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The entanglement of logic and set theory, constructively

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The entanglement of logic and set theory, constructively

Theories of sets such as Zermelo Fraenkel set theory are usually presented as the combination of two distinct kinds of principles: logical and set-theoretic principles. The set-theoretic principles are imposed "on top" of first order logic. This is in agreement with a traditional view of logic as universally applicable and topic neutral. Such a view of logic has been rejected by the intuitionists, on the ground that quantification over infinite domains requires the use of intuitionistic rather than classical logic.

In the following, I consider *constructive* set theories, which use intuitionistic rather than classical logic, and argue that they manifest a distinctive *interdependence* or an *entanglement* between sets and logic. In fact, Martin-Löf type theory identifies fundamental logical and set-theoretic notions. Remarkably, one of the motivations for this identification is the thought that classical quantification over infinite domains is problematic, while intuitionistic quantification is not. The approach to quantification adopted in Martin-Löf's type theory is subtly interconnected with its predicativity. I conclude by recalling key aspects of an approach to predicativity by Poincaré, which focuses on the issue of correct quantification over infinite domains and relate it back to Martin-Löf type theory.

Keywords: intuitionistic logic; set theory; Martin-Löf type theory; predicativity, infinity; quantification; Poincaré.

Introduction

Standard approaches to set theory such as Zermelo Fraenkel set theory (ZF) take classical first order logic as granted and specify what counts as a set by means of a collection of axioms imposed "on top" of the classical predicate calculus.[1] In fact, ZF set theory is often presented as if it were obtained by combining two wholly separable layers: the logical and the set-theoretic axioms. This view of set theory as composed of two layers, first the logic and then the specific set-theoretic axioms, is in agreement with traditional accounts of logic, according to which distinctive characteristics of logic are its *generality* and *universal applicability*. Logic, according to this view, is *topic neutral*: the particular

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[1] The theory ZF is composed of axioms such as, for example, the power set axiom, and schemata such as, for example, the replacement schema. In the following, I use "principle" or "axiom" to refer to either an axiom or a schema.

domain under consideration has no say on the logical principles appealed to when reasoning about its objects.

This view of logic as applicable irrespective of the relevant domain of discourse has not gone unchallenged. Traditionally, intuitionists have taken a critical perspective on the universal applicability of classical logic. A frequent objection is that classical logic is not the correct logic to employ when reasoning about *infinite domains*: for those domains we need to use *intuitionistic logic*. This criticism may be seen as underpinning the thought that different domains of investigation may require different logics. The intuitionist will typically claim that given the pervasive role of infinity in mathematics, *in mathematics* we ought to use intuitionistic logic.

My starting point in this article is the mathematical practice. Under the stimulus of computer application, contemporary mathematics has seen the flourishing of *constructive* mathematics, which employs intuitionistic rather than classical logic.[2] In the following, I review prominent constructive approaches to the concept of set and argue that they question the above image of the relation between sets and logic, rather suggesting a distinctive *interdependence* or an *entanglement* between sets and logic. I consider two kinds of theories: constructive variants of ZF, such as CZF, and Martin-Löf Type Theory, MLTT.[3] Constructive set theories of the ZF family question the very possibility of sharply distinguishing set-theoretical and logical principles. In MLTT one further *identifies* fundamental logical and set-theoretic notions: *proposition* and *set*.[4] Crucially, the identification of propositions and sets in MLTT is deeply interconnected with the computational vocation of this theory, which aims at offering not only a foundational system for constructive mathematics but also a theory for (computer) program construction. As argued below, MLTT's identification of propositions with sets, which goes under the name of Curry-Howard isomorphism, cannot be explained solely with Martin-Löf's desire to develop a computational system for constructive mathematics. A further motive for this

[2] There are a number of variants of mathematics carried out using intuitionistic logic. The best-known constructive practice, and the one I shall refer to in the following, is the mathematics initiated by Bishop (1967). See also (Beeson 1985, Bishop and Bridges 1985, Bridges & Richman 1987, Troelstra & van Dalen 1988). For an introduction and further bibliographic references, see (Bridges and Palmgren 2013).
[3] See (Mhyill 1975, Aczel 1978, Aczel and Rathjen 2008, Martin-Löf 1975, Martin-Löf 1982, Martin-Löf 1984).
[4] I discuss MLTT's notions of proposition and set in section 4.

identification of logical and set-theoretic notions is the desire to fully embrace the intuitionistic meaning of the logical constants.

It is perhaps surprising that in the opening of the book "Intuitionistic type theory" (Martin-Löf 1984), the author explains the choice to develop an *intuitionistic* type theory with a perceived difficulty with classical quantification over *infinite domains* and relates this back to the early 20th century debate on *predicativity*. Predicativity made its appearance in an exchange between Poincaré and Russell, prompted by the discovery of the set-theoretic paradoxes.[5] According to one rendering of this notion, a definition is impredicative if it defines an entity by reference to (e.g. generalization over) a totality to which the entity itself belongs, and is predicative otherwise. Martin-Löf's type theory is said to be predicative as higher order quantification over all propositions, all relations etc. is unavailable within this theory. There is, in fact, a deep connection between MLTT's predicativity and the Curry-Howard isormorphism, as further discussed below. There is therefore a deep connection between MLTT underlying intuitionistic logic and this theory's predicativity.

A strong correlation between predicativity and the issue of legitimate quantification over infinite domains is a theme which appears prominently in the reflection on predicativity by Poincaré (1909, 1912)[6]. Poincaré saw a strong nexus between the insurgence of the paradoxes, impredicativity and what he considered as incorrect treatment of infinite domains. Adopting a traditional view of infinity as potential rather than actual, Poincaré thought that paradoxes like Russell's arise because we treat as actual or completed potentially infinite domains. He therefore proposed predicative restrictions as a way of ensuring a paradox-free treatment of infinite domains. While Poincaré's discussion on predicativity seems to offer a way of explaining the perceived difficulties with classical quantification over infinite domains, the identification of propositions with sets in Martin-Löf's type theory offers a possible way out from these difficulties.

The article is organised as follows. In section 1, I review the standard approach to set theory exemplified by ZF set-theory. I then review significant aspects of constructive Zermelo Fraenkel set theory, CZF, and compare this system with ZF set theory (section 2). The principal claim is that constructive variants of ZF place under strain the very

[5] See e.g. (Poincaré 1905, Poincaré 1906, Poincaré 1909, Poincaré 1912, Russell 1906a, Russell 1906b, Russell 1908). See also (Feferman 2005, Crosilla 2017).

[6] This is also a prominent theme in (Weyl 1918).

separation between set-theoretic and logical principles. Starting from section 3, I discuss Martin-Löf type theory and consider the relation between the logical and set-theoretic notions of proposition and set. In section 5, I discuss the relation between this identification and the computational vocation of type theory. In section 6, I review Martin-Löf's discussion of quantification over infinite domains. Finally, in section 7, I highlight the main traits of an analysis of predicative definitions by the late Poincaré and briefly discuss MLTT's treatment of quantification.

1. Zermelo Fraenkel set theory and its family

In introducing Zermelo Fraenkel set theory, we usually present it as if it were the combination of two wholly distinct layers: the logical and the set-theoretic axioms. The first layer takes the form of the first order predicate calculus, detailing the behaviour of the logical constants and the quantifiers: it specifies legitimate ways of reasoning. The set-theoretic axioms, instead, determine the subject matter of the theory and state which sets the theory is about. More precisely, the set-theoretic axioms may be divided in three kinds of principles: some introduce primitive sets (e.g. the empty set and the set of natural numbers), others specify operations for forming new sets from already given ones (e.g. union, power set, etc.) and others set out the identity criteria for sets (extensionality), as well as properties of the universe of sets (foundation). The two components of ZF, logical and set-theoretic axioms, are typically perceived as separate: the set-theoretic axioms are imposed "on top" of the first order predicate calculus. In fact, standard presentations of ZF focus primarily on the concept of set: sets have a foundational and unifying role to play, as, arguably, any mathematical object can be defined in terms of them.[7] While classical logic is presupposed and given for granted, we need a formal logical calculus to remove possible ambiguities, especially in the formulation of the separation schema.[8] Given this way of presenting set theory, it is tempting to think that the set-theoretic principles we use to codify ZF's concept of set are clearly distinguishable from the background logical ones: their role is to systematize the concept of set, rather than clarify how to reason about sets. This may be further substantiated by observing that a number of variants of ZF set theory have been considered in the literature. These are obtained by

[7] See e.g. (Suppes 1960, Herbaceck and Jech 1999).

[8] The separation schema allows for the definition of subsets of a given set, A, by collecting together all the elements of A that satisfy a given property, expressed by a formula in the language of set theory.

taking a different combination of set-theoretic axioms on the basis of the *same* first order logic. As a way of example, a remarkable subsystem of ZF, Kripke Platek set theory, features prominently in generalised recursion theory, while extensions of ZF by large cardinal axioms are at the forefront of research in set theory. The set-theoretic axioms may be read as specifying which kind of sets one is concerned with (e.g. ZF sets, admissible sets, large cardinals etc). The very possibility of defining *different* set theories on the basis of the *same* logical system may be taken to suggest some form of independence of the logical from the set-theoretic principles.

This way of presenting set theory is in agreement with a traditional conception of logic as universally applicable, wholly general and topic neutral. According to this view, the laws of logic govern the forms of thought irrespective of the particular domain of discourse we are concerned with, they are universally applicable and do not belong to the sphere of some special science.[9] In particular, the logical laws hold of any domain, irrespective of the nature of its elements and, especially, its cardinality.

Very different is the view expressed by the intuitionist. For the latter, there is substantial agreement between intuitionistic and classical forms of mathematics when we consider finite sets; however, a profound divergence appears when we consider infinite sets.[10] From an intuitionistic perspective, classical concepts relating to infinite or transfinite sets become meaningless and may even give rise to inconsistency. For this reason mathematics requires the adoption of intuitionistic rather than classical logic. As further discussed below, a stark opposition between the finitary and the infinitary cases is also at the heart of Poincaré analysis of impredicativity (1909, 1912). A similar thought is clearly expressed by Weyl, who also draws a correlation between the use of *classical logic* on infinite domains and the paradoxes of set theory:

> […] classical logic was abstracted from the mathematics of finite sets and their subsets. … Forgetful of this limited origin, one afterwards mistook that logic for something above and prior to all mathematics, and finally applied it, without justification, to the mathematics of infinite sets. This is the fall and original sin of set theory, for which it is justly punished by the antinomies. It is not that such

[9] See e.g. (Frege 1953).

[10] See, for example, (Brouwer 1912).

contradictions showed up that is surprising, but that they showed up at such a late stage of the game. (Weyl 1946)

More recently, Michael Dummett (1991, 1993) has also argued that quantification over infinite domains requires not classical but intuitionistic logic. As further discussed below, the desire to make sense of quantification over infinite domains figures also in the constructive approach to the concept of set put forth by Martin-Löf.

2. Constructive ZF

Recent decades have seen the rise of variants of ZF which use intuitionistic rather than classical logic, as, for example, Intuitionistic Zermelo Fraenkel set theory (IZF) and Constructive Zermelo Fraenkel set theory (CZF).[11] These theories have been proposed as ways of formalising, in the familiar ZF language, the concept of set that is implicit in the work of constructive mathematicians. Prima facie, here too one has two layers: the logic, intuitionistic in this case, and the set-theoretic axioms. Formulating variants of ZF based on intuitionistic logic, however, requires a careful reconsideration of the ZF axioms, so to avoid to accidentally slide back into classical logic. Particular care is required, for example, in formulating the *foundation* axiom, which, in its standard form, states that every non-empty set has a least element with respect to the membership relation. In fact, on the basis of the other principles of ZF and intuitionistic logic, foundation implies the principle of excluded middle.[12] Similarly, adding the full *axiom of choice* to an intuitionistic variant of ZF gives back the principle of excluded middle, which becomes now provable from the axiom of choice (on the basis of the axioms of ZF and intuitionistic logic). Therefore, the axiom of choice needs to be omitted if we aim at formulating an *intuitionistic* set theory.[13]

[11] Thy system IZF was introduced in (Friedman 1973). It is an impredicative intuitionistic variant of ZF set theory. For bibliographic references on CZF, see footnote 2.

[12] To be more precise, the axiom of foundation implies the principle of excluded middle on the basis of intuitionistic logic and the other axioms of a system, such as IZF, with the unrestricted separation schema. In the case of systems such as CZF, which only admit separation for bounded formulas, one obtains a bounded form of excluded middle: $A \vee \neg A$ for every bounded formula A. Note that a formula, A, is bounded if every occurrence of quantifiers in A is of the form $\exists x \in D$ or $\forall x \in D$, with D a set. See (Aczel and Rathjen 2008, Crosilla 2015).

[13] One may add weaker choice principles such as countable and dependent choice to constructive variants of ZF without giving rise to a classical set theory. It is interesting to observe that while the full axiom of choice cannot be added to intuitionistic systems in the style of ZF, as it gives rise to a classical theory, the situation is different in the case of MLTT. Here a statement expressing

The case of constructive versions of ZF is suggestive of a form of interaction between the logical and the set-theoretic levels: if we are not careful in formulating our set-theoretic principles, we might engender a sudden change in logic, from intuitionistic to classical logic. Let us consider again the case of foundation. The foundation axiom states that the membership relation is well-founded, and thus describes a property of sets: well-foundedness. As a consequence, non-well-founded sets, such as sets which are members of themselves, are ruled out from ZF set theory. Foundation would therefore seem to squarely fit within the set-theoretic component of ZF set theory. For this reason it is surprising that its statement (on the basis of intuitionistic logic and the other axioms of ZF) already implies a logical principle, the principle of excluded middle. In fact, constructively unacceptable instances of the principle of excluded middle can be derived from foundation on the basis of a weak subsystem of CZF, prompting the thought that the axiom of foundation, in its usual formulation, is "inherently classical".

One may be tempted to dismiss this case as constituting insufficient proof of a genuine interplay between sets and logic, since we can replace foundation with a schema, *set induction*, which is classically equivalent to foundation, but does not give rise to the above difficulties in an intuitionistic context. Set induction legitimates reasoning by induction on the membership relation, but it has the advantage of not enforcing classical logic. The thought is that adequate care in formulating our axioms and schemata may suffice to reinstate, even in an intuitionistic context, the separation between logical and set-theoretic principles that we typically envisage in ZF and its classical variants. However, a different situation arises in the case of the axiom of choice, since no equivalent formulation has been proposed which remains neutral with respect to the logic. Only substantial weakening of the axiom of choice (such as countable or dependent choice) may be added to constructive versions of ZF without enforcing the shift to classical logic.[14]

Since the axiom of choice is usually considered set-theoretic rather than logical, it becomes difficult to draw a clear demarcation between set-theoretic and logical principles within constructive theories of the ZF family. This prompts the thought that the set-theoretic component of these theories already embodies a thoroughly *classical* concept of set. These observations are indicative that the very distinction between sets

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the axiom of choice (i.e. an intensional form of the axiom of choice) is in fact provable in virtue of the Curry-Howard isomorphism. See (Martin-Löf 2006) for discussion.

[14] See e.g. (Aczel and Rathjen 2008, Crosilla 2015).

and logic may be more difficult to formulate than it is often thought, and should not be taken for granted. There seems to be, on the contrary, a deep interaction between logic and our concept of set. This is after all unsurprising, as an interaction must arise between the two, since sets act as domains of quantification and the concept of set plays a fundamental role in the specification of the semantics of the quantifiers. In fact, it is in relation to the role of sets as domains of quantification that Martin-Löf type theory introduces a characteristic identification of crucial logical and set-theoretic notions, as further discussed below.

3. Martin-Löf type theory and the Curry-Howard correspondence

In Martin-Löf Type Theory we witness a profound interaction between logic and sets (Martin-Löf 1975, Martin-Löf 1984). Martin-Löf Type Theory is often considered the *canonical* constructive theory, playing a similar role for constructive mathematics as ZFC does for classical set theory. At the heart of MLTT is an *identification* of two kinds of entities: *propositions* and *sets*. This identification is the Curry-Howard isomorphism. In fact, a weakening of it, often termed Curry-Howard (or formulas-as-types) correspondence is typical of a number of intuitionistic type theories. I discuss MLTT's notions of propositions and sets in section 4 below. In the following, I first describe the Curry-Howard correspondence in a particularly simple case.

The Curry-Howard correspondence exploits a structural similarity between intuitionistic logic on the one side and constructive type theories on the other side. The correspondence holds already for the implicational fragment of intuitionistic logic, which is seen to correspond to a formalism known as the *typed lambda calculus*.[15] In this particularly simple case, the Curry-Howard correspondence is motivated by the observation that the behaviour of the intuitionistic implication is analogous to the behaviour of a function or program that given inputs (of some type) produces outputs (of some type).[16] The crucial thought is that the domain and range of such a function are types whose elements are (intuitionistic) proofs of formulas. More precisely, the formulas A

[15] See (Church 1940, Barendregt 1981).

[16] It is important to remark that the notion of type utilised in constructive typed theories does not coincide with the familiar notion of set from classical ZF, as further clarified in section 4. Furthermore, the notion of function is different from that of ZF. In ZF, a function is a graph, i.e. a set of ordered pairs. In constructive type theories one has a primitive notion of function, and functions may also be thought of as algorithms or procedures that given inputs of some type produce outputs of some type.

and B are seen as corresponding to the types of their proofs, which we can denote by $\underline{A}$ and $\underline{B}$ (respectively). An intuitionistic proof of an implication, $A \rightarrow B$, behaves as a *function* which transforms a proof of the formula A, into a proof of the formula B. That is, an intuitionistic proof of $A \rightarrow B$ may also be seen as corresponding to a function which takes as input an element of the type, $\underline{A}$, i.e. a proof of the formula A, and gives as output an element of the type, $\underline{B}$, i.e. a proof of the formula B.[17] Therefore, given the correspondence of A and B with the types of their proofs, the formula $A \rightarrow B$ can be seen as corresponding to the type collecting the proofs of this implication. The latter is the collection of all functions with domain $\underline{A}$ and range $\underline{B}$, and is also written $\underline{A} \rightarrow \underline{B}$.

The significance of this structural correspondence between formulas (and their proofs) and types (and their elements) lies in that the typed lambda calculus is the core of prominent functional programming languages. It is also at the heart of constructive type theories such as Martin-Löf type theory and the calculus of constructions. These can be read simultaneously as foundational systems for constructive mathematics and as high level programming languages.[18] A detailed description of MLTT is beyond the aims of this note, but a brief reminder of its main traits is necessary to clarify the identification of propositions with sets.[19]

4. Sets, propositions and their identification

Sets in MLTT are particular kinds of types, those types which function as *domains of quantification*: (typed) collections of "objects" we can meaningfully quantify over. Martin-Löf traces the origins of his concept of set back to Russell's concept of type as *range of significance of a propositional function* (1984, p. 22). As further clarified below, Martin-Löf substantially departs from Russell, since MLTT is an intuitionistic rather than a classical type theory. A characteristic of type theories such as MLTT is that they are not defined by first setting out intuitionistic predicate logic, and then introducing appropriate set-theoretic principles "on top" of the logical principles. Sets and logic are introduced *simultaneously* through ***rules*** which define both the logical and the relevant set-theoretic

[17] One may also say that in constructive type theories we are interested in *proofs* of formulas as ways of witnessing their truth. For example, a proof of an implication, $A \rightarrow B$, may be seen as an effective way of transforming a witness of the truth of A (i.e. a proof of A) into a witness of the truth of B (i.e. a proof of B). See (Sundholm 1994) for further discussion.

[18] The calculus of constructions is the formal system that underlies the core of the proof assistant Coq (Coquand 1985, Coquand and Huet 1988).

[19] See e.g. (Martin-Löf 1984, Nordstrom, Petersson and Smith 1990).

notions. In particular, sets in MLTT are introduced by rules which specify what counts as an element of a set and when two such elements are equal. In virtue of the Curry-Howard isomorphism, the very same rules can also be read as specifying what is a proposition in terms of what counts as a proof of that proposition.

Similarly as in the case of ZF, we may distinguish three groups of rules: those which introduce new sets (e.g. the set of natural numbers), rules which combine previously given sets (e.g. the Cartesian product) and rules which determine relevant identity criteria between sets (equality rules). Within the formalism itself, one has four kinds of rules for each new set or set constructor; for example, the natural number set is defined by four rules and so is the Cartesian product. The rules are: *formation*, *introduction*, *elimination* and *equality*. The *formation rules* declare which sets there are. For example, they state that there is a set N of natural numbers, or that if A and B are sets, then so are A×B and A→B. One has also rules for the so-called generalised Cartesian product, Π(A,B), and the generalised disjoint union, Σ(A,B), which endow MLTT with particular expressive strength.[20]

Key rules are the *introduction* and *elimination* rules. The purpose of the introduction rules is to declare what is a set, by specifying its (*canonical) elements*.[21] For example, in the case of the natural numbers, N, a canonical element is either 0 or it is of the form suc(n), for n canonical element of N (where "suc" stands for the successor operation). In the case of the set A × B (the Cartesian product of the sets A and B), a canonical element is a pair, whose first component is an element of A and whose second component is an element of B. The elimination rules explain how to use the elements of a set, that is, they explain how to define functions on the set defined by the introduction rules. Given the identification between propositions and sets, it is unsurprising that most rules governing set-introduction and set-elimination are in fact similar to the natural deduction rules for the introduction and elimination of the connectives and quantifiers in *intuitionistic* predicate logic. In fact, as further discussed below, MLTT's rules are more

[20] The rules for the generalised Cartesian product and the generalised disjoint union make use of the notion of family of sets over a given set: for example, for set A, B(x) is a family of sets over A if B(x) is a set for each element x of A. Under the Curry-Howard correspondence, the generalised Cartesian product, Π(A,B), and the disjoint union, Σ(A,B), of a family of sets correspond to the universal and existential quantifiers (respectively). These constructors therefore allow us to extend the Curry-Howard correspondence from propositional to full predicate logic.

[21] Canonical elements are prototypical or standard elements of a set. Sets typically contain also non-canonical elements, which can be brought to canonical form through application of the elimination rules.

informative than the corresponding natural deduction rules, as they also carry information on proofs. The equality rules, finally, furnish the relevant identity criteria for sets, by showing how a function defined through the elimination rule acts on the canonical elements of the set which were specified in the introduction rules.

Propositions play a similar role within Martin-Löf's system as formulas of ZF: for example, we may combine propositions by means of the logical operations to give rise to more complex propositions. Propositions may therefore be understood as the key logical notion within type theory. A substantial difference with formulas of ZF, however, is that propositions are *mathematical objects*, that is, in MLTT we reason about them within the "object language", rather than in the "metalanguage".[22] In MLTT one identifies propositions and sets. As a consequence, the rules for sets may also be read as rules for propositions. For instance, the introduction and elimination rules for $A \times B$ may also be read as the natural deduction rules for conjunction introduction and elimination (respectively).[23] The crucial difference with the usual natural deduction rules for intuitionistic logic, however, is that rules in MLTT are more informative. For simplicity, let us consider the introduction rules only. If we read an introduction rule as introducing a set, then, as we have seen above, the rule specifies the (canonical) elements of that set (and when two such elements are equal). In virtue of the Curry-Howard correspondence between propositions and sets, we may read the same set-introduction rule as introducing a proposition, whose proofs belong to the set. The rule now is read as specifying (canonical) proofs of the proposition.[24] For instance, if we read $A \times B$ not as a set but as a proposition, i.e. as the conjunction $A \wedge B$, its introduction rules first of all state that if A and B are true then so is $A \wedge B$. More importantly, the rules also specify a (canonical) proof of this proposition: a pair whose components are a proof of A and a proof of B. This additional information (compared with standard natural deduction rules) is at the heart of MLTT's computational interpretation.[25] Clearly, if we are only interested in recovering

[22] More precisely, in type theory one distinguishes between *propositions* and *judgements*. The first ones are objects within the theory; we combine them by means of the logical operations. The premises and conclusion of a logical inference are instead judgements (Martin-Löf, 1984, p. 3). For example, if A and B are propositions, the judgment "$A \vee B$ is true" asserts the truth of the proposition "$A \vee B$".

[23] Note that the Cartesian product of two sets, A and B, is obtained by applying the rules for the Σ type constructor, which corresponds to the existential quantifier.

[24] The notion of canonical proof may be explained in terms of direct proof (Martin-Löf 1987).

[25] See, for example, the discussion in (Martin-Löf 1982).

the natural deduction rules, we can ignore or "forget" the additional information.[26] For instance, to ascertain the truth of the proposition A ∧ B, it suffices to know that there is a proof of it (or that the set of its proofs is non-empty): we can ignore the additional information detailing the specific form of its proof.

In (Martin-Löf 1975, 1984), the Curry-Howard isomorphism is presented by observing that propositions and sets follow the same pattern of introduction and elimination rules and, on this ground, may be identified. The idea further developed in (Martin-Löf 1982) is that the correspondence makes the computational or algorithmic content of constructive proofs fully explicit, so that we can use MLTT as a very general programming language.[27] This is discussed next.

5. Computational content and predicativity

The Curry-Howard correspondence is a fundamental component of a number of intuitionistic type theories, including the type lambda calculus mentioned above. Most discussions of the Curry-Howard correspondence focus on a weaker claim compared with MLTT's identification of propositions and sets:

(1) every proposition gives rise to a set, the set of its proofs.

This turns out to suffice for a computational understanding of constructive type theories. As a way of clarifying the significance of (1), we can read a proposition as setting out a task or a problem.[28] For example, the proposition stating that every even natural number is the double of some natural number, may be seen as setting out the following task: find the half of any even natural number. If we give a *constructive* proof of this proposition, we obtain a step-by-step specification of a procedure that enables us to compute the half of any even natural number. In other terms, a constructive proof of this proposition produces an algorithm which calculates a solution to the problem set by that proposition. By (1), to a proposition corresponds a set, the set of its proofs. But this means that a proof of the above mathematical statement on even natural numbers, gives rise to a function within type theory. A function in constructive type theory is, after all, an algorithm. In this case, an algorithm that takes even natural numbers as inputs and produces natural

[26] See also (Sambin, Valentini 1998).
[27] This thought has been brought substantially forward in recent years through the development of proof assistants such as Nuprl, Agda and Coq.
[28] See (Martin-Löf 1984).

numbers as output, their half. In other terms, by working in a constructive type theory that satisfies (1), we produce proofs of mathematical statements and, crucially, these proofs are algorithms that can be used to obtain real computer programs.

A distinctive characteristic of Martin-Löf's type theory is that it strengthens statement (1) to a full *identification* of propositions and sets. Not only each proposition gives rise to a set, but each proposition *is* a set:

> […] there appears to be no fundamental difference between propositions and types. Rather the difference is one of point of view: in the case of a proposition, we are not so much interested in what its proofs are as in whether it has a proof, that is, whether it is true or false, whereas in the case of a type, we are of course interested in what its objects are and not only in whether it is empty or nonempty. (1975, p 77)[29]

The identification of propositions and sets that is characteristic of MLTT has far-reaching consequences for the ensuing concept of set that this theory codifies. In fact, combining such an identification with a form of impredicativity gives rise to Girard's paradox.[30] Impredicativity here takes the form of quantification over propositions. More precisely, we may consider an impredicative extension of MLTT obtained by postulating that the collection of all propositions, that we call "Prop", is itself a set. This gives rise to inconsistency for the following reason: if Prop is a set, i.e. a domain of quantification, and sets are identified with propositions, then Prop acts as the set of all sets or as a universal domain of quantification. A set of all sets figured in a first inconsistent version of MLTT, quickly rectified after the discovery of Girard's paradox. One may then read Girard's paradox as proving the incompatibility of the above form of impredicativity with the identification of propositions and sets. Given such an incompatibility, there are therefore two options: (a) to hold to the identification of propositions and sets and renounce to impredicative quantification, or (b) to renounce to the above identification and allow instead impredicativity.[31] Martin-Löf type theory opts for (a), as discussed in the next section. Option (b) has been pursued with the calculus of constructions, which is an impredicative constructive type theory. To avoid Girard's paradox, the calculus of constructions distinguishes between propositions and sets, which are introduced by

[29] Note that above I have followed (Martin-Löf 1984) amd called "set" what in this quotation Martin-Löf calls type.
[30] See (Girard 1972). See also (Coquand 1986, Jacobs 1989) for discussion.
[31] See (Coquand 1989) for an analysis.

separate sets of rules. However, to preserve the computational character of constructive proofs, the calculus of constructions does comply with the Curry-Howard correspondence in the weaker form of (1): instead of the principle that every proposition *is* a set, we simply assume that *for every proposition, there is a corresponding set*, i.e. the set of its proofs. Since propositions and sets are now distinct, this seems sufficient to avoid Girard's paradox while preserving the computational advantages of the Curry-Howard correspondence. In addition, one also gains more expressive power compared with MLTT, due to impredicative quantification.

6. Martin-Löf type theory, infinite domains and predicativity

In MLTT, one takes the narrower route, which avoids impredicativity. Two main components seem to be in play in determining this choice: (i) the thought that a thorough intuitionistic approach requires compliance with the Curry-Howard isomorphism, and (ii) the worry often expressed by the intuitionist that classical quantification over infinite domains lacks meaning. As to the first point, Martin-Löf (2008) clarifies that the Curry-Howard isomorphism:

> […] was to me from the beginning the natural completion of the Brouwer-Heyting-Kolmogorov interpretation of intuitionistic first-order predicate logic, just drawing the full consequences, so to say, of the Brouwer-Heyting-Kolmogorov interpretation.

The thought is that full adherence to intuitionistic logic requires the identification of propositions and sets.[32] By Girard's paradox, compliance with this strong form of the Curry-Howard correspondence requires also compliance with predicativity. In fact, the two issues, (i) and (ii), are intimately connected. Martin-Löf (2008) writes:

> Now the paradox that Girard discovered in 1971 (Girard 1972), in the first version of constructive type theory, then called intuitionistic type theory, showed that Curry-Howard is incompatible with impredicativity […]. So one of them has to go: either Curry-Howard has to go or there is some problem with impredicativity, with which there had been problems from the very beginning: when the notion itself was

[32] A detailed discussion of this point would require an analysis of the most distinctively philosophical component of MLTT, its meaning explanations. Due to space constraint, this will need to be postponed to another occasion. See also (Sundholm 1986).

> introduced by Russell in 1906 (Russell 1906), it was precisely because it was a problematic notion.

In his 1984 book (page 1), Martin-Löf also refers back to the origins of predicativity and mentions the perceived difficulty with classical quantification over infinite domains. The author begins by recalling the difficulties caused by the introduction of the axiom of reducibility in *Principia Mathematica* and explains that this left us with the simple theory of types, whose official justification rests on the interpretation of propositions as truth values.

> The laws of the classical propositional logic are then clearly valid, and so are the quantifier laws, as long as quantification is restricted to finite domains. However, it does not seem possible to make sense of quantification over infinite domains, like the domain of natural numbers, on this interpretation of the notions of proposition and propositional function. For this reason, among others, what we develop here is an intuitionistic theory of types, which is also predicative (or ramified).

Like the intuitionist, Martin-Löf claims that there are problems with classical quantification over infinite domains, and proposes the shift to intuitionistic logic as a solution. At page 11 of (Martin-Löf 1984), we read:

> Because of the difficulties of justifying the rules for forming propositions by means of quantification over infinite domains, when a proposition is understood as a truth value, this explanation is rejected by the intuitionists and replaced by saying that a proposition is defined by laying down what counts as a proof of that proposition, and that a proposition is true if it has a proof, that is, if a proof of it can be given.

This then brings us back to the Curry-Howard correspondence (Martin-Löf 1984, p. 13):

> If we take seriously the idea that a proposition is defined by laying down how its canonical proofs are formed […] and accept that a set is defined by prescribing how its canonical elements are formed, then it is clear that it would only lead to unnecessary duplication to keep the notions of proposition and set (and the associated notions of proof of a proposition and element of a set) apart. Instead, we simply identify them, that is, treat them as one and the same notion. This is the

formulae-as-types (propositions-as-sets) interpretation on which intuitionistic type theory is based.[33]

The thought that we need to change logic to make sense of quantification over infinite domains clearly disagrees with the traditional view of classical logic as universally applicable and topic neutral which was discussed in section 1. In fact, the infinitary character of mathematical domains has here a say on which logic we chose to employ in our mathematical practice. A remarkable aspect of Martin-Löf's approach to the issue of quantification over infinite domains is that it enforces not only a shift to intuitionistic logic, but also the theory's compliance with predicativity.

The complaint that classical quantification is only suitable in the case of finite domains is a prominent element of the intuitionistic criticism of classical logic. Although Martin-Löf refers to Russell, concerns for quantification over infinite domains are at the heart of the other key historical figure of predicativity, Poincaré (1909, 1912). Martin-Löf does not expand on what makes classical quantification over infinite domains problematic. In the following, I wish to briefly recall the main traits of Poincaré's reflection on predicativity (Poincaré 1909, 1912).[34] I will extract a few significant points from Poincaré's analysis which may help us clarify the perceived difficulty with classical quantification over infinite domains and the concomitant requirement of compliance with predicativity. I will then go back to the case of Martin-Löf's type theory.

7. Poincaré's logic of infinity

The focus of Poincaré (1909, 1912) is the question whether standard ways of reasoning can be applied to infinite collections. The discussion centres on problematic impredicative definitions, which typically specify an entity, say *X*, by reference to *all* the elements of a collection to which *X* belongs. Poincaré's worry is that in case the collection we use to define *X* is infinite, it may have an element of instability: it seems that by reference to it we may give rise to a "new" element of it, namely *X*.

Let us clarify this point with an example. Suppose we define an entity, *X*, by

[33] A similar view is presented in (Martin-Löf 1975, p. 76-7), where the author writes: "it will not be necessary to introduce the notion of proposition as a separate notion because we can represent each proposition by a certain type, namely, the type of proofs of that proposition."

[34] A thorough analysis of Poincaré's contribution to the debate on predicativity may be found in (Heinzmann 1985). See also (Feferman 2005) for a survey on predicativity.

postulating a relation *R* between this entity and all the elements of an infinite collection defined by a general condition *G*. Furthermore, suppose that *X* itself satisfies condition *G*. Poincaré claims that *X*'s definition is (in general) illegitimate. The thought is that since *X* is defined by postulating a relation, *R*, with *all* of the *G*s, *X'*s definition would require us to first fix the extension of *G*, i.e. the collection of all the objects that satisfy *G*. However, as *X* turns out to be an element of the extension of *G*, it is as if *X*'s definition would generate a new element, *X,* of *G's* extension. Poincaré also expresses this by claiming that the definition of *X* may *modify* *G*'s extension and *disorders* *G* itself.

Poincaré's talk of sets being "modified" and "disordered" by the introduction of "new" elements is clearly problematic from a contemporary classical point of view. It seems to introduce a dynamic component to the concept of set that is extraneous to the classical conception. More should be said on this issue. In the following, I focus on two main components that can be extracted from Poincaré's discussion and may offer a way of making sense of the perceived difficulty with quantification over infinite domains.

The first component has to do with a potentialist conception of infinity that is at the heart of Poincaré's discussion. According to Poincaré, infinity is unboundedness, or the possibility of extending a finite set at will. Furthermore, the case of infinite domains is importantly different from the case of finitary domains. This contrasts with a common understanding of Cantorian set theory: the analogy between finite and infinite sets is often taken to be at the heart of Cantorian set theory's acceptance of actual infinity. Poincaré rejected actual infinity and thought that the illegitimate assumption of actual infinity gave rise to well-known paradoxes such as Russell's: we treat an infinite (and hence unbounded) set as if it had been completed. Let us consider once more the example above of the definition of *X*. As X is defined in terms of a relation with all of the G's, it seems that we would need to fix *G*'s extension in order to make sense of the relation *R* between *X* and all the *G*'s. However, for Poincaré, if *G*'s extension is an infinite set, this is problematic. The thought seems to be that in the finitary case we may assume that the elements of *G*'s extension are already available somehow independently of the defining condition *G*. Therefore fixing *G*'s extension would seem to be in general unproblematic. However, when dealing with an infinite set, we cannot presuppose, for Poincaré, the availability of all the elements of the set independently of its definition. This is because of the unbounded nature of infinite sets. The consequence is that a finitary defining condition is now the only tool we have for characterising the set; an enumeration of all its elements is not an option. In the case of an infinite domain, therefore, we have no more than the definition of the set to help us making sense of quantification over it: we cannot presuppose that all the elements of the set are already available. To further clarify this point, it is useful to consider the second component of Poincaré's analysis of impredicativity, its concept of set.

The concept of set underlying Poincaré's reflection is very different from the familiar concept of set of ZF set theory. One may consider Poincaré's concept of set as essentially a variant of the traditional concept of set as extension of a concept.[35] Simplifying considerably, in the following, I take a set to be the *extension* of a definition or a condition (i.e. the collection of all the entities that satisfy that definition or condition). If we take this concept of set and combine it with a potentialist view of infinity as described above, we are brought to consider infinite sets whose extension is given exclusively in terms of the defining condition, say *G*. More specifically, an infinite set is not obtained by separating or selecting all the elements of the set from a given universe of sets; it arises from the possibility of always extending an initial finite fragment of it. Now, to make sense of quantification over an infinite set, its definition needs to be sufficiently informative. In the example of *X* and *G* above, *G* itself needs to provide enough information so that we can define *X* by postulating a relation *R* between *X* and *all* of the *G*'s. This may be expressed by saying that *G* needs to offer a stable or *invariant* definition of its extension, one which determines once and for all what belongs to the set and which relations hold between the elements of the set. The possibility of "new" elements appearing at a "later stage" and "disordering" the set should not occur. Poincaré (1912) hints at a genetic definition of sets, starting from a description of some initial elements, which then are used to construct (the description of) new ones, and so on. In this way we definitely prescribe what belongs to a set, and the possibility of subsequently modifying the extension of its definition does not arise. In contemporary logical terms, going beyond Poincaré's remarks, we may say that a definition of an infinite set needs to give an *invariant* or absolute specification of all of its elements, starting from uncontroversial initial elements. If we have such an invariant specification of a set, generalization over it becomes meaningful even if the set is infinite, since each element of the set can be specified (at least in principle) and no "new" elements will accidentally slip in and disorder the set.

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[35] See also (Parsons 2002).

8. Back to Martin-Löf's type theory

Let us go back now to Martin-Löf's type theory. Here sets are defined by introduction and elimination rules which specify what a set is in terms of its elements (and give suitable identity conditions). It is tempting to think that sets in type theory make Poincaré's requirement of genetic definition of a set from its elements fully precise: a set's definition gives not only a general membership condition, say *G*, that selects all and only the entities that satisfy *G*, but it gives rules that specify its canonical elements. For example, the most fundamental infinite set, the set of the natural numbers, is given by introduction rules that describe the canonical elements of this set: 0 and the successor of any element of the natural numbers set.[36]

Let us see now how quantification works in MLTT, including the case of infinite domains, such as the set of all the natural numbers. Given the Curry-Howard correspondence, quantification over a set is obtained by an application of the rules for the generalised Cartesian product, Π (universal quantifier), and the generalised disjoint union, Σ (existential quantifier). For example, a universally quantified statement of the form $(\forall x \in N)\ B(x)$ arises as particular reading of the generalised Cartesian product Π(N,B). An element of this type is a function that for each element *n* of N produces an element of B(*n*). By the Curry-Howard correspondence, this is a function that transforms a proof that *n* is an element of N (a natural number) to a proof of B(*n*). Martin-Löf (1984, p. 34) stresses the agreement of this understanding of the universal quantifier with the usual BHK interpretation of the quantifiers, according to which a proof of the above statement is a *method* which takes an *arbitrary* element *n* of N into a proof of B(*n*). Now, this reading of the universal quantifier does not require the prior full determination of the domain of quantification, N. For quantification to make sense, we do not need to presuppose that the extension of the condition defining a set (in this case the set of natural numbers) be fully determined. All what is required is a uniform method that taken an *arbitrary* element *n* of N produces a proof of B(*n*). This would suggest that we can make sense, intuitionistically, of a universal statement, even in the case in which the domain is infinite.

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[36] Also the real numbers can be represented in type theory. For example, by applying the rules for the constructor Σ (and other constructors) to the natural number set, one can express the notion of set of Cauchy real numbers in MLTT.

Conclusion

In this article, I have reviewed a traditional picture of set theory, often implicit in introductory expositions of ZF set theory, for which classical first order logic is taken for granted, and a collection of specific set-theoretic axioms is introduced "on top" of it. I have argued that a comparison with constructive theories of sets questions the very possibility of drawing a clear demarcation between logical and set-theoretic principles. I have discussed two distinct cases: first of all the case of set theories, like CZF, which share their language with ZF, and secondly the case of MLTT. In the first case, the surprising observation is that the axioms of foundation and choice, which appear at first sight to be squarely set-theoretic, also act as logical principles.

In the case of MLTT, one witnesses an even stronger form of interaction between logical and set theoretic notions, in the form of the Curry-Howard isormorphism which identifies propositions with sets. This identification is at the heart of MLTT's predicativity and suggests an entanglement of logic and set theory. I have argued that a worry about classical quantification over infinite domains appears to be one of the motivations for MLTT's use of intuitionistic logic, and, consequently, its compliance with the Curry-Howard isomorphism. This bears similarities to the traditional intuitionist rejection of classical quantification over infinite sets. For the intuitionist, mathematics is primarily concerned with the infinite, and the latter requires the use of intuitionistic logic. I have noted that this profoundly differs from a traditional conceptions of classical logic as universally applicable and topic neutral.

A concern for generalizations over infinite domains is also at the heart of Poincaré's discussion on predicativity. Poincaré departs from a potentialist view of infinity and combines it with a concept of set as extension of a definition. This brings him to require a regimentation of definitions in mathematics, to avoid problematic impredicative definitions. The French mathematician hints at a genetic definition of sets by a step by step construction from its elements. I have suggested that MLTT may be seen as making precise sense of Poincaré's suggestion of predicative and genetic construction of sets from their elements.

To conclude, the above analysis questions not only the traditional view of classical logic as universally applicable and topic neutral, but the very possibility of clearly demarcating logic from set theory. The entanglement between logic and set theory prompted by constructive set theories deserves to be investigated in greater depth. Its analysis is bound to shed light on the relation between logic and sets beyond the constructive case.

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