of  $R^n$ , the  $n$ -dimensional Euclidean space. Let  $e = (1, 1, \dots, 1)^T \in R^n$  be the vector of ones. For two vectors  $u, v \in R^n$ ,  $u \leq v$  means  $u_i \leq v_i$  for every  $i = 1, \dots, n$ , and in particular,  $v \geq 0$  means  $v \in R_+^n$ . For a set  $S \subseteq \{1, 2, \dots, n\}$ ,  $|S|$  denotes the cardinality of  $S$ , and  $S_c = \{1, 2, \dots, n\} \setminus S$  is the complement of  $S$ . For a matrix  $A$  with columns  $a_j$ ,  $1 \leq j \leq n$ , we use  $A_S$  to denote the submatrix of  $A$  with columns  $a_j, j \in S$ . Similarly,  $x_S$  denotes the subvector of  $x$  with components  $x_j, j \in S$ . For  $x \in R^n$ , let  $\|x\|_1 = \sum_{i=1}^n |x_i|$  denote the  $\ell_1$ -norm of  $x$ . For  $A \in R^{m \times n}$ , we use  $\mathcal{R}(A^T)$  to denote the range space of  $A^T$ , i.e.,  $\mathcal{R}(A^T) = \{A^T u : u \in R^m\}$ .

In this section, we develop a necessary and sufficient condition for  $x$  to be the unique least  $\ell_1$ -norm nonnegative solution to a linear system. Note that when  $x$  is the unique optimal solution to the problem (2), there is no other nonnegative solution  $w \neq x$  such that  $\|w\|_1 \leq \|x\|_1$ . Thus the uniqueness of the solution  $x$  is equivalent to

$$\{w : Aw = b, w \geq 0, \|w\|_1 \leq \|x\|_1\} = \{x\}.$$

Since  $x \geq 0$  and  $w \geq 0$ , we have  $\|w\|_1 = e^T w$  and  $\|x\|_1 = e^T x$ . Thus the above relation can be further written as  $\{w : Aw = Ax, e^T w \leq e^T x, w \geq 0\} = \{x\}$ . Consider the following linear programming (LP) problem with the variable  $w \in R^n$  :

$$\min\{0^T w : Aw = Ax, e^T w \leq e^T x, w \geq 0\}, \quad (3)$$

which is feasible (since  $w = x$  is always a feasible solution), and the optimal value of the problem is finite (equal to zero). From the above discussion, we immediately have the following observation.

**Lemma 2.1**  *$x$  is the unique least  $\ell_1$ -norm nonnegative solution to the system  $Ax = b$  if and only if  $w = x$  is the unique optimal solution to the problem (3), i.e.,  $(w, t) = (x, 0)$  is the unique optimal solution to the following problem:*

$$\min\{0^T w : Aw = Ax, e^T w + t = e^T x, (w, t) \geq 0\} \quad (4)$$

where  $t$  is a slack variable introduced into (3).

Note that the dual problem of (4) is given by

$$\begin{aligned} \max \quad & (Ax)^T y + (e^T x)\beta \\ \text{s.t.} \quad & A^T y + \beta e \leq 0, \\ & \beta \leq 0, \end{aligned} \tag{5}$$

where  $y$  and  $\beta$  are variables. Throughout this section, we use  $(s, r) \in R_+^{n+1}$  to denote the slack variables of the problem (5), i.e.,

$$s = -(A^T y + \beta e) \geq 0, \quad r = -\beta \geq 0.$$

Let us recall a fundamental theorem for LP problems. Let  $B \in R^{m \times n}$  be a given matrix, and  $p \in R^m$  and  $c \in R^n$  be two given vectors. Consider the linear program (LP)

$$\min\{c^T x : Bx = p, \quad x \geq 0\}, \tag{6}$$

and its dual problem

$$\max\{p^T y : B^T y + s = c, \quad s \geq 0\}. \tag{7}$$

Any optimal solution pair  $(x, (y, s))$  to the problems (6) and (7) satisfies the so-called complementary slackness condition:  $x^T s = 0, x \geq 0$  and  $s \geq 0$ . Moreover, if a solution pair  $(x, (y, s))$  satisfies that  $x + s > 0$ , it is called a strictly complementary solution pair. For any feasible linear programming problems (6) and (7), there always exists a pair of strictly complementary solutions.

**Lemma 2.2** ([38]) (i) (Optimality condition)  $(x, (y, s))$  is a solution pair of the LP problems (6) and (7) if and only if it satisfies the following conditions:  $Bx = p, B^T y + s = c, x \geq 0, s \geq 0$ , and  $x^T s = 0$ . (ii) (Strict complementarity) If (6) and (7) are feasible, then there exists a pair  $(x^*, (y^*, s^*))$  of strictly complementary solutions to (6) and (7).

We now prove the following necessary condition for the problem (2) to have a unique optimal solution.

**Lemma 2.3** If  $x$  is the unique least  $\ell_1$ -norm nonnegative solution to the system  $Ax = b$ , then there exists a vector  $\eta \in R^n$  satisfying

$$\eta \in \mathcal{R}(A^T), \quad \eta_i = 1 \text{ for } i \in J_+, \text{ and } \eta_i < 1 \text{ for } i \notin J_+, \tag{8}$$

where  $J_+ = \{i : x_i > 0\}$ .

*Proof.* Consider the problem (4) and its dual problem (5), both of which are feasible. By Lemma 2.2, there exists an optimal solution  $(w^*, t^*)$  to the problem (4) and an optimal solution  $(y^*, \beta^*)$  to (5) such that these two solutions constitute a pair of strictly complementary solutions. Let  $(s^*, r^*) = (-A^T y^* - \beta^* e, -\beta^*)$  be the value of the associated slack variables of the dual problem (5). Then by the strict complementarity, we have

$$(w^*)^T s^* = 0, \quad t^* r^* = 0, \quad w^* + s^* > 0, \quad t^* + r^* > 0. \tag{9}$$

Since  $x$  is the unique least  $\ell_1$ -norm nonnegative solution to  $Ax = b$ , by Lemma 2.1,  $(x, 0)$  is the unique optimal solution to the problem (4). Thus

$$(w^*, t^*) = (x, 0), \quad (10)$$

which implies that  $w_i^* > 0$  for all  $i \in J_+ =: \{i : x_i > 0\}$  and  $w_i^* = 0$  for all  $i \notin J_+$ . Thus it follows from (9) and (10) that

$$r^* > 0, \quad s_i^* = 0 \quad \text{for all } i \in J_+, \quad \text{and } s_i^* > 0 \quad \text{for all } i \notin J_+.$$

That is,

$$\beta^* < 0, \quad (A^T y^* + \beta^* e)_i = 0 \quad \text{for } i \in J_+, \quad (A^T y^* + \beta^* e)_i < 0 \quad \text{for } i \notin J_+,$$

which can be written as

$$\beta^* < 0, \quad \left[ A^T \left( \frac{y^*}{-\beta^*} \right) - e \right]_i = 0 \quad \text{for } i \in J_+, \quad \left[ A^T \left( \frac{y^*}{-\beta^*} \right) - e \right]_i < 0 \quad \text{for } i \notin J_+.$$

By setting  $\eta = A^T y^* / (-\beta^*)$ , the condition above is equivalent to

$$\eta \in \mathcal{R}(A^T), \quad \eta_i = 1 \quad \text{for } i \in J_+, \quad \text{and } \eta_i < 1 \quad \text{for } i \notin J_+,$$

as desired.  $\square$

Throughout this paper, the condition (8) is called the *range space property* (RSP) of  $A^T$  at  $x \geq 0$ . So Lemma 2.3 shows that this property is a necessary condition for the  $\ell_1$ -problem to have a unique optimal solution. We now prove another necessary condition.

**Lemma 2.4** *If  $x$  is the unique least  $\ell_1$ -norm nonnegative solution to the system  $Ax = b$ , then the matrix*

$$M = \begin{pmatrix} A_{J_+} \\ e_{J_+}^T \end{pmatrix} \quad (11)$$

*has full column rank, where  $J_+ = \{i : x_i > 0\}$ .*

*Proof.* Assume the contrary that the columns of matrix  $M$  defined by (11) is linearly dependent. Then there exists a vector  $u \in R^{|J_+|}$  such that

$$u \neq 0, \quad Mu = \begin{pmatrix} A_{J_+} \\ e_{J_+}^T \end{pmatrix} u = 0. \quad (12)$$

Let  $(w, t)$  be given by  $w = (w_{J_+}, w_{J_0}) = (x_{J_+}, 0)$  and  $t = 0$ , where  $J_0 = \{i : i \notin J_+\}$ . Then it is easy to see that such defined  $(w, t)$  is an optimal solution to the problem (4). On the other hand, let us define  $(\tilde{w}, \tilde{t})$  as follows:

$$\tilde{w} = (\tilde{w}_{J_+}, \tilde{w}_{J_0}) = (w_{J_+} + \lambda u, 0), \quad \tilde{t} = 0.$$

Since  $w_{J_+} = x_{J_+} > 0$ , there exists a small  $\lambda \neq 0$  such that

$$\tilde{w}_{J_+} = w_{J_+} + \lambda u \geq 0. \quad (13)$$

Substituting  $(\tilde{w}, \tilde{t})$  into the constraints of the problem (4), we see from (12) that  $(\tilde{w}, \tilde{t})$  satisfies all those constraints. Thus  $(\tilde{w}, \tilde{t})$  is also an optimal solution to the problem (4). It follows from

(13) that  $\tilde{w}_{J_+} \neq w_{J_+}$  since  $\lambda u \neq 0$ . Therefore, the optimal solution to the problem (4) is not unique. However, by Lemma 2.1, when  $x$  is the unique least  $\ell_1$ -norm nonnegative solution to the system  $Ax = b$ , the problem (4) must have a unique optimal solution. This contradiction shows that  $M$  has full column rank.  $\square$

The next result shows that the combination of the necessary conditions developed in Lemmas 2.3 and 2.4 is sufficient for the  $\ell_1$ -problem to have a unique optimal solution.

**Lemma 2.5** *Let  $x \geq 0$  be a solution to the system  $Ax = b$ . If the condition (8) (i.e., the RSP of  $A^T$ ) is satisfied at  $x$  and the matrix  $M$  given by (11) has full column rank, then  $x$  is the unique least  $\ell_1$ -norm nonnegative solution to the system  $Ax = b$ .*

*Proof.* By Lemma 2.1, to prove that  $x$  is the unique least  $\ell_1$ -norm nonnegative solution to the system  $Ax = b$ , it is sufficient to prove that the problem (4) has a unique optimal solution  $(x, 0)$ . First, the condition (8) implies that there exist  $\eta$  and  $y$  such that

$$A^T y = \eta, \quad \eta_i = 1 \text{ for } i \in J_+, \text{ and } \eta_i < 1 \text{ for } i \notin J_+.$$

By setting  $\beta = -1$ , the relation above can be written as

$$(A^T y)_i + \beta = 0 \text{ for } i \in J_+, \text{ and } (A^T y)_i + \beta < 0 \text{ for } i \notin J_+, \quad (14)$$

from which we see that  $(y, \beta)$  satisfies all constraints of the problem (5). We now further verify that it is an optimal solution to (5). By (14), the objective value of (5) at  $(y, \beta)$  is

$$\begin{aligned} (Ax)^T y + (e^T x)\beta &= x^T (A^T y) + (e^T x)\beta \\ &= \sum_{i \in J_+} x_i (A^T y)_i + \beta \sum_{i \in J_+} x_i \\ &= -\beta \sum_{i \in J_+} x_i + \beta \sum_{i \in J_+} x_i = 0. \end{aligned} \quad (15)$$

Since the optimal value of (4) is zero, by LP duality theory, the maximum value of the dual problem is also zero. Thus it follows from (15) that the point  $(y, \beta)$  satisfying (14) is an optimal solution to the problem (5).

We now prove that the optimal solution of (4) is uniquely determined under the assumption of the theorem. Assume that  $(w^*, t^*)$  is an arbitrary optimal solution to the problem (4), which of course satisfies all constraints of (4), i.e.,

$$Aw^* = Ax, \quad e^T w^* + t^* = e^T x, \quad (w^*, t^*) \geq 0. \quad (16)$$

Since  $(y, \beta)$ , satisfying (14), is an optimal solution of (5),  $((w^*, t^*), ((y, \beta), s))$  is a solution pair to (4) and (5). From (14), we see that the dual slack variables  $s_i = -((A^T y)_i + \beta) > 0$  for  $i \notin J_+$  and  $r = -\beta = 1 > 0$ . By complementary slackness property (Lemma 2.2(i)), we must have that

$$t^* = 0, \quad w_i^* = 0 \text{ for all } i \notin J_+.$$

By substituting these known components into (16) and noting that  $x_i = 0$  for  $i \notin J_+$ , we see that the remaining components of  $(w^*, t^*)$  satisfy

$$A_{J_+} w_{J_+}^* = Ax = A_{J_+} x_{J_+}, \quad e_{J_+}^T w_{J_+}^* = e^T x = e_{J_+}^T x_{J_+}, \quad w_{J_+}^* \geq 0.$$



Since the matrix  $M = \begin{pmatrix} A_{J_+} \\ e_{J_+}^T \end{pmatrix}$  has full column rank,  $w_{J_+}^* = x_{J_+}$  is the unique solution to the reduced system above. Therefore,  $(w^*, t^*)$  is uniquely given by  $(x, 0)$ . In other words, the only optimal solution to the problem (4) is  $(x, 0)$ . By Lemma 2.1,  $x$  must be the unique least  $\ell_1$ -norm nonnegative solution to the system  $Ax = b$ .  $\square$

By Lemmas 2.3, 2.4 and 2.5, we summarize the main result of this section as follows.

**Theorem 2.6** *Let  $x^*$  be a nonnegative solution to the system  $Ax = b$ . Then  $x^*$  is the unique least  $\ell_1$ -norm nonnegative solution to the system  $Ax = b$  if and only if the RSP (8) holds at  $x^*$  and the matrix  $M = \begin{pmatrix} A_{J_+} \\ e_{J_+}^T \end{pmatrix}$  has full column rank, where  $J_+ = \{i : x_i^* > 0\}$ .*

Clearly, when  $A_{J_+}$  has full column rank, so does the matrix  $M$  given by (11). The converse is not true, i.e., when the matrix  $M$  given by (11) has full column rank, it does not imply that the matrix  $A_{J_+}$  has full column rank in general. For instance,  $M = \begin{pmatrix} A_{J_+} \\ e_{J_+}^T \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$  has full column rank, but  $A_{J_+} = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$  does not. However, when the RSP (8) holds at  $x$ , we see that  $e_{J_+} = A_{J_+}^T u$  for some  $u \in R^n$ , in which case  $A_{J_+}$  has full column rank if and only if  $\begin{pmatrix} A_{J_+} \\ e_{J_+}^T \end{pmatrix}$  has full column rank. Thus Theorem 2.6 can be further stated as follows.

**Theorem 2.7** *Let  $x^*$  be a nonnegative solution to the system  $Ax = b$ . Then  $x^*$  is the unique least  $\ell_1$ -norm nonnegative solution to the system  $Ax = b$  if and only if the RSP (8) holds at  $x^*$  and the matrix  $A_{J_+}$  has full column rank, where  $J_+ = \{i : x_i^* > 0\}$ .*

The above results completely characterize the uniqueness of least  $\ell_1$ -norm nonnegative solutions to a system of linear equations, and thus the question (a) in Sect. 1 has been fully addressed. Note that  $A_{J_+} \in R^{m \times |J_+|}$ , so when it has full column rank, we must have  $\text{rank}(A_{J_+}) = |J_+| \leq m$ . Thus Theorem 2.7 shows that *if the  $\ell_1$ -problem has a unique optimal solution  $x$ , then  $x$  must be  $m$ -sparse*. We can use the results established in this section to address many other questions associated with  $\ell_0$ - and  $\ell_1$ -problems. This will be discussed in later sections of this paper.

We now close this section by giving two examples to show that our necessary and sufficient condition can be easily used to check the uniqueness of least  $\ell_1$ -norm nonnegative solutions of linear systems.

**Example 2.8** Consider the linear system  $Ax = b$  with

$$A = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & -1 & -1 & 6 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix},$$

to which  $x^* = (1/2, 1/2, 0, 0)^T$  is a nonnegative solution. It is easy to see that the submatrix  $A_{J_+}$  associated with this solution has full column rank. Moreover, by taking  $y = (1, -1, 0)^T$ , we have  $\eta = A^T y = (1, 1, 0, -7)^T \in \mathcal{R}(A^T)$ , which clearly satisfies (8). Thus the RSP of  $A^T$  holds at  $x^*$ . Therefore, by Theorem 2.7 (or Lemma 2.5),  $x^*$  is the unique least  $\ell_1$ -norm nonnegative solution to the system  $Ax = b$ .

**Example 2.9** Consider the linear system  $Ax = b$  with

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & -0.1 & 0 & -0.2 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix},$$

to which  $x^* = (1/2, 10/3, 10/3, 10/3)^T$  is a least  $\ell_1$ -norm nonnegative solution. By taking  $y = (11, -10, -12)^T$ , we have  $\eta = A^T y = (1, 1, 1, 1)^T \in \mathcal{R}(A^T)$ . Thus the RSP of  $A^T$  holds at  $x^*$ . However, the matrix

$$A_{J_+} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & -0.1 & 0 & -0.2 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

does not have full column rank. By Theorem 2.7,  $x^*$  is NOT the unique least  $\ell_1$ -norm nonnegative solution to the system  $Ax = b$ . In fact, we have another least  $\ell_1$ -norm nonnegative solution given by  $\tilde{x} = (1/2, 10, 0, 0)^T$  (for which the associated matrix  $A_{J_+}$  has full column rank, but the RSP of  $A^T$  does not hold at  $\tilde{x}$ ).

### 3 RSP-based efficiency analysis for $\ell_1$ -minimization

For linear systems without nonnegativity constraints, some sufficient conditions for the strong equivalence between  $\ell_0$ - and  $\ell_1$ -problems have been developed in the literature. If these sufficient conditions are applied directly to sparsest nonnegative solutions of linear systems, the resulting criteria would be very restrictive. For instance, by applying the mutual coherence condition, we immediately conclude that if a nonnegative solution  $x$  obeys  $\|x\|_0 < (1 + 1/\mu(A))/2$  where  $\mu(A)$  denotes the mutual coherence of  $A$  (i.e.,  $\mu(A) = \max_{i \neq j} a_i^T a_j / (\|a_i\|_2 \|a_j\|_2)$  where  $a_i, 1 \leq i \leq n$ , are the columns of  $A$ ), then  $x$  is the unique sparsest solution and the unique least  $\ell_1$ -norm solution to the linear system  $Ax = b$ . In this case, the unique sparsest nonnegative solution coincides with the unique sparsest solution and the unique least  $\ell_1$ -norm solution of the linear system. Clearly, such a sufficient condition is too restrictive. In fact, a sparsest nonnegative solution is usually not the sparsest one to the linear system, and the sparsest nonnegative ones can be also multiple (as shown by Example 3.4 in this section). Although some conditions have been developed specifically for the sparsest nonnegative solutions (see e.g. [16, 6, 29]), these conditions still imply the strong equivalence between  $\ell_0$ - and  $\ell_1$ -problems. They can only partially explain the efficiency of the  $\ell_1$ -method for solving  $\ell_0$ -problems. In this section, we show that Theorems 2.6 and 2.7 enable us to broadly understand the relationship between  $\ell_0$ - and  $\ell_1$ -problems and to deeply interpret the efficiency of the  $\ell_1$ -method through the RSP of  $A^T$ . First, we have the following property for sparsest nonnegative solutions.

**Lemma 3.1** *If  $x$  is a sparsest nonnegative solution to the linear system  $Ax = b$ , then  $M = \begin{pmatrix} A_{J_+} \\ e_{J_+}^T \end{pmatrix}$  has full column rank, where  $J_+ = \{i : x_i > 0\}$ .*

*Proof.* Let  $x$  be a sparsest nonnegative solution to the linear system and let  $J_+ = \{i : x_i > 0\}$ . Assume by contrary that the columns of the matrix  $M$  are linearly dependent. Then there exists a vector  $v \neq 0$  in  $\mathbb{R}^{|J_+|}$  such that

$$\begin{pmatrix} A_{J_+} \\ e_{J_+}^T \end{pmatrix} v = 0.$$

It follows from  $e_{J_+}^T v = 0$  and  $v \neq 0$  that  $v$  must have at least two nonzero components with different signs, i.e.,  $v_i v_j < 0$  for some  $i \neq j$ . Define the vector  $\tilde{v} \in R^n$  as follows:  $\tilde{v}_{J_+} = v$  and  $\tilde{v}_i = 0$  for all  $i \notin J_+$ . We consider the vector

$$y(\lambda) = x + \lambda \tilde{v}, \quad \lambda \geq 0.$$

Note that  $y(\lambda)_i = 0$  for all  $i \notin J_+$ , and that

$$Ay(\lambda) = Ax + A(\lambda \tilde{v}) = b + \lambda A_{J_+} v = b.$$

Thus  $y(\lambda)$  is also a solution to the linear system  $Ax = b$ . By the definition of  $\tilde{v}$ ,  $\tilde{v}$  has at least one negative component. Thus let

$$\lambda^* = \frac{x_{i_0}}{-\tilde{v}_{i_0}} = \min \left\{ \frac{x_i}{-\tilde{v}_i} : \tilde{v}_i < 0 \right\},$$

where  $\lambda^*$  must be a positive number and  $i_0 \in J_+$ . By such a choice of  $\lambda^*$  and the definition of  $y(\lambda^*)$ , we conclude that  $y(\lambda^*) \geq 0$ ,  $y(\lambda^*)_i = 0$  for  $i \notin J_+$ , and  $y(\lambda^*)_{i_0} = 0$  with  $i_0 \in J_+$ . Thus  $y(\lambda^*)$  is a nonnegative solution to the linear system  $Ax = b$ , which is sparser than  $x$ . This is a contradiction. Therefore,  $M$  must have full column rank.  $\square$

By Theorem 2.6 and Lemma 3.1, we immediately have the following result.

**Theorem 3.2**  *$\ell_0$ - and  $\ell_1$ -problems are equivalent if and only if the RSP (8) holds at an optimal solution of the  $\ell_0$ -problem. (In other words, a sparsest nonnegative solution  $x$  to the system  $Ax = b$  is the unique least  $\ell_1$ -norm nonnegative solution to the system if and only if the RSP (8) holds at  $x$ .)*

*Proof.* Assume that problems (1) and (2) are equivalent. So the  $\ell_0$ -problem has an optimal solution  $x$  that is the unique least  $\ell_1$ -norm nonnegative solution to the system  $Ax = b$ . Thus, by Theorem 2.6 (or Lemma 2.3), the RSP (8) must hold at  $x$ . Conversely, assume that the RSP (8) holds at an optimal solution  $x$  to the  $\ell_0$ -problem. Since  $x$  is a sparsest nonnegative solution to the system  $Ax = b$ , by Lemma 3.1, the matrix  $\begin{pmatrix} A_{J_+} \\ e_{J_+}^T \end{pmatrix}$  has full column rank. Thus by Lemma 2.5 (or Theorem 2.6) again,  $x$  must be the unique least  $\ell_1$ -norm nonnegative solution to the system  $Ax = b$ . Hence  $\ell_0$ - and  $\ell_1$ -problems are equivalent.  $\square$

The above result indicates that the RSP (8) at an optimal solution of  $\ell_0$ -problem is a necessary and sufficient condition for the equivalence between  $\ell_0$ - and  $\ell_1$ -problems. Thus all existing sufficient conditions for strong equivalence or equivalence between these two problems must imply the RSP (8), but the converse is clearly not true in general, as shown by the following example.

**Example 3.3** (When existing criteria fail, the RSP may still succeed).

$$A = \begin{pmatrix} 0 & -1 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{3}} & -1 & 0 & 0 \\ -1 & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

For this example, the system  $Ax = b$  does not have a solution  $x$  with  $\|x\|_0 = 1$ . So  $x^* = (1, 0, \sqrt{3}, 0, 0, 0)^T$  is a sparsest nonnegative solution to this linear system. Note that the mutual

coherence  $\mu(A) = \max_{i \neq j} a_i^T a_j / \|a_i\|_2 \|a_j\|_2 = \sqrt{2}/\sqrt{3}$ . Thus the mutual coherence condition  $\|x\|_0 < \frac{1}{2}(1 + 1/\mu(A)) = (\sqrt{2} + \sqrt{3})/(2\sqrt{2}) \approx 1.077$  fails for this example. The RIP [8] fails since the last two columns of  $A$  are linearly dependent. This example also fails to comply with the definition of the NSP. Let us now check the RSP of  $A^T$  at  $x^*$ . By taking  $y = (\frac{1}{2} + \sqrt{3}, \frac{1}{2}, -1)^T$ , we have

$$\eta = A^T y = \left( 1, -\left(\frac{1}{2} + \sqrt{3}\right), 1, -\frac{1}{2}, \frac{2\sqrt{3}-1}{2\sqrt{2}}, -\frac{2\sqrt{3}-1}{2\sqrt{2}} \right)^T \in \mathcal{R}(A^T),$$

where the first and third components of  $\eta$  are equal to 1 (corresponding to  $J_+ = \{1, 3\}$  determined by  $x^*$ ) and all other components of  $\eta$  are less than 1. Thus the RSP (8) holds at  $x^*$ . By Theorem 3.2,  $\ell_1$ -minimization guarantees to locate this solution.

This example indicates that even if the existing sufficient conditions fail, the RSP can still interpret the efficiency of the  $\ell_1$ -method for solving  $\ell_0$ -problems. To further understand the efficiency of the  $\ell_1$ -method, let us decompose the class of linear systems with nonnegative solutions, denoted by  $\mathcal{G}$ , into three subclasses. That is,  $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$  where  $\mathcal{G}_i$ 's are defined as follows:

$\mathcal{G}_1$  : The system  $Ax = b$  has a *unique* least  $\ell_1$ -norm nonnegative solution and a *unique* sparsest nonnegative solution.

$\mathcal{G}_2$  : The system  $Ax = b$  has a *unique* least  $\ell_1$ -norm nonnegative solution and *multiple* sparsest nonnegative solutions.

$\mathcal{G}_3$  : The system  $Ax = b$  has *multiple* least  $\ell_1$ -norm nonnegative solutions.

Clearly, every linear system with a nonnegative solution falls into one of these categories. Since many existing sufficient conditions (such as the mutual coherence, RIP and NSP) imply the strong equivalence between  $\ell_0$ - and  $\ell_1$ -problems, these conditions can apply only to (and explain the efficiency of the  $\ell_1$ -method only for) a subclass of linear systems in  $\mathcal{G}_1$ . However, the RSP (8) defined in this paper goes far beyond this scope of linear systems. An important feature of the RSP (8) is that it does not require a linear system to have a unique sparsest nonnegative solution in order to achieve the equivalence between  $\ell_0$ - and  $\ell_1$ -problems, as shown by the next example.

**Example 3.4** (The  $\ell_1$ -method may guarantee to solve an  $\ell_0$ -problem with multiple optimal solutions.) Consider the system  $Ax = b$  with

$$A = \begin{pmatrix} 0.2 & 0 & -0.3 & -0.1 & 0.5 & -0.25 \\ 0 & 0.2 & 0.5 & 0.2 & -0.9 & 0.05 \\ 0.2 & 0 & -0.3 & -0.1 & 0.5 & -0.25 \end{pmatrix}, \quad b = \begin{pmatrix} 0.1 \\ -0.1 \\ 0.1 \end{pmatrix}.$$

For this example, it is easy to verify that  $Ax = b$  has multiple sparsest nonnegative solutions:

$$x^{(1)} = (0, \frac{2}{5}, 0, 0, \frac{1}{5}, 0)^T, \quad x^{(2)} = (0, 0, 0, 4, 1, 0)^T, \quad x^{(3)} = (\frac{2}{9}, 0, 0, 0, \frac{1}{9}, 0)^T.$$

Since  $\|x^{(1)}\|_1 > \|x^{(3)}\|_1$  and  $\|x^{(2)}\|_1 > \|x^{(3)}\|_1$ , by Theorem 3.2, the RSP of  $A^T$  is impossible to hold at  $x^{(1)}$  and  $x^{(2)}$ . So we only need to check the RSP at  $x^{(3)}$ . Taking  $y = (5, 5/3, 0)^T$  yields  $\eta = A^T y = (1, 1/3, -2/3, -1/6, 1, -7/6)^T \in \mathcal{R}(A^T)$  where the first and fifth components are 1,

and all others are strictly less than 1. Thus the RSP (8) holds at  $x^{(3)}$ , which (by Theorem 3.2) is the unique least  $\ell_1$ -norm nonnegative solution to the linear system. So the  $\ell_1$ -method solves the  $\ell_0$ -problem, although this  $\ell_0$ -problem has multiple optimal solutions.

The following corollary is an immediate consequence of Theorem 3.2, which claims that when an  $\ell_0$ -problem has multiple sparsest nonnegative optimal solutions, only one of them can satisfy the RSP of  $A^T$ .

**Corollary 3.5** *For any underdetermined system of linear equations, there exists at most one sparsest nonnegative solution satisfying the RSP (8).*

Theorem 3.2, together with Example 3.4, shows that  $\ell_0$ - and  $\ell_1$ -problems can be equivalent provided that the RSP (8) is satisfied at an optimal solution to the  $\ell_0$ -problem, irrespective of the multiplicity of optimal solutions to the  $\ell_0$ -problem. The RSP-based analysis has shown that the success of the  $\ell_1$ -method can be guaranteed not only for a subclass of linear systems in  $\mathcal{G}_1$ , but also for a wide range of linear systems in  $\mathcal{G}_2$ . Note that for a linear system in  $\mathcal{G}_3$ , there is no guarantee for the success of the  $\ell_1$ -method when solving an  $\ell_0$ -problem, due to the multiplicity of  $\ell_1$ -minimizers in this case. As a result, the RSP-based analysis has actually identified the broadest class of  $\ell_0$ -problems (in  $\mathcal{G}_1 + \mathcal{G}_2$ ) that can be guaranteed to be solved by using the  $\ell_1$ -method. This analysis not only indicates the guaranteed efficiency of the  $\ell_1$ -method, but also sheds light on the restriction of this method for solving  $\ell_0$ -problems. So the question (b) in Sect. 1 has been addressed to a large extent by this analysis.

Since many existing conditions imply the strong equivalence between  $\ell_0$ - and  $\ell_1$ -problems, they can only explain the success of  $\ell_1$ -methods for solving some  $\ell_0$ -problems in  $\mathcal{G}_1$ . These strong-equivalence-based conditions cannot apply to any  $\ell_0$ -problem in  $\mathcal{G}_2$  which has multiple sparsest optimal solutions, and hence they cannot interpret the numerical efficiency of the  $\ell_1$ -method in these situations. Different from these existing methods, the RSP-based analysis has shown that the guaranteed success of the  $\ell_1$ -method not only takes place for problems in  $\mathcal{G}_1$ , but for a wide range of linear systems in  $\mathcal{G}_2$  as well. This does show that the actual success rate of the  $\ell_1$ -method for solving  $\ell_0$ -problems is remarkably higher than what the strong-equivalence-based theory can predict. So the RSP-based theory can efficiently interpret the actual performance of the  $\ell_1$ -method.

**Remark 3.6** We have seen from the above discussions that Theorem 3.2 is more powerful than the existing theory to interpret the guaranteed success of the  $\ell_1$ -method for solving  $\ell_0$ -problems, and it enables us to broadly understand the relationship between these two problems. However, Theorem 3.2 does not actually provide an explicit criterion for checking the tractability of  $\ell_0$ -problems, since the prior knowledge of the optimal solution to  $\ell_0$ -problems may not be available. Several tractability conditions for  $\ell_0$ -problems have been developed in the literature, such as the RIP of order  $2K$  and NSP of order  $2K$  (under which  $\ell_0$ - and  $\ell_1$ -problems are strongly equivalent). Thus one may ask whether *there is any possibility to derive certain equivalent conditions for  $\ell_0$ - and  $\ell_1$ -problems by using RSP type property without prior knowledge of the optimal solution to  $\ell_0$ -problems.* Our analysis in Sect. 4 will show that the RSP of  $A^T$  at individual points can be strengthened to guarantee the strong equivalence between  $\ell_0$ - and  $\ell_1$ -problems without prior knowledge of the optimal solution to  $\ell_0$ -problems, leading to a RSP

type tractability condition for  $\ell_0$ -problems. (See Theorem 4.3 and Corollary 4.4 in Sect. 4 for details.) However, a common feature of tractability conditions developed for  $\ell_0$ -problems so far is that all these conditions are very restrictive. Whether there exists a less restrictive tractability condition, which does not rely on any prior knowledge of optimal solutions to  $\ell_0$ -problems and can guarantee equivalence (instead of only strong equivalence) between  $\ell_0$ - and  $\ell_1$ -problems, remains a worthwhile research topic in this field.

**Remark 3.7** While we focus on the relationship between  $\ell_0$ - and  $\ell_1$ -problems in this paper, it is worth noting that our results can be easily generalized to interpret the relationship between the  $\ell_0$ - and weighted  $\ell_1$ -problems as well. More specifically, let us consider the weighted  $\ell_1$ -problem

$$\min\{\|Wx\|_1 : Ax = b, x \geq 0\}, \quad (17)$$

where  $W = \text{diag}(w)$  and  $w > 0$ . By the nonsingular linear transformation,  $u = Wx$ , the above weighted  $\ell_1$ -problem is equivalent to

$$\min\{\|u\|_1 : (AW^{-1})u = b, u \geq 0\}. \quad (18)$$

Clearly,  $x$  is the unique optimal solution to the weighted problem (17) if and only if  $u = Wx$  is the unique optimal solution to the  $\ell_1$ -problem (18), and  $u$  and  $x$  have the same supports. Thus any weighted  $\ell_1$ -problem with weight  $W = \text{diag}(w)$ , where  $w$  is a positive vector in  $R^n$ , is nothing but a normal  $\ell_1$ -problem with a scaled matrix  $AW^{-1}$ . As a result, applying Theorems 2.7 to the  $\ell_1$ -problem (18), we conclude that  $u$  is the unique optimal solution to (18) if and only if  $(AW^{-1})_{J_+(u)}$  has full column rank, and there exists a vector  $\zeta \in \mathcal{R}((AW^{-1})^T)$  such that  $\zeta_i = 1$  for  $u_i > 0$  and  $\zeta_i < 1$  for  $u_i = 0$ . By the one-to-one correspondence between solutions of (17) and (18), and by transforming back to the weighted  $\ell_1$ -problem using  $u = Wx$  and  $\eta = W\zeta$ , we immediately conclude that  $x$  is the unique optimal solution to the weighted  $\ell_1$ -problem (17) if and only if (i)  $A_{J_+}$  has full column rank where  $J_+ = \{i : x_i > 0\}$ , and (ii) there exists an  $\eta \in \mathcal{R}(A^T)$  such that  $\eta_i = w_i$  for  $x_i > 0$ , and  $\eta_i < w_i$  for  $x_i = 0$ . We may call the property (ii) above as the *weighted RSP of  $A^T$  at  $x$* . Thus the results in this paper can be easily generalized to weighted  $\ell_1$ -methods. For instance, the counterpart of Theorem 3.2 for the equivalence between  $\ell_0$ - and weighted  $\ell_1$ -problems can be also stated by using the above-mentioned weighted RSP property, and the efficiency of weighted  $\ell_1$ -methods for solving  $\ell_0$ -problems can be adequately understood from this new angle.

**Remark 3.8** The RSP-based analysis and results developed in this section can be also applied to the sparsest optimal solution to the linear program (LP)

$$\min\{c^T x : Ax = b, x \geq 0\}. \quad (19)$$

The sparsest optimal solution of (19) is meaningful. For instance, in production planning scenarios, the decision variables  $x_i \geq 0$ ,  $i = 1, \dots, n$ , represent what production activities/events that should take place and how much resources should be allocated to them in order to achieve an optimal objective value. The sparsest optimal solution of a linear program provides the smallest number of activities to achieve the optimal objective value. In many situations, reducing the number of activities is vital for efficient planning, management and resource allocations. We

denote by  $d^*$  the optimal value of (19), which can be obtained by solving the LP by simplex methods, or interior point methods. We assume that (19) is feasible and has a finite optimal value  $d^*$ . Thus the optimal solution set of the LP is given by  $\{x : Ax = b, x \geq 0, c^T x = d^*\}$ . So a sparsest optimal solution to the LP is an optimal solution to the  $\ell_0$ -problem

$$\min \left\{ \|x\|_0 : \begin{pmatrix} A \\ c^T \end{pmatrix} x = \begin{pmatrix} b \\ d^* \end{pmatrix}, x \geq 0 \right\}, \quad (20)$$

associated with which is the  $\ell_1$ -problem

$$\min \left\{ \|x\|_1 : \begin{pmatrix} A \\ c^T \end{pmatrix} x = \begin{pmatrix} b \\ d^* \end{pmatrix}, x \geq 0 \right\}. \quad (21)$$

Therefore all developed results for sparsest nonnegative solutions of linear systems in this paper can be directly applied to (20) and (21). For instance, from Theorems 2.7 and 3.2, we immediately have the following statements:  *$x$  is the unique least  $\ell_1$ -norm optimal solution to LP (19) if and only if the matrix  $H = \begin{pmatrix} A_{J_+} \\ c_{J_+}^T \end{pmatrix}$  has full column rank, and there exists a vector  $\eta \in R^n$  obeying*

$$\eta \in \mathcal{R}([A^T, c]), \eta_i = 1 \text{ for all } i \in J_+, \text{ and } \eta_i < 1 \text{ for all } i \notin J_+ \quad (22)$$

where  $J_+ = \{i : x_i > 0\}$ . Moreover, a sparsest optimal solution to LP (19) is the unique least  $\ell_1$ -norm optimal solution to the LP if and only if the range space property (22) holds at this optimal solution. Note that a degenerated optimal solution has been long studied since 1950s (see [22, 23] and the references therein). It is well known that finding a degenerated optimal solution requires extra effort than nondegenerated ones. Finding the most degenerated optimal solution or the sparsest optimal solution becomes even harder. By applying the RSP theory to (20) and (21), we may obtain a new understanding for the most degenerated or the sparsest optimal solutions of LPs.

## 4 Application to compressed sensing

One of the tasks in compressed sensing is to exactly recover a sparse vector (representing a signal or an image) via an underdetermined system of linear equations [7, 14, 8, 18]. In this section, we consider the exact recovery of an unknown sparse nonnegative vector  $x^*$  by  $\ell_1$ -minimization. For this purpose, we assume that an  $m \times n$  ( $m < n$ ) sensing matrix  $A$  and the measurements  $y = Ax^*$  are available. A nonnegative solution  $x$  to the system  $Ax = b$  is said to have a guaranteed recovery (or to be exactly recovered) by  $\ell_1$ -minimization if  $x$  is the unique least  $\ell_1$ -norm nonnegative solution to the linear system. To guarantee the success of recovery, the current compressed sensing theory assumes that the matrix  $A \in R^{m \times n}$  ( $m < n$ ) satisfies some conditions (e.g., RIP or NSP of order  $2K$ ) which imply the following properties: (i)  $x^*$  is the unique least  $\ell_1$ -norm nonnegative solution to the system  $Ax = y = Ax^*$  (where the components of  $y$  are measurements); (ii)  $x^*$  is the unique sparsest nonnegative solution to the system  $Ax = y$ . So the  $\ell_0$ - and  $\ell_1$ -problems involved must be strongly equivalent. Most of the recovering conditions developed so far are for the so-called uniform recovery.

## 4.1 Uniform recovery of sparse nonnegative vectors

The exact recovery of all  $K$ -sparse nonnegative vectors (i.e.,  $\{x : x \geq 0, \|x\|_0 \leq K\}$ ) by a single sensing matrix  $A$  is called the *uniform recovery* of  $K$ -sparse nonnegative vectors. To develop a RSP-based recovery theory, let us first introduce the following concept.

**Definition 4.1** (RSP of order  $K$ ). *Let  $A$  be an  $m \times n$  matrix with  $m < n$ .  $A^T$  is said to satisfy the range space property of order  $K$  if for any subset  $S \subseteq \{1, \dots, n\}$  with  $|S| \leq K$ ,  $\mathcal{R}(A^T)$  contains a vector  $\eta$  such that  $\eta_i = 1$  for all  $i \in S$ , and  $\eta_i < 1$  for all  $i \in S_c = \{1, 2, \dots, n\} \setminus S$ .*

We first show that if  $A^T$  has the RSP of order  $K$ , then  $K$  must be bounded by the spark of  $A$ , denoted by  $\text{Spark}(A)$ , which is the smallest number of columns of  $A$  that are linearly dependent (see e.g. [15, 5]).

**Lemma 4.2** *If  $A^T$  has the RSP of order  $K$ , then any  $K$  columns of  $A$  are linearly independent, so  $K < \text{Spark}(A)$ .*

*Proof.* Let  $S = \{s_1, \dots, s_K\}$ , with  $|S| = K$ , be an arbitrary subset of  $\{1, \dots, n\}$ . Suppose that  $A^T$  has the RSP of order  $K$ . We now prove that  $A_S$  has full column rank. It is sufficient to show that  $z_S = 0$  is the only solution to  $A_S z_S = 0$ . Indeed, let  $A_S z_S = 0$ . Then  $z = (z_S, z_{S_c} = 0) \in R^n$  is in the null space of  $A$ . By the RSP of order  $K$ , there exists a vector  $\eta \in \mathcal{R}(A^T)$  such that every component of  $\eta_S$  is 1, i.e.,  $\eta_{s_i} = 1$  for  $i = 1, \dots, K$ . By the orthogonality of the null and range spaces, we have

$$z_{s_1} + z_{s_2} + \dots + z_{s_K} = z_S^T \eta_S = z^T \eta = 0. \quad (23)$$

Now let  $k$  be an arbitrary number with  $1 \leq k \leq K$ , and  $S_k = \{s_1, s_2, \dots, s_k\} \subseteq S$ . Since  $|S_k| \leq |S| = K$ , it follows from the definition of RSP of order  $K$ , there exists a vector  $\tilde{\eta} \in \mathcal{R}(A^T)$  with  $\tilde{\eta}_{s_i} = 1$  for every  $i = 1, \dots, k$  and  $\tilde{\eta}_j < 1$  for every  $j \notin S_k$ . By the orthogonality again, it follows from  $z^T \tilde{\eta} = 0$  that

$$(z_{s_1} + \dots + z_{s_k}) + (\tilde{\eta}_{s_{k+1}} z_{s_{k+1}} + \dots + \tilde{\eta}_{s_K} z_{s_K}) = 0.$$

This is equivalent to

$$(z_{s_1} + \dots + z_{s_k}) + (z_{s_{k+1}} + \dots + z_{s_K}) + [z_{s_{k+1}}(\tilde{\eta}_{s_{k+1}} - 1) + \dots + z_{s_K}(\tilde{\eta}_{s_K} - 1)] = 0$$

which, together with (23), implies that

$$(\tilde{\eta}_{s_{k+1}} - 1)z_{s_{k+1}} + \dots + (\tilde{\eta}_{s_K} - 1)z_{s_K} = 0$$

where  $\tilde{\eta}_{s_i} < 1$  for  $i = k+1, \dots, K$ . Since such relations hold for every specified  $k$  with  $1 \leq k \leq K$ . In particular, for  $k = K-1$ , the relation above is reduced to  $(\tilde{\eta}_{s_K} - 1)z_{s_K} = 0$  which implies that  $z_{s_K} = 0$  since  $\tilde{\eta}_{s_K} < 1$ . For  $k = K-2$ , the relation above is of the form

$$(\tilde{\eta}_{s_{K-1}} - 1)z_{s_{K-1}} + (\tilde{\eta}_{s_K} - 1)z_{s_K} = 0$$

which, together with  $z_{s_K} = 0$  and  $\tilde{\eta}_{s_{K-1}} < 1$ , implies that  $z_{s_{K-1}} = 0$ . Continuing this process by considering  $k = K-3, \dots, 1$ , we deduce that all components of  $z_S$  are zero. Thus  $A_S$  has full column rank. By the definition of  $\text{Spark}(A)$ , we must have  $K < \text{Spark}(A)$ .  $\square$



The RSP of order  $K$  can completely characterize the uniform recovery of all  $K$ -sparse non-negative vectors by  $\ell_1$ -minimization, as shown by the next result.

**Theorem 4.3** *Let the measurements of the form  $y = Ax$  be taken. Then any  $x \geq 0$  with  $\|x\|_0 \leq K$  can be exactly recovered by the  $\ell_1$ -method (i.e.,  $\min\{\|z\|_1 : Az = y, z \geq 0\}$ ) if and only if  $A^T$  has the RSP of order  $K$ .*

*Proof.* Assume that the RSP of order  $K$  is satisfied. Let  $x^* \geq 0$  be an arbitrary vector with  $\|x^*\|_0 \leq K$ . Let  $S = J_+ = \{i : x_i^* > 0\}$ . Since  $|S| = \|x^*\|_0 \leq K$ , by the RSP of order  $K$ , there exists a vector  $\eta \in \mathcal{R}(A^T)$  such that  $\eta_i = 1$  for all  $i \in S$ , and  $\eta_i < 1$  for all  $i \in S_c$ . This implies that the RSP (8) holds at  $x^* \geq 0$ . Moreover, it follows from Lemma 4.2 that  $A_S$  has full column rank. Hence, by Theorem 2.7,  $x^*$  is the unique least  $\ell_1$ -norm nonnegative solution to the system  $Ax = y = Ax^*$ . So  $x^*$  can be exactly recovered by the  $\ell_1$ -method.

Conversely, assume that any  $x \geq 0$  with  $\|x\|_0 \leq K$  can be exactly recovered by the  $\ell_1$ -method. We now prove that the RSP of order  $K$  must be satisfied. Let  $S = J_+ = \{i : x_i > 0\}$ . Under the assumption,  $x$  is the unique optimal solution to the  $\ell_1$ -problem

$$\min\{\|z\|_1 : Az = y = Ax, z \geq 0\}.$$

By Theorem 2.7, the RSP (8) holds at  $x$ , i.e., there exists a vector  $\eta \in \mathcal{R}(A^T)$  such that  $\eta_i = 1$  for all  $i \in S = J_+$ , and  $\eta_i < 1$  otherwise. Since  $x$  can be any  $K$ -sparse nonnegative vectors, this implies that  $S = J_+$  can be any subset of  $\{1, \dots, n\}$  with  $|S| \leq K$ , and for every such a subset there exists accordingly a vector  $\eta$  satisfying the above property. By Definition 4.1,  $A^T$  has the RSP of order  $K$ .  $\square$

Let  $a_j, 1 \leq j \leq n$ , be the columns of  $A$  and let  $a_0 = 0$ . Let  $P$  denote the convex hull of  $a_j, 0 \leq j \leq n$ . Donoho and Tanner [16] introduced the following concept: The polytope  $P$  is outwardly  $K$ -neighborly if every subset of  $K$  vertices not including  $a_0 = 0$  spans a face of this polytope. They have shown that the polytope  $P$  is *outwardly  $K$ -neighborly* if and only if any nonnegative solution  $x$  to the system  $Ax = b$  with  $\|x\|_0 \leq K$  is the unique optimal solution to the  $\ell_1$ -problem. In other words, the outwardly  $K$ -neighborly property is a full geometric characterization of the uniform recovery of  $K$ -sparse nonnegative vectors. Some equivalent properties, such as the strictly half  $k$ -balanced and the strictly half  $k$ -thick, were also introduced by Zhang [46]. These are certain properties imposed on the range space of a matrix, and they are largely defined from a geometric point of view. Clearly, these properties are different from the RSP of order  $K$  which is derived from the LP strict complementarity theory. Moreover, Khajehnejad et al [29] characterized the uniform recovery by using the property of  $\mathcal{N}(A)$ , the null space  $A$ . They have showed that all nonnegative  $K$ -sparse vector can be exactly recovered if and only if for every vector  $w \neq 0$  in  $\mathcal{N}(A)$ , and every index set  $S \subseteq \{1, \dots, n\}$  with  $|S| = K$  such that  $w_{S_c} \geq 0$ , it holds that  $e^T w > 0$ . Different from the geometric description by Donoho and Tanner [16] and the null-space-based analysis by Khajehnejad et al [29], the RSP of order  $K$  introduced in this section provides an alternative full characterization of the uniform recovery from the perspective of the range space of  $A^T$ . Clearly, while from different perspectives, all the above-mentioned properties (outwardly  $K$ -neighborly, strictly half  $k$ -balanced, null space, and range space) are equivalent since all these properties are necessary and sufficient conditions for the uniform recovery of all  $K$ -sparse vectors.

As a result, all these properties imply the strong equivalence between  $\ell_0$ - and  $\ell_1$ -problems, so the RSP of order  $K$  is also a sufficient condition for the tractability of  $\ell_0$ -problems. It is easy to verify that if the matrix  $A$  has the RIP of order  $2K$ , or the NSP of order  $2K$ , then its transpose  $A^T$  must have the RSP of order  $K$ .

We now close this section by stressing the difference between the RSP of order  $K$  and the RSP (8). Such a difference can be easily seen from the following result.

**Corollary 4.4** *If  $A^T$  has the RSP of order  $K$ , then any  $\hat{x} \geq 0$  with  $\|\hat{x}\|_0 \leq K$  is both the unique least  $\ell_1$ -norm nonnegative solution and the unique sparsest nonnegative solution to the linear system  $Ax = y = A\hat{x}$ .*

*Proof.* By Theorem 4.3, under the RSP of order  $K$ , any  $\hat{x} \geq 0$  with  $\|\hat{x}\|_0 \leq K$  can be exactly recovered by  $\ell_1$ -minimization, i.e.,  $\hat{x}$  is the unique least  $\ell_1$ -norm nonnegative solution to the system  $Ax = y = A\hat{x}$ . We now prove that  $\hat{x}$  is also the sparsest nonnegative solution to this system. Assume that there exists another solution  $z \geq 0$  such that  $\|z\|_0 \leq \|\hat{x}\|_0$ . Let  $S = \{i : z_i > 0\}$ . Since  $|S| = \|z\|_0 \leq \|\hat{x}\|_0 \leq K$ , by the RSP of order  $K$ , there exists an  $\eta \in \mathcal{R}(A^T)$  such that  $\eta_i = 1$  for all  $i \in S$ , and  $\eta_i < 1$  for all  $i \in S_c$ . Thus the individual RSP (8) holds at  $z$ . By Lemma 4.2, any  $K$  columns of  $A$  are linearly independent. Since the number of the columns of  $A_S$ , where  $S = \{i : z_i > 0\}$ , is less than  $K$ , this implies that  $A_S$  has full column rank. By Theorem 2.7,  $z$  is also the unique least  $\ell_1$ -norm nonnegative solution to the system  $Ax = y = A\hat{x}$ . Thus  $z = \hat{x}$ , which implies that  $\hat{x}$  is the unique sparsest nonnegative solution to this system.  $\square$

This result shows that the RSP of order  $K$  is much more restrictive than the individual RSP (8) which is defined at a single point. The former requires that the RSP (8) hold at every  $K$ -sparse nonnegative solution. By contrast, the individual RSP (8) is only a local property, and it does not imply that the underlying linear system has a unique sparsest nonnegative solution, as we have shown in Sect. 3.

## 4.2 Non-uniform recovery of sparse nonnegative vectors

The purpose of uniform recovery is to exactly recover all  $k$ -sparse vectors. So some strong assumptions (such as the RIP, NSP and the RSP of certain orders) must be imposed on the matrix. These strong assumptions for achieving uniform recovery imply that the unknown sparse vector  $x$  must be the unique optimal solution to both  $\ell_0$ - and  $\ell_1$ -problems (hence, the strong equivalence between these two problems are actually required by the uniform recovery). In this subsection, we extend the uniform-recovery theory to the nonuniform one by using the RSP-based theory. So far, there exists some limited literature handling the non-uniform recovery of sparse signals. From a geometric perspective, Donoho and Tanner [16] introduced the so-called weak neighborliness conditions for nonuniform recovery by  $\ell_1$ -minimization, and they have shown under such a condition that most nonnegative  $K$ -sparse vectors can be exactly recovered by the  $\ell_1$ -method. Ayaz and Rauhut [1] focused on the non-uniform recovery of signals with given sparsity and given signal length by  $\ell_1$ -minimization. Different from their methods, we introduce the so-called Weak RSP of order  $K$  in this subsection, which is a range space property of  $A^T$  that can guarantee the exact recovery of some vectors which may have high sparsity level, going

beyond the scope of normal uniform recoveries.

Given a sensing matrix  $A$ , Theorem 2.7 claims that a vector  $x^*$  can be exactly recovered by  $\ell_1$ -minimization provided that the RSP(8) hold at  $x^*$  and that the matrix  $A_{J_+}$ , where  $J_+ = \{i : x_i^* > 0\}$ , has full-column rank. Such an  $x^*$  is not necessarily the unique sparsest nonnegative solution to the linear system as shown by Example 3.4, and it may not even be a sparsest nonnegative solution as well. For instance, let

$$A = \begin{pmatrix} 6 & 4 & 1.5 & 4 & -1 \\ 6 & 4 & -0.5 & 4 & 0 \\ 0 & -2 & 31.5 & -1 & -1.5 \end{pmatrix}, y = \begin{pmatrix} 4 \\ 4 \\ -1 \end{pmatrix} = Ax^*$$

where  $x^* = (1/3, 1/2, 0, 0, 0)^T$ . It is easy to see that  $\tilde{x} = (0, 0, 0, 1, 0)^T$  is the unique sparsest nonnegative solution to the system  $Ax = y$ , while  $x^*$  is the unique least  $\ell_1$ -norm nonnegative solution to the system  $Ax = y$ . Although  $x^*$  is not the sparsest nonnegative solution, it can be exactly recovered by the  $\ell_1$ -method. Because of this, it is interesting to develop a recovery theory without requiring that the targeted unknown sparse vector be a sparsest or be the unique sparsest solution to a linear system. This is also motivated by some practical applications. In fact, a real sparse signal or image may not be sparse enough to be recovered by the uniform recovery, and partial information for the unknown sparse vector may be available in some situations, for example, the support of an unknown vector may be known. The concept of RSP of order  $K$  can be easily adapted to handle these cases. So we introduce the following concept.

**Definition 4.5**(WRSP of order  $K$ ) *Let  $A$  be an  $m \times n$  matrix with  $m < n$ .  $A^T$  is said to satisfy the weak range space property (WRSP) of order  $K$  if the following two properties are satisfied:*

- (i) *There exists a subset  $S \subseteq \{1, \dots, n\}$  such that  $|S| = K$  and  $A_S$  has full column rank;*
- (ii) *For any subset  $S \subseteq \{1, \dots, n\}$  such that  $|S| \leq K$  and  $A_S$  has full column rank, the space  $\mathcal{R}(A^T)$  contains a vector  $\eta$  such that  $\eta_i = 1$  for  $i \in S$ , and  $\eta_i < 1$  otherwise.*

The WRSP of order  $K$  only requires that the individual RSP hold for those subsets  $S \subseteq \{1, \dots, n\}$  with  $|S| \leq K$  and  $A_S$  being full-column-rank, while the RSP of order  $K$  requires that the individual RSP hold for any subset  $S \subseteq \{1, \dots, n\}$  with  $|S| \leq K$ . So the WRSP of order  $K$  is less restrictive than the RSP of order  $K$ . By Theorem 2.6, we have the following result.

**Theorem 4.6** *Let the measurements of the form  $y = Ax$  be taken. Suppose that there exists a subset  $S \subseteq \{1, \dots, n\}$  such that  $|S| = K$  and  $A_S$  has full column rank. Then  $A^T$  has the WRSP of order  $K$  if and only if any  $x \geq 0$ , satisfying that  $\|x\|_0 \leq K$  and  $A_{J_+}$  has full-column-rank where  $J_+ = \{i : x_i > 0\}$ , can be exactly recovered by the  $\ell_1$ -minimization  $\min\{\|z\|_1 : Az = y = Ax, z \geq 0\}$ .*

*Proof.* Assume that  $A^T$  has the WRSP of order  $K$ . Let  $x$  be an arbitrary nonnegative vector such that  $\|x\|_0 \leq K$  and  $A_{J_+}$  has full-column-rank, and let  $S = J_+ = \{i : x_i > 0\}$ . Since  $A^T$  has the WRSP of order  $K$ , there exists an  $\eta \in \mathcal{R}(A^T)$  such that  $\eta_i = 1$  for  $i \in S = J_+$ , and  $\eta_i < 1$  otherwise. This implies that the RSP(8) holds at  $x$ . Since  $A_{J_+}$  has full column rank, by Theorem 2.7,  $x$  must be the unique least  $\ell_1$ -norm nonnegative solution to the linear system  $Az = y (= Ax)$ . In other words,  $x$  can be exactly recovered by  $\ell_1$ -minimization. Conversely, we assume that any

$x \geq 0$ , satisfying that  $\|x\|_0 \leq K$  and  $A_{J_+}$  has full-column-rank, can be exactly recovered by  $\ell_1$ -minimization. We now prove that  $A^T$  must have the WRSP of order  $K$ . In fact, let  $x \geq 0$  be a vector such that  $\|x\|_0 \leq K$  and  $A_{J_+}$  has full-column-rank. Denote by  $S = J_+ = \{i : x_i > 0\}$ . Since  $x$  can be exactly recovered by the  $\ell_1$ -method, it is the unique least  $\ell_1$ -norm nonnegative solution to the system  $Az = y = Ax$ . By Theorem 2.7, the RSP (8) holds at  $x$ , i.e., there exists an  $\eta \in \mathcal{R}(A^T)$  such that  $\eta_i = 1$  for  $i \in J_+ = S$ , and  $\eta_i < 1$  otherwise. Since  $x$  can be any vector such that  $\|x\|_0 \leq K$  and  $A_{J_+}$  has full column rank, this implies that the condition (ii) of Definition 4.5 holds, thus  $A^T$  has the WRSP of order  $K$ .  $\square$

We may further relax the concept of RSP and WRSP, especially when partial information available to the unknown vector. For instance, when  $\|x\|_0 = K$  is known, we may introduce the next two concepts.

**Definition 4.7** (PRSP of order  $K$ ). *We say that  $A^T$  has the partial range space property (PRSP) of order  $K$  if for any subset  $S$  of  $\{1, \dots, n\}$  with  $|S| = K$ , the range space  $\mathcal{R}(A^T)$  contains a vector  $\eta$  such that  $\eta_i = 1$  for all  $i \in S$ , and  $\eta_i < 1$  otherwise.*

**Definition 4.8** (PWRSP of order  $K$ ).  *$A^T$  is said to have partial weak range space property (PWRSP) of order  $K$  if for any subset  $S \subseteq \{1, \dots, n\}$  such that  $|S| = K$  and  $A_S$  has full column rank,  $\mathcal{R}(A^T)$  contains a vector  $\eta$  such that  $\eta_i = 1$  for all  $i \in S$ , and  $\eta_i < 1$  otherwise.*

Different from the RSP of order  $K$ , the PRSP of order  $K$  only requires that the individual RSP hold for the subset  $S$  with  $|S| = K$ . Similarly, the PWRSP of order  $K$  is also less restrictive than WRSP. Based on such definitions, we have the next result which follows from Theorem 2.7 straightaway.

**Theorem 4.9** (i) *The matrix  $A^T$  has the partial range space property (PRSP) of order  $K$  if and only if any  $x \geq 0$ , with  $\|x\|_0 = K$ , can be exactly recovered by the  $\ell_1$ -minimization  $\min\{\|z\|_1 : Az = y = Ax, z \geq 0\}$ .*

(ii)  *$A^T$  has the PWRSP of order  $K$  if and only if any  $x \geq 0$ , satisfying that  $\|x\|_0 = K$  and  $A_{J_+}$  has full-column-rank where  $J_+ = \{i : x_i > 0\}$ , can be exactly recovered by the  $\ell_1$ -minimization  $\min\{\|z\|_1 : Az = y = Ax, z \geq 0\}$ .*

When  $A_S$  has full column rank, we have  $|S| \leq m$ . Thus the WRSP and PWRSP of order  $K$  imply that  $K \leq m$ . Moreover, the PRSP of order  $K$  implies that  $K < \text{Spark}(A)$ . In fact, the proof of this fact is identical to that of Lemma 4.1. Theorems 4.6 and 4.9(ii) indicate that a portion of vectors with  $\|x\|_0 \leq m$  can be exactly recovered if a sensing matrix satisfies certain properties milder than the RSP of order  $K$  (and thus milder than RIP and NSP of order  $2K$ ). Since the PRSP, WRSP and PWRSP of order  $K$  do not require that the individual RSP hold for all subsets  $S$  with  $|S| \leq K$ , by Theorem 4.3, these properties are nonuniform-type recovering conditions developed through the range space property of  $A^T$ .

It is worth mentioning that when a priori information, such as the sign restriction, is available, Juditsky, Karzan, and Nemirovski [28] have developed some exact recovery criteria for  $\ell_1$ -minimization based on the so-called  $s$ -semigoodness. Clearly, their concepts and recovery conditions are remarkably different from the ones developed in this section.

## 5 Conclusions

In this paper, we have addressed several questions associated with the  $\ell_0$ - and  $\ell_1$ -problems with nonnegativity constraints. More specifically, through the range space property of  $A^T$ , we have characterized the conditions for the  $\ell_1$ -problem to have a unique optimal solution, for  $\ell_0$ - and  $\ell_1$ -problems to be equivalent, and for sparse vectors to be uniformly and non-uniformly recovered. We have shown the following main results: (i) A vector  $x \geq 0$  is the unique optimal solution to the  $\ell_1$ -problem if and only if the RSP holds at this vector, and the associated submatrix  $A_{J_+}$  has full column rank; (ii)  $\ell_0$ - and  $\ell_1$ -problems are equivalent if and only if the RSP (8) holds at an optimal solution of the  $\ell_0$ -problem; (iii) All  $K$ -sparse vectors can be exactly recovered by a single sensing matrix  $A$  if and only if  $A^T$  has the RSP of order  $K$ . From our analysis, we see that the RSP originates naturally from the strict complementarity property of linear programming problems. Via the RSP-based analysis, the relationship between  $\ell_0$ - and  $\ell_1$ -problems can be broadly understood. This analysis has indicated that the uniqueness of optimal solutions of the  $\ell_0$ -problem is not the reason for the problem to be computationally tractable, and the multiplicity of optimal solutions of the  $\ell_0$ -problem is also not the reason for the problem to be hard. The RSP may hold in both situations.

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## References

- [1] U. Ayaz and H. Rauhut, Nonuniform sparse recovery with subgaussian matrices, Technical Report, University of Bonn, 2011.
- [2] J. Bardsley and J. Nagy, Covariance-preconditioned iterative methods for nonnegativity constrained astronomical imaging, *SIAM J. Matrix Anal. Appl.*, 27 (2006), pp. 1184–1198.
- [3] P.S. Bradley, O.L. Mangasarian and J.B. Rosen, Parsimonious least norm approximation, *Comput. Optim. Appl.*, 11 (1998), pp. 5–21.
- [4] P.S. Bradley, U.M. Fayyad and O.L. Mangasarian, Mathematical programming for data mining: formulations and challenges, *INFORMS J. Computing*, 11 (1999), pp. 217–238.
- [5] A.M. Bruckstein, D.L. Donoho and M. Elad, From sparse solutions of systems of equations to sparse modeling of signals and images, *SIAM Rev.*, 51 (2009), pp. 34–81.
- [6] A.M. Bruckstein, M. Elad and M. Zibulevsky, On the uniqueness of nonnegative sparse solutions to underdetermined systems of equations, *IEEE Trans. Inform. Theory*, 54 (2008), pp. 4813–4820.
- [7] E. Candès, *Compressive sampling*, in Proceedings of the International Congress of Mathematicians, Madrid, Spain, 2006.
- [8] E. Candès, The restricted isometry property and its implications for compressed sensing, *Comptes Rendus de l'Académie des Sciences, Paris, Série I*, 346 (2008), pp. 589–592.

- [9] E. Candès and T. Tao, Decoding by linear programming, *IEEE Trans. Inform. Theory*, 51 (2005), pp. 4203–4215.
- [10] E. Candès, J. Romberg and T. Tao, Robust uncertainty principles: Exact reconstruction from highly incomplete frequency information, *IEEE Trans. Inform. Theory*, 52 (2006), pp. 489–509.
- [11] E.J. Candès, M.B. Wakin and S.P. Boyd, Enhancing sparsity by reweighted  $\ell_1$  minimization, *J. Fourier Anal. Appl.*, 14 (2008), pp. 877–905.
- [12] D. Chen and R. Plemmons, *Nonnegativity constraints in numerical analysis*, in Symposium on the Birth of Numerical Analysis, 2007.
- [13] I. Daubechies, R. DeVore, M. Fornasier and C.S. Güntürk, Iteratively reweighted squares minimization for sparse recovery, *Commun. Pure. Appl. Math.*, 63 (2010), pp. 1–38.
- [14] D. L. Donoho, Compressed sensing, *IEEE Trans. Inform. Theory*, 52 (2006), pp. 1289–1306.
- [15] D. L. Donoho and M. Elad, Optimality sparse representation in general (non-orthogonal) dictionaries via  $\ell_1$  minimization, *Proc. Natl. Acad. Sci.*, 100 (2003), pp. 2197–2202.
- [16] D.L. Donoho and J. Tanner, Sparse nonnegative solutions of underdetermined linear equations by linear programming, *Proc. Natl. Acad. Sci.*, 102 (2005), pp. 9446–9451.
- [17] D.L. Donoho and J. Tanner, Counting the faces of randomly-projected hypercubes and orthants with applications, *Discrete Comput. Geom.*, 43 (2010), pp. 522–541.
- [18] M. Elad, *Sparse and Redundant Representations: From Theory to Applications in Signal and Image Processing*, Springer, New York, 2010.
- [19] M. Elad and A. Bruckstein, A generalized uncertainty principle and sparse representation in pairs of bases, *IEEE Trans. Inform. Theory*, 48 (2002), pp. 2558–2567.
- [20] A. Feuer and A. Nemirovsky, On sparse representation in pairs of bases, *IEEE Trans. Inform. Theory*, 49 (2003), pp. 1579–1581.
- [21] J.J. Fuchs, On sparse representations in arbitrary redundant bases, *IEEE Trans. Inform. Theory*, 50 (2004), pp.1341–1344.
- [22] H.J. Greenberg, An analysis of degeneracy, *Naval Res. Logist. Quart.*, 33 (1986), pp. 635–655.
- [23] H.J. Greenberg, The use of the optimal partition in a linear programming solution for postoptimal analysis, *Oper. Res. Lett.*, 15 (1994), pp. 179–185.
- [24] R. Gribonval and M. Nielsen, Sparse decompositions in unions of bases, *IEEE Trans. Inform. Theory*, 49 (2003), pp. 3320–3325.

- [25] R. He, W. Zheng, B. Hu and X. Kong, *Nonnegative sparse coding for discriminative semi-supervised learning*, in Proceedings of IEEE conference on Computer Vision and Pattern Recognition (CVPR), 2011, pp. 2849–2856.
- [26] P.O. Hoyer, Nonnegative sparse coding, in *Proc. of the 12th IEEE Workshop*, 2002, pp. 557–565.
- [27] A. Juditski and A. Nemirovski, On verifiable sufficient conditions for sparse signal recovery via  $\ell_1$  minimization, *Math. Program., Ser. B*, 127 (2011), pp. 57–88.
- [28] A. Juditsky, F. Kilinc Karzan, and A. Nemirovski, Verifiable conditions of  $\ell_1$ -recovery for sparse signals with sign restrictions, *Math. Program., Ser. B*, 127 (2011), 89–122.
- [29] M.A. Khajehnejad, A.G. Dimakis, W. Xu and B. Hassibi, Sparse recovery of nonnegative signals with minima expansion, *IEEE Trans. Signal Proc.*, 59 (2011), pp. 196–208.
- [30] D. Lee and H. Seung, Learning the parts of objects by nonnegative matrix factorization, *Nature*, 401 (1999), pp. 788–791.
- [31] O.L. Mangasarian, Minimum-support solutions of polyhedral concave programs, *Optimization*, 45 (1999), pp. 149–162.
- [32] O.L. Mangasarian, *Machine learning via polyhedral concave minimization*, in Applied Mathematics and Parallel Computing-Festschrift for Klaus Ritter (H. Fischer, B. Riedmueller and S. Schaeffler eds.), Springer, Heidelberg, 1996, pp. 175–188.
- [33] B.K. Natarajan, Sparse approximate solutions to linear systems, *SIAM J. Comput.*, 24 (1995), pp. 227–234.
- [34] P.D. O’Grady and S.T. Rickard, *Recovery of nonnegative signals from compressively samples observations via nonnegative quadratic programming*, SPASR’09, 6 April, 2009.
- [35] H. Park and H. Kim, Nonnegative matrix factorization based on alternating nonnegativity constrained least squares and the active set method, *SIAM J. Matrix Anal. Appl.*, 30 (2008), pp. 713–730.
- [36] P. Pauca, J. Piper and R. Plemmons, Nonnegative matrix factorization for spectral data analysis, *Linear Algebra Appl.*, 416 (2006), pp. 29–47.
- [37] B. Recht, M. Fazel and P.A. Parrilo, Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization, *SIAM Rev.*, 52 (2010), pp. 471–501.
- [38] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, New York, 1986.
- [39] M. Slawski and M. Hein, *Sparse recovery for protein mass spectrometry data*, Technical Report, Saarland University, Germany, 2010.
- [40] A. Szlam, Z. Guo and S. Osher, *A split Bregman method for nonnegative sparsity penalized least squares with applications to hyperspectral demixing*, in IEEE Inter. Conf. on Imaging Processing, 2010, pp. 1917–1920.

- [41] J.A. Tropp, Greed is good: Algorithmic results for sparse approximation, *IEEE Trans. on Information Theory*, 50 (2004), pp. 2231–2242.
- [42] N. Vo, B. Moran and S. Challa, *Nonnegative-least-square classifier for face recognition*, in Proc. of the 6th Inter. Symposium on Neural Networks: Advances in Neural Networks, 2009, pp. 449–456.
- [43] F. Wang and P. Li, *Compressed nonnegative sparse coding*, in IEEE 10th Int. Conf. on Data Mining (ICDM), Dec. 2010, pp. 1103–1108.
- [44] M. Wang, W. Xu and A. Tang, A unique nonnegative solution to an underdetermined system: from vectors to matrices, *IEEE Trans. Signal Proc.*, 59 (2011), pp. 1107–1016.
- [45] Z. Wen, W. Yin, D. Goldfarb and Y. Zhang, A fast algorithm for sparse reconstruction based on shrinkage, subspace optimization, and continuation, *SIAM J. Sci. Comput.*, 32 (2010), 1832–1857.
- [46] Y. Zhang, *A simple proof for recoverability of  $\ell_1$ -minimization (II): the nonnegative case*, Technical Report, Rice University, 2005.
- [47] Y. Zhang, Theory of compressive sensing via  $\ell_1$ -minimization: A non-RIP analysis and extensions, *J. Oper. Res. Soc., China*, 1 (2013), pp. 79–105.
- [48] Y.B. Zhao, An approximation theory of matrix rank minimization and its application to quadratic equations, *Linear Algebra Appl.*, (437) 2012, pp. 77–93.
- [49] Y.B. Zhao and D. Li, Reweighted  $\ell_1$ -minimization for sparse solutions to underdetermined linear systems, *SIAM J. Optim.*, 22 (2012), pp. 1065–1088.