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Goodwin, Simon M.; Pengelly, Rachel; Stewart, David I.; Thomas, Adam R.

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MONOGAMOUS SUBVARIETIES OF THE NILPOTENT CONE

SIMON M. GOODWIN, RACHEL PENGELLY, DAVID I. STEWART, AND ADAM R. THOMAS

In memory of Gary, who influenced us greatly

Abstract. Let G be a reductive algebraic group over an algebraically closed field $\Bbbk$ of prime characteristic not 2, whose Lie algebra is denoted $\mathfrak{g}$. We call a subvariety $\mathfrak{X}$ of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ *monogamous* if for every $e \in \mathfrak{X}$, the $\mathfrak{sl}_2$-triples (e, h, f) with $f \in \mathfrak{X}$ are conjugate under the centraliser $C_G(e)$. Building on work by the first two authors, we show there is a unique maximal closed G-stable monogamous subvariety $\mathcal{V} \subset \mathcal{N}$ and that it is an orbit closure, hence irreducible. We show that $\mathcal{V}$ can also be characterised in terms of Serre's G-complete reducibility.

1. Introduction

Let $\Bbbk$ be an algebraically closed field of characteristic $p \neq 2$, and G a simple algebraic $\Bbbk$-group with Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Theorems of Jacobson–Morozov and Kostant say that if $\Bbbk$ is of characteristic 0, then for any nilpotent $e \in \mathfrak{g}$ there exists an $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{g}$ which is unique up to conjugacy by the centralizer of e in G, see [Mor42,Jac51,Kos59]. We continue the investigation into generalising Kostant's uniqueness theorem to fields of small characteristic. Let $\mathfrak{X}$ be a subset of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$. We say that $\mathfrak{X}$ is *monogamous* if the following property holds:

Let (e, h, f) and (e, h', f') be $\mathfrak{sl}_2$-triples with $e, f, f' \in \mathfrak{X}$. Then (e, h, f) is $C_G(e)$-conjugate to (e, h', f').

The main theorem of [ST18] proves that $\mathcal{N}$ is monogamous if and only if $p > h(G)$, where $h(G)$ is the Coxeter number for G. When G is of classical type, the first two authors [GP22] found a unique maximal G-stable closed subvariety of $\mathcal{N}$ that is monogamous. This paper completes the story by treating the exceptional types. Define the following subset of $\mathcal{N}$:

$$\mathcal{V} := \left\{ x \in \mathcal{N} \;\middle|\; \begin{array}{l} x^{[p]} = 0, \\ x \text{ is not regular in a Levi subalgebra with a factor of type } A_{p-1}, \text{ and} \\ x \text{ is not subregular if } G \text{ is of type } G_2 \text{ and } p = 3. \end{array} \right\}$$

Theorem 1.1. *Let G be a simple algebraic group over an algebraically closed field $\Bbbk$ of characteristic $p > 2$. Then $\mathcal{V}$ is the unique maximal G-stable closed monogamous subvariety of $\mathcal{N}$. Furthermore, $\mathcal{V}$ is irreducible, being the closure of a single orbit as specified in Tables 1 and 2 below.*

In [Ste10b], a close relationship was found between uniqueness of $\mathfrak{sl}_2$-subalgebras and the existence of so-called non-G-cr $\mathfrak{sl}_2$-subalgebras. The notion of G-complete reducibility for subgroups of G is due to Serre [Ser05], and the natural generalisation to subalgebras of $\mathfrak{g}$ was introduced by McNinch [McN07]. Given a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, we say that $\mathfrak{h}$ is *G-completely reducible* (G-cr for short) if for every parabolic subalgebra $\mathfrak{p}$ such that $\mathfrak{h} \subseteq \mathfrak{p}$ we have that there exists some Levi subalgebra $\mathfrak{l}$ of $\mathfrak{p}$ with $\mathfrak{h} \subseteq \mathfrak{l}$.

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We say $\mathfrak{X} \subseteq \mathcal{N}$ is *A_1-G-cr* if every subalgebra generated by an $\mathfrak{sl}_2$-triple (e, h, f) with $e, f \in \mathfrak{X}$ is G-cr.

Theorem 1.2. *Let G be a simple algebraic group over an algebraically closed field $\Bbbk$ of characteristic $p > 2$. Then $\mathcal{V}$ is the unique maximal G-stable closed A_1-G-cr subvariety of $\mathcal{N}$.*

The proof follows very quickly from Theorem 1.1; see Section 4.

Remark 1.3. It would be interesting to know more about the geometry of the nilpotent variety $\mathcal{V}$. In type A, Donkin [Don90] showed that the closure of all orbits are normal. Orbit closures in the remaining classical types are considered by Xiao and Shu [XS15]. For exceptional types $G_2, F_4, \ldots, E_8$, results of Thomsen [Tho00] show that our varieties $\mathcal{V}$ are in fact Gorenstein normal varieties with rational singularities as long as $p \geq 5, 11, 7, 11, 13,$ respectively.

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2. Preliminaries

Throughout, $\Bbbk$ is an algebraically closed field of characteristic $p > 2$ and G is a simple $\Bbbk$-group with $\mathfrak{g} = \mathrm{Lie}(G)$. There is an inherited $[p]$-map on $\mathfrak{g}$ and we use $x^{[p]}$ to denote the image of $x \in \mathfrak{g}$ under this map. The variety of all nilpotent elements in $\mathfrak{g}$, often called the nilpotent cone, is denoted by $\mathcal{N}$. The restricted nullcone is the subvariety of $\mathcal{N}$ consisting of elements x such that $x^{[p]} = 0$ and we denote it by $\mathcal{N}_p$. The distribution of nilpotent elements among $\mathfrak{sl}_2$-subalgebras of $\mathfrak{g}$ is insensitive to central isogeny, and so we assume that whenever G is classical, it is one of $\mathrm{SL}(V)$, $\mathrm{Sp}(V)$ or $\mathrm{SO}(V)$ and write $G = \mathrm{Cl}(V)$ for brevity; if G is exceptional, we take it to be simply connected.

Recall that a prime p is bad for G if $p = 2$ and G is of type B, C or D; if $p \leq 3$ and G is exceptional; or if $p \leq 5$ and G is of type E_8; otherwise it is good. In some examples we require a choice of base for the root system associated to $\mathfrak{g}$; we use Bourbaki notation [Bou05]. Finally, we fix a maximal torus T of G.

2.1. Nilpotent orbits and Hasse diagrams.

The orbits for the action of G on $\mathcal{N}$ are called nilpotent orbits. There are finitely many such and they are classified. In case G is of exceptional type, we describe an orbit $\mathrm{O} = G \cdot x$ by a label indicating a Levi subalgebra in which e is distinguished; for these labels we refer to [LS12].

When $G = \mathrm{Cl}(V)$, the classification or orbits in terms of actions on V is well-known and can be found in [Jan04, Section 1], but we recap it here for ease of reference. Set $m = \dim V$. If $G = \mathrm{SL}(V)$, orbits are parameterised by partitions of m according to the Jordan decomposition of their elements' actions on V; we write $x \sim (\lambda_1, \ldots, \lambda_r)$ where $\lambda_1 \geq \cdots \geq \lambda_r$ is the partition of m corresponding to x. In types B and C orbits are parameterised by partitions of m with an even number of even parts and an even number of odd parts, respectively. In type D it is slightly more complicated. A partition is called very even if it only has even parts and they all occur with even

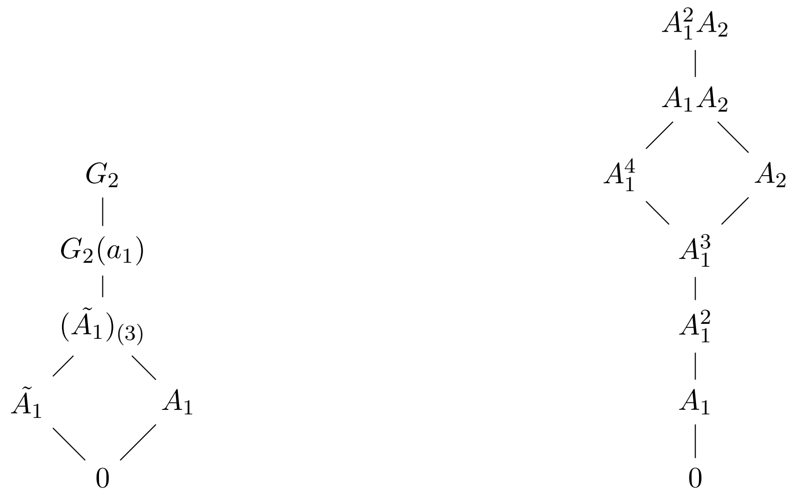

FIGURE 1. Full Hasse diagram for G_2 when $p = 3$ and partial Hasse diagram for E_8 when $p = 3$.

multiplicity. There is one orbit for each partition of m with an even number of even parts that is not very even; and two orbits for every very even partition of m.

To check that $\mathcal{V}$ is a closed subvariety of $\mathcal{N}$ we require information about the Hasse diagrams for the closure relation on nilpotent orbits. For classical types, apart from type D, the closure order on orbits is precisely the dominance order on partitions. In type D we start with the Hasse diagram for the dominance order on partitions with an even number of even parts. Then we replace each very even partition λ with two nodes λ_1, λ_2 and replace each edge from λ to μ with two edges from λ_i to μ. For exceptional types the picture is actually incomplete in general. But if p is good for G, the existence of Springer morphisms implies that the Hasse diagrams remain the same as those in characteristic 0; [Spa82, Thèoréme III 5.2]. In bad characteristic, there are not even the same number of nilpotent and unipotent orbits; for certain bad primes there are more nilpotent orbits than in characteristic 0. To deal with this, Hesselink gives a partition of $\mathcal{N}$ into strata, one for each nilpotent orbit of G in characteristic 0. These strata are locally closed and G-stable.

The following forthcoming theorem of Premet [Pre] is sufficient for our needs; it reduces the determination of the Hasse diagram of nilpotent orbit closures to establishing the extra edges needed to accommodate when the Hesselink strata split into multiple orbits.

Theorem 2.1. *Let $S_i \subset \mathcal{N}$ be a Hesselink stratum and let O_i be the orbit of maximal dimension contained in S_i. Then the Hasse diagram of the closure relation on the nilpotent orbits $\{\mathrm{O}_i\}$ is the same as the Hasse diagram for the closure relation on all nilpotent orbits in characteristic* 0.

Since $p > 2$, there are precisely two cases where the Hesselink strata contains more than one orbit, namely when $p = 3$ and G is of type G_2 or E_8. In these cases precisely one stratum of each splits into two orbits. Away from those cases then, the Hasse diagrams are the same as in characteristic 0; these can be found in [Car93, Section 13.4]. However, two edges are missing in the E_8 diagram: there should be edges between the pairs of labels $(E_6+A_1, E_8(b_6))$ and (A_3+A_1, A_3) (see [Spa82, p. 249]). The Hasse diagram for G_2 when $p = 3$ can be deduced from [Stu71] and is reproduced in the left of Figure 1.

G	m	λ	
A_{m-1}	$a(p-1)+r$	$((p-1)^a, r)$	
$B_{\frac{m-1}{2}}$	$p+a(p-1)+r\ (r>0)$	$(p,(p-1)^a, r-1, 1)$	a even
		$(p,(p-1)^{a-1}, p-2, r+1)$	a odd
	$p+a(p-1)$	$(p,(p-1)^a)$	
	$\leq p$	(m)	
$C_{\frac{m}{2}}$	$a(p-1)+r$	$((p-1)^a, r)$	
$D_{\frac{m}{2}}$	$p+a(p-1)+r$	$(p,(p-1)^a, r)$	a even
		$(p,(p-1)^{a-1}, p-2, r, 1)$	a odd
	$\leq p$	$(m-1, 1)$	

TABLE 1. Partition λ corresponding to the orbit O_λ such that $\mathcal{V} = \overline{\mathrm{O}}_\lambda$ in the classical types, where $a \geq 0$ and $0 \leq r < p-1$.

G	p	O	G	p	O	G	p	O	G	p	O
G_2	3	$\tilde{A}_1^{(3)}$	E_6	3	A_1^3	E_7	3	A_1^4	E_8	3	A_1^4
	5	$G_2(a_1)$		5	$D_4(a_1)$		5	$A_3A_2A_1$		5	A_3^2
	≥ 7	G_2		7	$E_6(a_3)$		7	$E_7(a_5)$		7	A_4^2
F_4	3	$A_1\tilde{A}_1$		11	$E_6(a_1)$		11	$E_7(a_3)$		11	$E_8(a_6)$
	5	$F_4(a_3)$		≥ 13	E_6		13	$E_7(a_2)$		13	$E_8(a_5)$
	7	$F_4(a_2)$					17	$E_7(a_1)$		17	$E_8(a_4)$
	11	$F_4(a_1)$					≥ 23	E_7		19	$E_8(a_3)$
	≥ 13	F_4								23	$E_8(a_2)$
										29	$E_8(a_1)$
										≥ 31	E_8

TABLE 2. Orbit O such that $\mathcal{V} = \overline{\mathrm{O}}$ in the exceptional types.

We can now prove part of Theorem 1.1.

Lemma 2.2. *The subset $\mathcal{V} \subseteq \mathcal{N}$ is a closed G-stable subvariety; moreover, it is the closure of a single orbit in every case, as specified in Tables 1 and 2.*

Proof. Suppose $G = \mathrm{Cl}(V)$ with $\dim V = m$. An orbit corresponding to a partition λ of m is contained in the restricted nullcone if and only if the largest part of λ is at most p. Let $G = \mathrm{SL}(V)$ or $\mathrm{Sp}(V)$ (resp. $\mathrm{SO}(V)$), and let $x \in \mathcal{N}$ with partition represented by λ. Then x is not regular in a Levi subalgebra with a factor of type A_{p-1} if λ contains no parts of size p (resp. at most one part of size p). Now every orbit represented in Table 1 represents a single orbit in $\mathcal{V}$: for G of type D, each λ given in Table 1 is not very even. Observe that any other orbit in $\mathcal{V}$ must be represented by a partition lower than λ in the dominance ordering, and hence is contained in $\overline{\mathrm{O}}_\lambda$; and vice-versa, by definition of $\mathcal{V}$.

Now suppose G is of exceptional type. We use the tables in [Ste16] to determine the orbits in the restricted nullcone. Note that the A_7 orbit is not restricted in E_8 when $p = 3$ so we may appeal to Theorem 2.1 and inspect the Hasse diagrams. □

Let X be a node in the Hasse diagram for the closure order on nilpotent orbits of $\mathfrak{g}$. We call X a *neighbour of* $\mathcal{V}$ if $\mathcal{V}$ does not contain the orbit corresponding to X but there is an edge from X to some orbit contained in $\mathcal{V}$. We say Y is a *minimal neighbour of* $\mathcal{V}$ if Y is a neighbour of $\mathcal{V}$ and the closure of Y contains no other neighbours of $\mathcal{V}$.

Example 2.3. Let G be of type E_8, with $p = 3$. The bottom of the Hasse diagram is as shown on the right of Figure 1. By Lemma 2.2, $\mathcal{V}$ is the closure of the $4A_1$ orbit. Therefore A_2 and $A_2 + A_1$ are the only neighbours of $\mathcal{V}$. As A_2 is in the closure of the $A_2 + A_1$ orbit, A_2 is the only minimal neighbour of $\mathcal{V}$.

Lemma 2.4. *Let G be of exceptional type. Table 3 lists the minimal neighbours of $\mathcal{V}$.*

G	p	min. orbs.	G	p	min. orbs.	G	p	min. orbs.	G	p	min. orbs.
G_2	3	$G_2(a_1)$	E_6	3	A_2	E_7	3	A_2	E_8	3	A_2
	5	G_2		5	A_4, D_4		5	A_4, D_4		5	A_4, D_4
F_4	3	A_2, $\tilde{A}_2$		7	D_5		7	D_5, A_6		7	D_5, A_6
	5	C_3, B_3		11	E_6		11	E_6		11	E_6, D_7
	7	$F_4(a_1)$					13	$E_7(a_1)$		13	$E_7(a_1)$
	11	F_4					17	E_7		17	E_7
										19	$E_8(a_2)$
										23	$E_8(a_1)$
										29	E_8

TABLE 3. Minimal neighbours of $\mathcal{V}$.

Proof. When the Hasse diagram is known, the result follows by inspection. For G of type E_8 with $p = 3$ we need to rule out A_7 as a minimal neighbour of $\mathcal{V}$. However, since an element in this orbit is regular in a Levi subalgebra of type A_7, it must be connected to A_6, hence cannot be a minimal neighbour. □

2.2. G-cr subalgebras.

Proposition 2.5. *Suppose $e \in \mathcal{N}_p$. If e is contained in an $\mathfrak{sl}_2$-triple then there exists a G-cr subgroup $X \leq G$ of type A_1 such that $\operatorname{Lie}(X)$ contains e.*

Proof. If $G = \mathrm{SL}(V)$ then $e^{[p]} = 0$ implies e has Jordan blocks of size at most p, which means e is regular in a Levi subalgebra of type $A_{r_1} \times \cdots \times A_{r_i}$ with each $r_i \leq p - 1$. The image of $X = \mathrm{SL}_2$ under the completely reducible representation given by $L(r_1) \oplus \cdots \oplus L(r_i)$ satisfies the demands of the theorem, where r_j now represents a (restricted) high weight. So assume G is not of type A. Then if p is good for G, it is very good, and the result follows from [McN05, Propositions 33, 52].

So we may assume p is bad, and therefore that G is exceptional. As before, the orbits of $\mathcal{N}_p$ can be worked out from the tables in [Ste16] and there are not very many. By inspection, it follows that the label of every restricted nilpotent class is denoted by sums of A_r for $r < p$ and $D_4(a_1)$ if $G = E_8$, $p = 5$ or is $G_2(a_1)$ when $G = G_2$, $p = 3$; note that the class $(\tilde{A}_1)_{(3)}$ is excluded since it is not contained in an $\mathfrak{sl}_2$-triple.

We first deal with the final case. The subsystem subgroup $A_2 < G_2$ contains an A_2-irreducible subgroup X of type A_1. By [Ste10a, Theorem 1], all simple subgroups of G_2 are G_2-cr when $p = 3$.

The restriction of the nontrivial 7-dimensional G_2-module to X is $L(2)^2 + L(0)$. It follows that the nilpotent elements contained in $\mathrm{Lie}(X)$ have Jordan blocks of size $3^2, 1$ and thus are in the $G_2(a_1)$ class by [Ste16, Table 4].

In the remaining cases, every class is a distinguished element in $\mathfrak{l} = \mathrm{Lie}(L)$ for some Levi subgroup L with simple factors only of type A_r with $r < p$ or D_4. By [Ser05, Proposition 3.2], a subgroup X of L is G-cr if and only if it is L-cr. Furthermore a subgroup X of a central product $L = L_1 L_2$ is L-cr if and only if the projection of X to both L_1 and L_2 is L-cr. Therefore, it suffices to deal with the cases where L is simple and simply connected of type A_r ($r < p$) or D_4–but these cases have already been tackled. □

If X is G-cr then so is $\mathrm{Lie}(X)$ by [McN07, Theorem 1]; so we get the following.

Corollary 2.6. *Suppose $e \in \mathcal{N}_p$. Then there exists a G-cr subalgebra $\mathfrak{s} \cong \mathfrak{sl}_2$ of $\mathfrak{g}$ containing e.*

The following is used a couple of times, and is [McN07, Lemma 4].

Lemma 2.7. *Let L be a Levi factor of a parabolic subgroup of G. Suppose that we have a Lie subalgebra $\mathfrak{s} \subset \mathfrak{l} = \mathrm{Lie}(L)$. Then $\mathfrak{s}$ is G-cr if and only if $\mathfrak{s}$ is L-cr.*

Proposition 2.8. *Suppose $e \in \mathcal{N}$ is distinguished in a Levi subalgebra $\mathfrak{l} = \mathrm{Lie}(L)$ with a factor of type A_{p-1}. Then there is an $\mathfrak{sl}_2$-triple (e, h, f) such that $\mathfrak{s} := \langle e, h, f\rangle$ is non-G-cr and $f \in \overline{L \cdot e}$.*

Proof. By Lemma 2.7 it suffices to treat the case that $L = \mathrm{SL}(V)$ with $\dim V = p$. In that case, let $\mathfrak{s} = \langle e, h, f\rangle$ be the image of $\mathfrak{sl}_2$ under the representation given by the p-dimensional baby Verma module $Z_0(0)$; cf. [Jan98, Section 5.4]. As $V \downarrow X = Z_0(0)$ is a non-trivial extension of the irreducible module $L(p-2)$ by the trivial module we have that $\mathfrak{s}$ is not L-cr. It is easy to see that one of e or f has a full Jordan block on V and is therefore regular. But the whole of $\mathcal{N}(L)$ is the closure of a regular nilpotent element so we are done. □

Lemma 2.9. *Let (e, h, f) be an $\mathfrak{sl}_2$-triple with $e, f \in \mathcal{N}$. Suppose that e and f are distinguished in Levi subalgebras of $\mathfrak{g}$ with no factors of type A_{p-1}. If $\mathfrak{s} := \langle e, f\rangle$ is G-cr then $\mathfrak{s}$ is a p-subalgebra.*

Proof. Suppose $\mathfrak{s}$ is not a p-subalgebra. Then by [ST18, Lemma 4.3], $\mathfrak{s}$ must be L-irreducible in a Levi subalgebra $\mathfrak{l} = \mathrm{Lie}(L)$ with $L = L_1 L_2 \dots L_r$ and L_1 of type A_{p-1}, say. Therefore, the projection $\bar{\mathfrak{s}}$ of $\mathfrak{s}$ to $\mathfrak{l}_1 = \mathrm{Lie}(L_1)$ is also L_1-irreducible, so that $\bar{\mathfrak{s}}$ acts irreducibly on the p-dimensional natural L_1-module. However, the classification of p-dimensional irreducible $\mathfrak{sl}_2$-modules in [Jan98, Section 5.4] shows that the image of e or f in $\bar{\mathfrak{s}}$ is regular in L_1, a contradiction. □

3. Monogamy of $\mathcal{V}$

We start with an observation that $\mathcal{V}$ can be characterised using the following partial order on $\mathcal{N}$.

Definition 3.1. Let $x, y \in \mathcal{N}$. We say $x \preceq y$ (resp. $x \prec y$) if $\mathrm{rank}(\mathrm{ad}(x)^{p-1}) \leq \mathrm{rank}(\mathrm{ad}(y)^{p-1})$ (resp. $\mathrm{rank}(\mathrm{ad}(x)^{p-1}) < \mathrm{rank}(\mathrm{ad}(y)^{p-1})$).

Note that $\mathrm{rank}(\mathrm{ad}(x)^{p-1})$ can be calculated from the adjoint Jordan blocks of x of size at least p, and if G is exceptional, this can be done by reference to [Ste16, Section 3.1]. Now if G is classical and $e, f \in \mathcal{V}$ one always has $e \in G \cdot f$. Therefore, the next lemma follows from a simple case-by-case check, using Tables 1 & 2, the Hasse diagrams for nilpotent orbit closures and [Ste16, Section 3.1].

Lemma 3.2. *Let $x, y \in \mathcal{N}$ such that $x \in \mathcal{V}$, and $y \notin \mathcal{V}$. Then $x \prec y$.*

Remark 3.3. Comparing ranks of $(p-1)$-th powers is necessary for the partial order to differentiate nilpotent orbits contained in $\mathcal{V}$. For example, let be G of type E_6, $p = 5$, and take $x, y \in \mathcal{N}$ to be representatives of the $D_4(a_1)$ and A_4 classes respectively. Then we have $x \in \mathcal{V}$ and $y \notin \mathcal{V}$. Using [Ste16, Table 16] we see that $\mathrm{rank}(\mathrm{ad}(x)) = \mathrm{rank}(\mathrm{ad}(y)) = 78$, however $\mathrm{rank}(\mathrm{ad}(x)^{p-1}) = 11 < 15 = \mathrm{rank}(\mathrm{ad}(y)^{p-1})$.

Let $\mathfrak{X} \subseteq \mathcal{N}$. We say that $\mathfrak{X}$ is *partially monogamous* if the following holds.

Whenever (e, h, f) and (e, h', f') are two $\mathfrak{sl}_2$-triples with $e, f, f' \in \mathfrak{X}$ and $f, f' \preceq e$, then f and f' are conjugate under the action of $C_G(e)$.

Lemma 3.4. *Let $\mathfrak{X}$ be a subvariety of $\mathcal{N}_p$. Then $\mathfrak{X}$ is monogamous if and only if it is partially monogamous.*

Proof. One direction is trivial. Suppose $\mathfrak{X}$ is partially monogamous but not monogamous. Then there exist $\mathfrak{sl}_2$-triples (e, h, f) and (e, h', f') with $e, f, f' \in \mathfrak{X}$ such that (e, h, f) is not $C_G(e)$-conjugate to (e, h', f'). Since $\mathfrak{X}$ is partially monogamous it follows that either $f \not\preceq e$ or $f' \not\preceq e$; without loss of generality we assume the former. Thus $\mathrm{rank}(\mathrm{ad}(e)^{p-1}) < \mathrm{rank}(\mathrm{ad}(f)^{p-1})$, and in particular, e and f are not conjugate.

Let $(f, \tilde{h}, \tilde{e})$ be an $\mathfrak{sl}_2$-triple with f conjugate to $\tilde{e}$, which exists by Proposition 2.5. Then the two $\mathfrak{sl}_2$-triples $(f, -h, e)$ and $(f, \tilde{h}, \tilde{e})$ satisfy $f, e, \tilde{e} \in \mathfrak{X}$ and $e, \tilde{e} \preceq f$. But as $\mathfrak{X}$ is partially monogamous, we must have that f conjugate to $\tilde{e}$, which is in turn conjugate to e, a contradiction. ☐

Theorem 1.1 for classical types follows from Lemma 2.2 and the main theorem of [GP22]. For the remainder of this section we suppose G is of exceptional type.

3.1. **Bad characteristic.** We first treat the case when p is bad for G. Fix $0 \neq e \in \mathcal{V}$ for the remainder of this section. We use the representatives as in [LS04], presented in [Ste16]. If G is of type G_2 and $p = 3$, then the element e with label $(\tilde{A}_1)_{(3)}$ cannot be extended to an $\mathfrak{sl}_2$-triple by [ST18, Theorem 1.7]. So we exclude that case from now on.

Lemma 3.5. *The normaliser $N_G(\langle e \rangle)$ (and centraliser $C_G(e)$) is smooth if and only if the class of e does not occur in the following table.*

G	p	*class of e*
G_2	3	$G_2(a_1)$
F_4	3	$F_4,\ \tilde{A}_2A_1$
E_6	3	$E_6,\ E_6(a_1),\ E_6(a_3),\ A_5,\ A_2^2A_1,\ A_2^2$
E_8	3	$E_8,\ E_8(a_1),\ E_8(a_3),\ E_7,\ E_6A_1,\ E_8(b_6),\ A_7,\ E_6,\ E_6(a_3)A_1,\ A_5A_1,\ A_2^2A_1^2,\ A_2^2A_1$
	5	$E_8,\ A_4A_3$

Proof. Every element e has a cocharacter τ for which $\mathrm{im}(\tau)$ is contained in $N_G(\langle e \rangle)$ but not $C_G(e)$. Therefore, the dimension of $N_G(\langle e \rangle)$ is precisely $\dim C_G(e) + 1$. Similarly, $\dim \mathfrak{n}_{\mathfrak{g}}(\langle e \rangle) = \dim \mathfrak{c}_g(\langle e \rangle) + 1$ thanks to the existence of $\mathfrak{sl}_2$-triples. Therefore $N_G(\langle e \rangle)$ is smooth precisely when $C_G(e)$ is smooth.

It is straightforward to use Magma to calculate the dimension of $\mathfrak{c}_{\mathfrak{g}}(e)$. Comparing these dimensions with the dimension of $C_G(e)$ presented in [LS04, Tables 22.1.1–22.1.5] completes the proof. ☐

We may now deduce an important reduction.

Proposition 3.6. *There exists an $\mathfrak{sl}_2$-triple $(e, \overline{h}, \overline{f})$ with $\overline{f} \in \mathcal{V}$ and $\overline{h} \in \mathfrak{t} = \mathrm{Lie}(T)$. Moreover, if (e, h, f) is also an $\mathfrak{sl}_2$-triple then h is $C_G(e)$-conjugate to $\overline{h}$.*

Proof. We know from Proposition 2.5 that there is an $\mathfrak{sl}_2$-triple $(e, \overline{h}, \overline{f})$ with $\overline{f}$ in the same nilpotent class as e. By Lemma 3.5, the group $N_G(\langle e \rangle)$ is smooth. Therefore, all maximal tori in $\mathfrak{n}_{\mathfrak{g}}(\langle e \rangle)$ are $N_G(\langle e \rangle)$-conjugate. A computation in Magma shows that $\mathfrak{n}_{\mathfrak{g}}(\langle e \rangle) \cap \mathfrak{t}$ is a maximal torus of $\mathfrak{n}_{\mathfrak{g}}(\langle e \rangle)$. So we may assume that $\overline{h}$ is contained in $\mathfrak{t}$.

For the final part, first note that since $[h, e] = 2e$ we have $[h^{[p]}, e] = \mathrm{ad}(h)^p e = 2e$ thanks to Fermat's Little Theorem. Therefore $\mathfrak{h} = \langle h^{[p]^r} \mid r = 0, 1, \ldots \rangle$ is an abelian p-closed subalgebra of $\mathfrak{n}_{\mathfrak{g}}(\langle e \rangle)$. It follows from [SF88, Chapter 2, Corollary 4.2] that $\mathfrak{h} = \mathfrak{t}' \oplus \mathfrak{n}'$ where $\mathfrak{t}'$ is the set of semisimple elements of $\mathfrak{h}$. Since $\mathfrak{t}'$ is a torus, the above argument shows that up to $N_G(\langle e \rangle)$-conjugacy we may assume that $\mathfrak{t}'$ is contained in $\mathfrak{t}$. In particular, $\overline{h} \in \mathfrak{t}'$.

Because $\mathfrak{c}_{\mathfrak{g}}(\langle e \rangle)$ has codimension 1 in $\mathfrak{n}_{\mathfrak{g}}(\langle e \rangle)$ and $\overline{h} \notin \mathfrak{c}_{\mathfrak{g}}(\langle e \rangle)$ we see that the torus $\mathfrak{t}'$ decomposes as $\mathfrak{t}' = \mathfrak{c}_{\mathfrak{t}'}(e) \oplus \langle \overline{h} \rangle$. Furthermore, $\mathfrak{n}' \subset \mathfrak{c}_{\mathfrak{g}}(\langle e \rangle)$. It follows that $h = \overline{h} + h'$ for some $h' \in \mathfrak{c}_{\mathfrak{g}}(e) \cap \mathfrak{c}_{\mathfrak{g}}(\overline{h})$.

Since $h = [e, f]$ and $\overline{h} = [e, \overline{f}]$ we also have $h' \in \mathrm{im}(\mathrm{ad}(e))$. Thus

$$h' \in W = \mathfrak{c}_{\mathfrak{g}}(\langle e, h \rangle) \cap \mathrm{im}(\mathrm{ad}(e)).$$

Another Magma check shows that every element in W is p-nilpotent.

In particular, all eigenvalues of h' are 0. Since $h = \overline{h} + h'$ and $[h, f] = -2f$ we must have $[\overline{h}, f] = -2f$. Therefore, $f \in F = \ker(\mathrm{ad}(\overline{h}) + 2I_{\dim \mathfrak{g}})$ and so $h = [e, f] \in \mathrm{im}(\mathrm{ad}(e))(F)$. Note that $\overline{f} \in F$ also, so $\overline{h} \in \mathrm{im}(\mathrm{ad}(e))(F)$ and hence $h' \in \mathrm{im}(\mathrm{ad}(e))(F)$.

Thus $h' \in W \cap \mathrm{im}(\mathrm{ad}(e))(F)$. A final easy check in Magma shows that $W \cap \mathrm{im}(\mathrm{ad}(e))(F) = 0$, as required. □

We now describe an ad-hoc method to prove that if (e, h, f') is an $\mathfrak{sl}_2$-triple with $f' \in \mathcal{V}$ and $f' \preceq e$ then f' is uniquely determined up to $C := (C_G(e) \cap C_G(h))$-conjugacy. In principle, this can be implemented by hand, but for speed and accuracy we have used the Magma algebra system. Applying Proposition 3.6 and Lemma 3.4 then completes the proof that $\mathcal{V}$ is monogamous.

Setup:

By Proposition 3.6, there exists an $\mathfrak{sl}_2$-triple (e, h, f) with $h \in \mathfrak{t} = \mathrm{Lie}(T)$ and $f \in \mathcal{V}$ in the same nilpotent class as e. Let (e, h, f') be an $\mathfrak{sl}_2$-triple with $f' \in \mathcal{V}$ and $f' \preceq e$. Since

$$[h, f'] = -2f' \tag{1}$$

we have $f' \in F := \ker(\mathrm{ad}(h) + 2I_{\dim(\mathfrak{g})})$. We setup a generic element of the subspace F, namely $\tilde{f} = \sum x_i v_i \in \mathfrak{g}$ where the x_i are variables and $v_1, \ldots, v_{\dim(F)}$ is a basis for F. One can view $\tilde{f}$ as describing a $\dim(F)$-dimensional subvariety $\mathcal{F}$ of $\mathfrak{g}$.

Step 1: The equation

$$[e, \tilde{f}] = h, \tag{2}$$

yields a set of linear equations among the x_i. We use these to constrain $\tilde{f}$ and thus reduce the dimension of $\mathcal{F}$. Now every element of $\mathcal{F}$ forms an $\mathfrak{sl}_2$-triple with e.

Example 3.7. We give an example where Step 1 is sufficient. Let G be of type E_7, $p = 3$ and $e = e_{\alpha_2} + e_{\alpha_5} + e_{\alpha_7}$. Then e is a representative of the $(A_1^3)^{(1)}$ orbit and $e \in \mathcal{V}$ by Lemma 2.2. On this occasion it is obvious that (e, h, f) is an $\mathfrak{sl}_2$-triple with $h = h_2 + h_5 + h_7 \in \mathfrak{t}$ and $f = e_{-\alpha_2} + e_{-\alpha_5} + e_{-\alpha_7}$.

Let $F := \ker(\mathrm{ad}(h) + 2I_{\dim(\mathfrak{g})})$. A straightforward calculation shows that the space F is 27-dimensional with a basis of root vectors $v_1 = e_{r_1}, \ldots, v_{27} = e_{r_{27}}$ for some set of roots $r_1, \ldots, r_{27}$; in particular $r_{12} = -\alpha_2$, $r_{13} = -\alpha_5$ and $r_{14} = -\alpha_7$.

We let $\tilde{f} = \sum_i x_i v_i$ as above. We then compute $[e, \tilde{f}] = h$. For $i \neq 12, 13, 14$ we find that the left hand side has a coordinate of the form λx_i for $\lambda = 1$ or 2. Thus $x_i = 0$ for $i \neq 12, 13, 14$. On the other hand the coordinate of h_2 is seen to be equal to $x_{14} + 2$. Thus x_{14} must be 1. Similarly, the coordinates of h_5 and h_7 are $x_{13} + 2$ and $x_{12} + 2$, respectively. We have therefore determined all the variables in $\tilde{f}$ and found it must be f, as required.

<u>Step 2:</u> Consider the action of C on $\mathcal{F}$ by applying elements $g \in C$ to $\tilde{f}$. Find a set of variables $\{x_i \mid i \in Z\}$ such that every C-orbit in $\mathcal{F}$ contains a representative with $x_i = 0$ for $i \in Z$. Thus we may assume that these variables are zero in $\tilde{f}$, further reducing $\mathcal{F}$.

Example 3.8. We give an example where Steps 1 and 2 are sufficient. Let G be of type G_2 and $p = 3$. Consider $e = e_{10}$ which is a representative of the $\tilde{A}_1$ orbit, thus contained in $\mathcal{V}$ by Lemma 2.2.

Clearly, if $h = h_{10}, f = e_{-10}$, then (e, h, f) is an $\mathfrak{sl}_2$-triple with $f \in \mathcal{V}$. Define $F := \ker(\mathrm{ad}(h) + 2I_{\dim(\mathfrak{g})})$. This is 3-dimensional and we build $\tilde{f}$ as above:

$$\tilde{f} = x_1 e_{-11} + x_2 e_{-10} + x_3 e_{21}.$$

After Step 1 we find

$$\tilde{f} = x_1 e_{-11} + e_{-10} + x_3 e_{21}.$$

Now we apply elements of $C = C_G(e) \cap C_G(h)$ to $\tilde{f}$. First consider $x_{-01}(t) \in C$. We calculate that

$$x_{-01}(t) \cdot \tilde{f} = (t + x_1)e_{-11} + e_{-10} + x_3 e_{21}.$$

Therefore, by setting $t = -x_1$, we see that every C-orbit in $\mathcal{F}$ contains a representative with $x_1 = 0$. We're down to

$$\tilde{f} = e_{-10} + x_3 e_{21}.$$

Finally, conjugation by $x_{31}(t)$ sends $\tilde{f}$ to $e_{-10} + (t + x_3)e_{21}$. Thus we may conclude that $\tilde{f} = f$, as required.

<u>Step 3:</u> Finally, we impose the condition that $\tilde{f}$ should represent an element $f' \in \mathcal{V}$ with $f' \preceq e$. In particular, every element in $\mathcal{V}$ is p-nilpotent. Therefore, the equation

$$\mathrm{ad}(\tilde{f})^p = 0. \tag{3}$$

yields further polynomial equations we want the x_i to satisfy.

Forcing $\mathcal{F}$ to only contain elements f' with $f' \preceq e$ is slightly more subtle since we cannot simply calculate the 'rank' of $M = \mathrm{ad}(\tilde{f})^{p-1}$. Let $R = \mathrm{rank}(\mathrm{ad}(e)^{p-1})$ and ϵ be a map evaluating the remaining variables to choices in $\Bbbk$ (so each $f' \in \mathcal{F}$ is simply some $\epsilon(\tilde{f})$). We find a subset $r_1, \ldots, r_R$ of rows and subset $c_1, \ldots, c_R$ of columns such that, up to the reordering of rows and columns, the corresponding submatrix S of M is upper triangular and all diagonal entries are elements of $\mathbb{F}_p^*$.

Then any element $f' \in \mathcal{F}$ will satisfy $\operatorname{rank}(\operatorname{ad}(f')^{p-1}) \geq R$. We only want those elements $f' \preceq e$ which means $\operatorname{rank}(\operatorname{ad}(f')^{p-1}) \leq R$. Thus, given any row r of M we must have $\epsilon(r)$ is in the span of $\epsilon(r_1), \ldots, \epsilon(r_R)$. In particular, a row r' of M with zeroes at all columns $c_1, \ldots, c_R$ must evaluate to zero. This final set of conditions is enough to force all remaining variables to be 0.

Example 3.9. We give an example where we require Step 3. Let G be of type G_2 and $p = 3$. Consider $e = e_{01}$ which is a representative of the A_1 orbit, thus contained in $\mathcal{V}$ by Lemma 2.2.

Take $h = h_{01}, f = e_{-01}$, then (e, h, f) is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ with $f \in \mathcal{V}$. Define $F := \ker(\operatorname{ad}(h) + 2I_{\dim(\mathfrak{g})})$. This is 5-dimensional and we build $\tilde{f}$ as above:

$$\tilde{f} = x_1 e_{-32} + x_2 e_{-01} + x_3 e_{-10} + x_4 e_{11} + x_5 e_{32}.$$

After Step 1 we find

$$\tilde{f} = x_1 e_{-32} + e_{-01} + x_3 e_{-10} + x_4 e_{11} + x_5 e_{32}.$$

There are no elements of $C = C_G(e) \cap C_G(h)$ which we can use to reduce $\tilde{f}$, so we move onto Step 3.

The equation $\operatorname{ad}(\tilde{f})^p = 0$ gives many relations amongst the remaining variables but none that allow us to conveniently reduce $\tilde{f}$. Consider the matrix $M = \operatorname{ad}(\tilde{f})^{p-1}$. The first, eighth, tenth and thirteenth column of M consist only of zeroes, so we remove them, leaving the matrix M' as follows.

$$\begin{pmatrix}
x_1x_5 & 0 & 0 & x_5 & 2x_4^2 & 0 & 0 & x_4x_5 & 0 & x_5^2 \\
0 & 2x_4 & 0 & 0 & 0 & x_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 2x_4 & 0 & 0 & 0 & x_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2x_1x_5 + x_3x_4 & 0 & 0 & x_3x_5 + x_4^2 & 0 & 0 \\
0 & 2x_1x_4 + 2x_3^2 & 0 & 0 & 0 & x_1x_5 + 2x_3x_4 & 0 & 0 & x_3x_5 + x_4^2 & 0 \\
0 & x_3 & 0 & 0 & 0 & 2x_4 & 0 & 0 & x_5 & 0 \\
0 & 0 & x_3 & 0 & 0 & 0 & x_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x_1x_4 + x_3^2 & 0 & 0 & 2x_1x_5 + x_3x_4 & 0 & 0 \\
x_1 & 0 & 0 & 1 & x_3 & 0 & 0 & x_4 & 0 & x_5 \\
0 & 2x_1 & 0 & 0 & 0 & 2x_3 & 0 & 0 & 2x_4 & 0 \\
0 & 0 & 2x_1 & 0 & 0 & 0 & x_3 & 0 & 0 & 0 \\
x_1^2 & 0 & 0 & x_1 & x_1x_3 & 0 & 0 & 2x_3^2 & 0 & x_1x_5 \\
0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & x_3 & 0
\end{pmatrix}$$

We calculate that $R = \operatorname{rank}(\operatorname{ad}(e)^{p-1}) = 1$. Therefore, if $\epsilon(\tilde{f}) = f' \preceq e$ for some evaluation map ϵ, we must have that the rank of $\epsilon(M')$ is at most one. Observe that $M'_{10,4} = 1$ and so the rank of $\epsilon(M')$ is at least one. It follows that every row of $\epsilon(M')$ must be a multiple of the tenth row of $\epsilon(M')$.

Consider the sixth row of M'. This only has nonzero entries in columns 2, 6 and 9, namely $x_3, 2x_4$ and x_5. Since the tenth row is zero in columns 2, 6 and 9 it must be the case that the sixth row of $\epsilon(M')$ is zero. Hence $x_3 = x_4 = x_5 = 0$.

Similarly, row 11 of $\epsilon(M')$ must be zero. Thus $x_1 = 0$, and we may conclude that $\tilde{f} = f$.

3.2. **Good characteristic.** Suppose p is a good prime for G. As in the bad characteristic case, we describe an algorithm to deduce that $\mathcal{V}$ is monogamous. In good characteristic there is a considerable amount of theory at our disposal. In particular, every $e \in \mathcal{N}$ has an associated cocharacter: that is a homomorphism $\tau : \mathbb{G}_m \to G$ such that under the adjoint action, we have $\tau(t)\cdot e = t^2 e$ and τ evaluates in the derived subgroup of the Levi subgroup in which e is distinguished.

Lemma 3.10. *Suppose p is good for G. Let (e, h_1, f_1) and (e, h_2, f_2) be $\mathfrak{sl}_2$-triples with $e, f_1, f_2 \in \mathcal{V}$. Then we may assume that $h_1 = h_2 = h$, up to $C_G(e)$-conjugacy. More precisely, there exists a cocharacter τ associated to e such that $\operatorname{Lie}(\tau(\mathbb{G}_m)) = \langle h\rangle$. Furthermore if $\mathfrak{g} = \bigoplus_i \mathfrak{g}(i)$ is the grading of $\mathfrak{g}$ with respect to τ we have*

$$f - f' \in \bigoplus_{r>0} \mathfrak{g}_e(-2 + rp),$$

where $\mathfrak{g}_e(i) := \mathfrak{c}_{\mathfrak{g}}(e) \cap \mathfrak{g}(i)$.

Proof. We start by proving that h_i is toral. By Lemma 2.9, the subalgebra $\mathfrak{s}_i = \langle e, h_i, f_i\rangle$ is either a p-subalgebra or non-G-cr. In the former case, we are done. In the latter case, the argument in the proof of [ST18, Lemma 6.1] applies, showing h_i is toral.

Now we apply [ST18, Proposition 2.8]. This yields cocharacters τ_i associated to e such that $\operatorname{Lie}(\tau_i(\mathbb{G}_m)) = \langle h_i\rangle$. By [Jan03, Lemma 5.3], any two cocharacters associated to e are $C_G(e)$-conjugate. Therefore, h_1 and h_2 are $C_G(e)$-conjugate and so up to $C_G(e)$-conjugacy we may assume they are equal. Set $h = h_1 = h_2$.

Since $[e, f_1 - f_2] = h - h = 0$ we know $f_1 - f_2 \in \mathfrak{c}_{\mathfrak{g}}(e)$. Furthermore, $[h, f_1 - f_2] = -2(f_1 - f_2)$ and hence $f_1 - f_2 \in \bigoplus_r \mathfrak{g}(-2 + rp)$. The conclusion follows by noting that $\mathfrak{c}_{\mathfrak{g}}(e)$ is contained in the nonnegative graded part of $\mathfrak{g}$. ☐

Fix $0 \neq e \in \mathcal{V}$ for the remainder of this section. Choose a cocharacter τ associated to e such that $h \in \operatorname{Lie}(\tau(\mathbb{G}_m)) \subset \mathfrak{t}$ with $[h, e] = 2e$. In practice, we use the representatives and associated cocharacters given in [LT11]. We know from Pommerening [Pom77, Pom80] and Lemma 3.10 that there exists a unique $\overline{f} \in \mathfrak{g}(-2)$ such that $(e, h, \overline{f})$ is an $\mathfrak{sl}_2$-triple. Furthermore, if (e, h, f) is another $\mathfrak{sl}_2$-triple then $f = \overline{f} + f'$ with $f' \in \bigoplus_{r>0} \mathfrak{g}_e(-2 + rp)$. Therefore, we need to prove that if $f \in \mathcal{V}$ then up to $C = C_G(e) \cap C_G(h)$-conjugacy we have $f = \overline{f}$, i.e. that $f' = 0$.

To do this we use the ad-hoc method from Section 3.1. Indeed, by Lemma 3.4 it suffices to prove that $f = \overline{f}$ when $f \preceq e$. We now apply Steps 1–3 starting with the space $F = f + \bigoplus_{r>0} \mathfrak{g}_e(-2+rp)$.

Example 3.11. We give a final example, this time in good characteristic. Let G be of type E_7 and $p = 7$. Consider $e = e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}}$ which is a representative of the $(A_5)^{(2)}$ orbit; thus $e \in \mathcal{V}$ by Lemma 2.2. Furthermore, by [LT11, p. 108], e has the associated cocharacter $\tau = \begin{smallmatrix} 2 & 2 & 2 & 2 & 2 & -5 \\ & & -9 & & & \end{smallmatrix}$. It follows that $h = 2h_1 + 6h_3 + 5h_4 + 6h_5 + 2h_6 \in \operatorname{Lie}(\tau(\mathbb{G}_m))$.

The unique $\overline{f} \in \mathfrak{g}(-2)$ such that $(e, h, \overline{f})$ is an $\mathfrak{sl}_2$-triple is then given by $\overline{f} = 2e_{-\substack{100000\\0}} + 6e_{-\substack{010000\\0}} + 5e_{-\substack{001000\\0}} + 6e_{-\substack{000100\\0}} + 2e_{-\substack{000010\\0}}$.

Let $F = f + \bigoplus_{r>0} \mathfrak{g}_e(-2 + rp)$, which is 6-dimensional. We build a generic element $\tilde{f}$ of F as in Section 3.1 with six variables. Following Step 1 by enforcing the linear equations from $[e, \tilde{f}] = h$ yields

$$\tilde{f} = \overline{f} + x_1 e_{-\substack{123211\\2}} + x_2 e_{-\substack{001100\\1}} + x_2 e_{-\substack{011000\\1}} + x_3 e_{-\substack{000001\\0}} + x_4 e_{\substack{111111\\0}} - x_5 e_{\substack{122110\\1}} + x_5 e_{\substack{112210\\1}} + x_6 e_{\substack{234321\\2}}.$$

On this occasion $C := C_G(e) \cap C_G(h)$ is finite and we move onto Step 3.

Let $M = \mathrm{ad}(\tilde{f})^{p-1}$. We calculate that $R = \mathrm{rank}(\mathrm{ad}(e)^{p-1}) = 13$. So if $\epsilon(\tilde{f}) = f' \preceq e$ for some evaluation map ϵ, we must have that the rank of $\epsilon(M)$ is at most 13.

Ordering the basis of $\mathfrak{g}$ as in Magma, we use the 13×13 submatrix S of M corresponding to the rows r and columns c where

$$r = \{75, 125, 62, 94, 87, 129, 120, 97, 42, 82, 23, 34, 108\},$$
$$c = \{37, 100, 24, 52, 50, 109, 92, 60, 14, 40, 5, 9, 72\}.$$

The submatrix S is upper triangular and all diagonal entries are elements of $\mathbb{F}_p^*$. The only other nonzero entries in S can be found in row one, which is

$$(1 \;\; 0 \;\; 4x_2 \;\; 0 \;\; 0 \;\; 0 \;\; 5x_5 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0).$$

We find that 42 rows of M have zero entries in every column in c, so each of these rows must be zero. An example of such a row is the eighth row of M. In row 8 we find $x_4, 3x_5$ and $-x_6$ in columns $11, 15$ and 70 respectively. It follows that $x_4 = x_5 = x_6 = 0$. Similarly the 133rd row of M then allows us to deduce that $x_1 = x_2 = x_3 = 0$. Thus $\tilde{f} = f$ as required.

4. Proof of Theorems 1.1 and 1.2

Proposition 2.5 shows that for each $e \in \mathcal{V}$ there exists an $\mathfrak{sl}_2$-triple (e, h, f) with $\mathfrak{s} = \langle e, h, f\rangle = \mathrm{Lie}(X)$ for a G-cr subgroup $X < G$ of type A_1. Thus f must be G-conjugate to e and hence $f \in \mathcal{V}$. We have demonstrated in Section 3 that any other $\mathfrak{sl}_2$-triple (e, h', f') with $f' \in \mathcal{V}$ is $C_G(e)$-conjugate to (e, h, f). Therefore $\mathfrak{s}' = \langle e, h', f'\rangle$ is G-conjugate to $\mathfrak{s}$ and hence G-cr.

Now all that remains is to prove that $\mathcal{V}$ is the maximal closed G-stable subvariety of $\mathcal{N}$ satisfying both the monogamy and A_1-G-cr conditions.

For G of classical type, it follows from [GP22, Theorem 1] that $\mathcal{V}$ is maximal with respect to being monogamous. For the A_1-G-cr property, the ingredients are there but let us spell out the details, as these essentially make up the strategy for the groups of exceptional type used below.

Proposition 4.1. *Let G be a simple algebraic group of classical type. Then $\mathcal{V}$ is the maximal closed G-stable A_1-G-cr subvariety of $\mathcal{N}$.*

Proof. Suppose $\mathfrak{X}$ is a closed G-stable A_1-G-cr variety. Let $e \in \mathfrak{X} \setminus \mathcal{V}$. If e is distinguished in a Levi subalgebra $\mathfrak{l} = \mathrm{Lie}(L)$ with L having a factor of type A_{p-1} then Proposition 2.8 shows that e is contained in an $\mathfrak{sl}_2$-triple generating a non-G-cr subalgebra (these non-G-cr subalgebras are also exhibited in [GP22, Section 2.4]).

Therefore we may assume that $e^{[p]} \neq 0$. The discussion before Proposition 2.2 in *ibid.* exhibits an $\mathfrak{sl}_2$-triple (e, h, f) with $f^{[p]} = 0$ and f in $\overline{G \cdot e}$, thus $f \in \mathfrak{X}$. By Lemma 2.9, the $\mathfrak{sl}_2$-subalgebra generated by (e, h, f) must be G-cr. □

For the remainder of the section we assume G is of exceptional type.

Proposition 4.2. *The variety $\mathcal{V}$ is the maximal closed G-stable subvariety of $\mathcal{N}$ satisfying both the monogamy and A_1-G-cr conditions.*

Proof. Let $\mathfrak{X}$ be a G-stable closed subvariety of $\mathcal{N}$ satisfying either the monogamy or A_1-G-cr condition and $\mathfrak{X} \not\subset \mathcal{V}$. By Lemma 2.4, we may assume that $\mathfrak{X}$ contains an orbit from Table 3.

First suppose that there exists $e \in \mathfrak{X}$ which is distinguished in a Levi subalgebra $\mathfrak{l} = \mathrm{Lie}(L)$ with a factor of type A_{p-1}. Then Propositions 2.5 and 2.8 furnish us with two $\mathfrak{sl}_2$-triples (e, h, f) and (e, h', f') such that the first generates a G-cr subalgebra and the second generates a non-G-cr subalgebra. Moreover, f is in the same G-class as e and f' is in the closure of the G-class of e. Hence $\mathfrak{X}$ does not satisfy either condition, a contradiction.

Now suppose there is some minimal neighbour $e \in \mathfrak{X}$ such that $e^{[p]} \neq 0$. By consideration of Table 3 we see in each such case, e is distinguished in some Levi subalgebra $\mathfrak{l} = \mathrm{Lie}(L)$ of $\mathfrak{g}$ for which p is good for L and L has no factors of type A_{p-1}.

From [PS19, Section 2.4] we find an $\mathfrak{sl}_2$-triple (e, h, f) of $\mathfrak{l}$ with $f^{[p]} = 0$. Since L has no factor of type A_{p-1} it follows that $f \in \mathcal{V} \subseteq \mathfrak{X}$. Furthermore, $\mathfrak{s} = \langle e, f\rangle \cong \mathfrak{sl}_2$ is a non-L-cr subalgebra by Lemma 2.9. Hence by Lemma 2.7, $\mathfrak{X}$ does not satisfy the A_1-G-cr condition. Proposition 2.5 yields an $\mathfrak{sl}_2$-triple (f, h', e') which generates a G-cr $\mathfrak{sl}_2$-subalgebra. Therefore, f is contained in two non-conjugate $\mathfrak{sl}_2$-triples. Thus $\mathfrak{X}$ does not satisfy the monogamy condition either.

Finally, let G be of type G_2 and $p = 3$. The only minimal neighbour of $\mathcal{V}$ is the subregular orbit $G_2(a_1)$. A representative for this orbit is $e = e_{\alpha_2} + e_{-3\alpha_1-\alpha_2}$. This is regular in $\mathfrak{m} = \mathrm{Lie}(M)$ where M is the standard subsystem subgroup of type A_2 corresponding to the subsystem $\pm\alpha_2, \pm(3\alpha_1 + 2\alpha_2), \pm(3\alpha_1 + \alpha_2)$.

As in the proof of Proposition 2.8, there exists an $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{m}$ such that $\mathfrak{s} = \langle e, f\rangle$ is non-M-cr. Furthermore, f is in the orbit labelled A_1 (both as an A_2-orbit and G_2-orbit). We claim that $\mathfrak{s}$ is non-G-cr. By Proposition 2.5, the element f is contained in an $\mathfrak{sl}_2$-triple generating a G-cr subalgebra and by the claim, the $\mathfrak{sl}_2$-triple $(f, -h, e)$ generates a non-G-cr subalgebra. Hence $\mathfrak{X}$ does not satisfy either condition.

For the claim, note that $\mathfrak{s}$ is certainly G-reducible since it is non-M-cr. All G-cr $\mathfrak{sl}_2$-subalgebras which are G-reducible are contained in a Levi subalgebra. In this low-rank case, it immediately follows that all such $\mathfrak{sl}_2$-subalgebras are G-conjugate to either $\mathfrak{l}_1 = \langle e_{\pm\alpha_1}\rangle$ or $\mathfrak{l}_2 = \langle e_{\pm\alpha_2}\rangle$. Therefore a G-cr $\mathfrak{sl}_2$-subalgebra only contains nilpotent elements in the A_1 or $\tilde{A}_1$ classes. The claim follows since $\mathfrak{s}$ contains e which is in the subregular class.

□

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School of Mathematics, University of Birmingham, Birmingham, B15 2TT, UK

Email address: s.m.goodwin@bham.ac.uk (Goodwin)

Department of Mathematics, The University of Manchester, Oxford Road, Manchester, M13 9PL, UK

Email address: rachel.pengelly@manchester.ac.uk (Pengelly)

Department of Mathematics, The University of Manchester, Oxford Road, Manchester, M13 9PL, UK

Email address: david.i.stewart@manchester.ac.uk (Stewart)

Department of Mathematics, University of Warwick, Coventry, CV4 7AL, UK

Email address: adam.r.thomas@warwick.ac.uk (Thomas)