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Baker, Simon; Yu, Han

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# Root Sets of Polynomials and Power Series with Finite Choices of Coefficients

Simon Baker<sup>1</sup> · Han Yu<sup>2</sup>

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**Abstract** Given  $H \subseteq \mathbb{C}$  two natural objects to study are the set of zeros of polynomials with coefficients in  $H$ ,

$$\left\{ z \in \mathbb{C} : \exists k > 0, \exists (a_n) \in H^{k+1}, \sum_{n=0}^k a_n z^n = 0 \right\},$$

and the set of zeros of a power series with coefficients in  $H$ ,

$$\left\{ z \in \mathbb{C} : \exists (a_n) \in H^{\mathbb{N}}, \sum_{n=0}^{\infty} a_n z^n = 0 \right\}.$$

In this paper, we consider the case where each element of  $H$  has modulus 1. The main result of this paper states that for any  $r \in (1/2, 1)$ , if  $H$  is  $2 \cos^{-1}(\frac{5-4|r|^2}{4})$ -dense in  $S^1$ , then the set of zeros of polynomials with coefficients in  $H$  is dense in  $\{z \in \mathbb{C} : |z| \in [r, r^{-1}]\}$ , and the set of zeros of power series with coefficients in  $H$  contains the annulus  $\{z \in \mathbb{C} : |z| \in [r, 1)\}$ . These two statements demonstrate

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✉ Han Yu  
hy25@st-andrews.ac.uk  
Simon Baker  
simonbaker412@gmail.com

<sup>1</sup> Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

<sup>2</sup> School of Mathematics and Statistics, University of St Andrews, St Andrews KY16 9SS, UK

quantitatively how the set of polynomial zeros/power series zeros fill out the natural annulus containing them as  $H$  becomes progressively more dense.

**Keywords** Root set · Littlewood polynomials · Unimodular polynomials

**Mathematics Subject Classification** Primary 30B30; Secondary 30C15 · 11C08

## 1 Introduction

Let  $H \subseteq \mathbb{C}$  be a finite set. Given such a  $H$  we define the root set of polynomials with coefficients in  $H$  to be:

$$R(H) := \left\{ z \in \mathbb{C} : \exists k > 0, \exists (a_n) \in H^{k+1}, \sum_{n=0}^k a_n z^n = 0 \right\}.$$

Similarly, we define the root set of power series with coefficients in  $H$  to be

$$R^*(H) := \left\{ z \in \mathbb{C} : \exists (a_n) \in H^{\mathbb{N}}, \sum_{n=0}^{\infty} a_n z^n = 0 \right\}.$$

The study of the sets  $R(H)$  and  $R^*(H)$  can be dated back to Littlewood [6] who studied the case where  $H = \{-1, 1\}$ . Since then many related works have appeared, most notable amongst these are the number theoretic results of Beaucoup, Borwein, Boyd and Pinner [3], and Borwein, Erdélyi and Littmann [4], who studied the distribution of roots and multiple roots. Related work also appeared in Bousch [5], where it was shown that  $R(\{-1, 1\})$  is dense in  $\{z : |z|^4 \in [1/2, 2]\}$ . In Shmerkin and Solomyak [8] some measure theoretic and topological properties of  $R(\{-1, 0, 1\})$  are studied in detail.

In what follows, we will adopt the following notational conventions:

$$S^r := \{z \in \mathbb{C} : |z| = r\}, \quad B(z, r) := \{z' \in \mathbb{C} : |z' - z| < r\},$$

and given some interval  $I$  in  $\mathbb{R}$  let

$$A_I := \{z \in \mathbb{C} : |z| \in I\}.$$

In this paper, we focus on the case where  $H$  is a subset of the unit circle  $S^1$ . Under this assumption it is straightforward to show that

$$R(H) \subseteq A_{[1/2, 2]} \text{ and } R^*(H) \subseteq A_{[1/2, 1]}.$$

Intuitively, one might expect that if we allowed  $H$  to become a progressively more dense subset of  $S^1$ , then  $R(H)$  and  $R^*(H)$  would begin to fill out their respective annuli. The main result of this paper shows that this intuition is correct.

Before stating this result we need to define a metric on  $S^1$  to properly quantify the density of  $H$ . Given  $e^{i\theta}, e^{i\theta'} \in S^1$  let  $d(e^{i\theta}, e^{i\theta'}) = \min\{|\theta - \theta'|, |2\pi - (\theta - \theta')|\}$ . This metric measures the interior angle of the sector of  $S^1$  determined by the two radii  $e^{i\theta}$  and  $e^{i\theta'}$ .

**Theorem 1.1** Fix  $r \in (1/2, 1)$ . Suppose  $H \subseteq S^1$  is  $2 \cos^{-1}(\frac{5-4r^2}{4})$ -dense. Then  $A_{[r,1]} \subseteq R^*(H)$  and  $R(H)$  is dense in  $A_{[r,r^{-1}]}$ .

The sets  $R(H)$  and  $R^*(H)$  are related by the following formula.

**Proposition 1.2** Let  $H \subseteq \mathbb{C}$  be any finite set, then the following relations hold:

$$R(H) = \frac{1}{R(H)},$$

and

$$\overline{R(H)} \cap B(0, 1) = R^*(H) \cap B(0, 1).$$

In the statement of Proposition 1.2,  $\overline{A}$  denotes the closure of a set  $A$ , and  $\frac{1}{A}$  denotes the set  $\{z \in \mathbb{C} : z^{-1} \in A\}$ .

*Proof* Given  $z \in \mathbb{C}$  suppose there is polynomial  $P \in H[x]$  such that,

$$P(z) = \sum_{n=0}^k a_n z^n = 0.$$

We can construct another polynomial  $Q \in H[x]$  such that  $Q(1/z) = 0$ . Just consider  $Q(x) = x^k P(\frac{1}{x})$  with  $k = \deg P$ . Therefore, whenever  $z \in R(H)$  we also have  $1/z \in R(H)$ . This proves our first relation.

Now we shall show that,

$$\overline{R(H)} \cap B(0, 1) = R^*(H) \cap B(0, 1).$$

Without loss of generality we can assume that  $H \subseteq \{z : |z| \leq 1\}$ . If  $z^* \in \overline{R(H)} \cap B(0, 1)$  then we can find a sequence  $(z_i) \in R(H)^{\mathbb{N}}$  with:

$$z_i \rightarrow z^*.$$

Moreover, since  $z^* \in B(0, 1)$  there exists a positive number  $M$  such that  $1 < M < \frac{1}{|z^*|}$ . Now let us consider any polynomial in  $H[x]$

$$P(x) = \sum_{n=0}^k a_n x^n.$$

The following result holds for  $|z_i| \leq M|z^*|$

$$\begin{aligned}
 |P(z^*) - P(z_i)| &\leq \sum_{n=0}^k |a_n| |(z^*)^n - z_i^n| \\
 &= \sum_{n=0}^k |a_n| |z^* - z_i| |(z^*)^{n-1} + (z^*)^{n-2}z_i + \dots + z_i^{n-1}| \\
 &\leq \sum_{n=0}^k |a_n| |z^* - z_i| n(M|z^*|)^{n-1} \\
 &\leq |z^* - z_i| \sum_{n=0}^k |a_n| n(M|z^*|)^{n-1}.
 \end{aligned}$$

Since  $|a_n| \leq 1$  and  $M|z^*| < 1$  the latter summation can be bounded uniformly with respect to  $k$ , namely:

$$\sum_{n=0}^k |a_n| n(M|z^*|)^{n-1} \leq C,$$

where  $C > 0$  is a constant that only depends upon  $z^*$ .

Each  $z_i$  is the root of some polynomial  $P_i \in H[x]$ , in which case by the above, for  $i$  sufficiently large we have

$$|P_i(z^*)| = |P_i(z^*) - P_i(z_i)| \leq C|z^* - z_i|. \tag{1}$$

For the sequence  $(P_i)$  there is either a uniform upper bound for the degrees of the  $P_i$ , or there exists a subsequence along which the degrees tend to infinity. In the first case, there must exist a polynomial  $Q \in H[x]$  and a subsequence  $(P_{i_j})$  such that  $P_{i_j} = Q$  for all  $i_j$ . By (1) we must then have  $Q(z^*) = 0$ . Suppose  $\deg Q = L$ , then

$$T(x) = Q(x) \sum_{n=0}^{\infty} x^{n(L+1)}$$

is a power series with digits in  $H$ . For this particular power series we clearly have  $T(z^*) = 0$ . Therefore, in the first case we have  $z^* \in R^*(H)$ . Now suppose there exists a subsequence  $(P_{i_j})$  such that  $\deg P_{i_j} \rightarrow \infty$ . Via a diagonalisation argument, one can assume without loss of generality that there exists a sequence  $(a_n) \in H^{\mathbb{N}}$  and an increasing sequence of natural numbers  $(l_n)$ , such that for all  $i_j \geq l_n$  the coefficient of the degree  $n$  term of  $P_{i_j}$  is  $a_n$ . In other words, as the  $i_j$  become sufficiently large the lower order terms of the  $P_{i_j}$ 's start to coincide. It follows from (1) then that for this sequence  $(a_n)$  we must have

$$\sum_{n=0}^{\infty} a_n(z^*)^n = 0.$$

Therefore,  $z^* \in R^*(H)$  and  $\overline{R(H)} \cap B(0, 1) \subseteq R^*(H) \cap B(0, 1)$ .

Now suppose  $z^* \in R^*(H) \cap B(0, 1)$ . Then there is a sequence  $(a_n) \in H^{\mathbb{N}}$  such that

$$\sum_{n=0}^{\infty} a_n(z^*)^n = 0.$$

This series is absolutely and uniformly convergent in  $B(0, c)$  for any  $0 < c < 1$ . Since  $z^* \in B(0, 1)$  it is contained in one of these sets for  $c$  sufficiently close to 1. We see that the function:

$$P(x) = \sum_{n=0}^{\infty} a_n x^n$$

is holomorphic on the interior of the unit disc, and therefore, the roots of  $P$  must form a discrete set. Since  $z^*$  belongs to the root set of  $P$  there must exist  $r > 0$  such that

$$\{z \in \mathbb{C} : P(z) = 0\} \cap B(z^*, r) = \{z^*\}.$$

Now, suppose that  $z^* \notin \overline{R(H)}$ , then there exists a ball  $B(z^*, r')$  such that  $B(z^*, r') \subseteq B(z^*, r)$  and

$$R(H) \cap \overline{B(z^*, r')} = \emptyset.$$

We can then consider the following integral with  $P_N(x) = \sum_{n=0}^N a_n x^n$

$$I_N = \int_{\partial B(z^*, r')} \frac{P'_N(x)}{P_N(x)} dx.$$

By our conditions on  $r'$  we see that  $P_N$  has no zeros in  $\overline{B(z^*, r')}$  for all  $N \in \mathbb{N}$ . Therefore, by the argument principle (see [1, p. 152]) we must have  $I_N \equiv 0$ . One can also assume that  $r'$  is sufficiently small so that  $P_N$  converges to  $P$  absolutely and uniformly. Therefore,

$$0 = \lim_{N \rightarrow \infty} I_N = \int_{\partial B(z^*, r')} \frac{P'(x)}{P(x)} dx.$$

However, it follows from another application of the argument principle, and the fact that  $P(x)$  has a single zero in  $\overline{B(z^*, r')}$  at  $z^*$ , that the above integral cannot be 0. This contradiction implies  $z^* \in \overline{R(H)}$  and our proof is complete. □

It is natural to wonder whether there exists a set  $H$  such that the sets  $R(H)$  and  $R^*(H)$  fill up their ambient annuli, that is  $A_{[1/2,2]}$  and  $A_{(1/2,1)}$  respectively. In fact such a  $H$  cannot exist. For any  $H \subseteq S^1$  there exists  $z \in \mathbb{C}$  with modulus  $1/2$  and  $\delta > 0$ , such that  $R(H) \cap B(z, \delta) = \emptyset$  and  $R^*(H) \cap B(z, \delta) = \emptyset$ . This is because of the following simple reasoning. Since  $H$  is a finite set there exists  $z \in \mathbb{C}$  such that  $|z| = 1/2$  and

$$|a_i + a_j z| > 1/2$$

for all  $a_i, a_j \in H$ . Equivalently

$$|a_i + a_j z| > \frac{|z|^2}{1 - |z|} \tag{2}$$

for all  $a_i, a_j \in H$ . By continuity, equation (2) holds under small perturbations of  $z$ . Therefore, there must exist  $\delta > 0$ , such that for all  $z' \in B(z, \delta)$  we have

$$|a_i + a_j z'| > \frac{|z'|^2}{1 - |z'|}.$$

Since

$$\left| \sum_{n=2}^k a_n (z')^n \right| \leq \frac{|z'|^2}{1 - |z'|}$$

for all  $(a_n) \in H^k$  and  $k \in \mathbb{N}$ , it follows that  $z'$  cannot be the zero of a power series or a polynomial. Therefore we must have  $R(H) \cap B(z, \delta) = \emptyset$  and  $R^*(H) \cap B(z, \delta) = \emptyset$ .

### 2 Proof of Theorem 1.1

We now turn our attention to proving Theorem 1.1. We start with the following technical proposition.

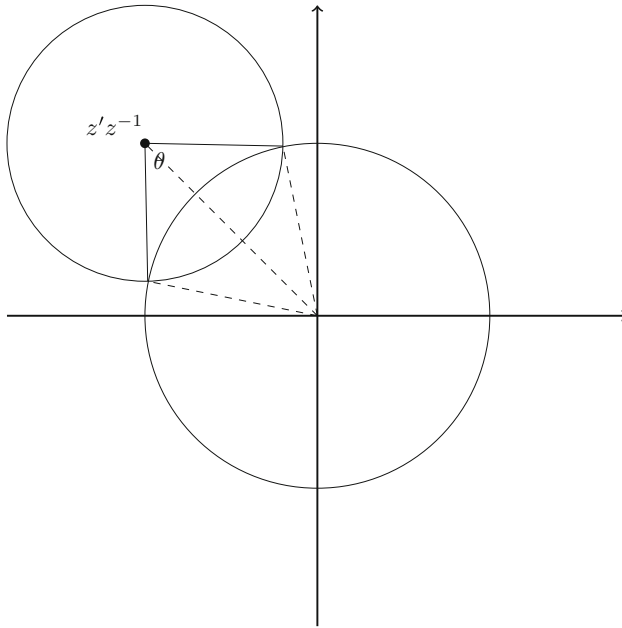
**Proposition 2.1** *Let  $z \in A_{(1/2,1)}$ . Suppose  $H$  is  $2 \cos^{-1}(\frac{5-4|z|^2}{4})$ -dense, then for any  $z' \in \overline{B(0, 2)}$  there exists  $a \in H$  such that  $z^{-1}(z' - a) \in \overline{B(0, 2)}$ .*

*Remark 2.2* It is useful to point out that the conclusion of this proposition is equivalent to:

$$\overline{B(0, 2)} \subset \bigcup_{a \in H} a + \overline{zB(0, 2)}.$$

*Proof* Let us start by fixing  $z' \in \overline{B(0, 2)}$ . Consider the point  $z'z^{-1}$ . Clearly  $z'z^{-1} \in \overline{B(0, 2|z|^{-1})}$ . Let

$$S(z'z^{-1}, |z|^{-1}) := \{\omega \in \mathbb{C} : |\omega - z'z^{-1}| = |z|^{-1}\}$$



**Fig. 1** A diagram of  $S_{z'z^{-1}}^{|z|^{-1}}$  intersecting  $\overline{B(0, 2)}$

be the circle centered at  $z'z^{-1}$  with radius  $|z|^{-1}$ .

Since  $z \in A_{(1/2,1)}$  we must have  $S(z'z^{-1}, |z|^{-1}) \cap \overline{B(0, 2)} \neq \emptyset$ . In fact this intersection must contain an arc of  $S(z'z^{-1}, |z|^{-1})$ . This arc is parameterised by two radii of  $S(z'z^{-1}, |z|^{-1})$  with interior angle  $\theta$ . See Fig. 1 for a diagram describing the intersection of  $S(z'z^{-1}, |z|^{-1})$  with  $\overline{B(0, 2)}$ . It is easy to see that the angle  $\theta$  is minimised when  $z'z^{-1}$  is as far from the origin as possible, i.e., when  $z'$  has modulus 2. Employing elementary techniques from geometry we can see that the angle  $\theta$  is at least twice the size of a particular angle of the triangle whose sides have length  $|z|^{-1}$ , 2, and  $2|z|^{-1}$  (see Fig. 1). Therefore, we can use the well-known cosine rule from trigonometry to show that  $\theta$  is always bounded below by

$$2 \cos^{-1} \left( \frac{5 - 4|z|^2}{4} \right).$$

Since  $H$  is  $2 \cos^{-1} \left( \frac{5-4|z|^2}{4} \right)$ -dense as a subset of  $S^1$ , there must exist  $a \in H$  such that  $z'z^{-1} - az^{-1}$  is contained in the arc of  $S(z'z^{-1}, |z|^{-1})$  which intersects  $\overline{B(0, 2)}$ . In particular, for this choice of  $a$  we have  $z^{-1}(z' - a) \in \overline{B(0, 2)}$ . □

Theorem 1.1 now follows almost immediately from Proposition 2.1.

*Proof of Theorem 1.1* By the relations given in Proposition 1.2 to prove Theorem 1.1 it is sufficient to only prove the statement relating to  $R^*(H)$ . Fix  $r \in (1/2, 1)$  and let  $H$  be a  $2 \cos^{-1} \left( \frac{5-4r^2}{4} \right)$ -dense subset of  $S^1$ . Note that  $H$  is automatically  $2 \cos^{-1} \left( \frac{5-4|z|^2}{4} \right)$ -dense for any  $z \in A_{[r,1)}$ . So we can apply Proposition 2.1 for any  $z \in A_{[r,1)}$ . Let us



now fix  $z \in A_{[r,1]}$  and apply Proposition 2.1 when  $z' = 0$ . So there exists  $a_0 \in H$  such that  $x_0 := z^{-1}(-a_0) \in \overline{B(0, 2)}$ . Rearranging yields

$$0 = a_0 + x_0z.$$

Applying Proposition 2.1 again with  $x_0$  in the place of  $z'$  yields  $a_1$  and  $x_1 := z^{-1}(x_0 - a_1)$ , such that  $x_1 \in \overline{B(0, 2)}$  and

$$0 = a_0 + a_1z + x_1z^2. \tag{3}$$

One can then apply Proposition 2.1 with  $z' = x_1$ . Repeating this procedure indefinitely yields a sequence  $(a_n)$  and  $(x_n)$  such that  $x_{n+1} = z^{-1}(x_n - a_{n+1})$  for all  $n \in \mathbb{N}$ . The terms in  $(x_n)$  remain in  $\overline{B(0, 2)}$ . Therefore, we are able to repeatedly apply the substitution  $x_{n+1} = z^{-1}(x_n - a_{n+1})$  in (3) and we obtain

$$0 = \sum_{n=0}^{\infty} a_n z^n.$$

Therefore,  $z \in R^*(H)$ . Since  $z$  was arbitrary we have  $A_{[r,1]} \subseteq R^*(H)$ . □

The proof of Theorem 1.1 was based upon ideas from  $\beta$ -expansions. The argument given relied upon adapting methods from [2,7]. The proof can easily be adapted to show that under the hypothesis of the theorem, for every  $z' \in \overline{B(0, 2)}$  there exists  $(a_n) \in H^{\mathbb{N}}$  such that  $\sum_{n=0}^{\infty} a_n z^n = z'$ .

### 3 Some Further Problems

There are some more challenging problems related to root sets  $R(H)$ ,  $R^*(H)$ . We mentioned in the beginning of this paper that there exist various results of multiple roots [3,4]. We say that a  $z \in \mathbb{C}$  is a multiple root of a holomorphic function  $f$  of order  $k$  if for all integers  $i = 0, 1, 2, \dots, k$

$$f^{(i)}(z) = 0.$$

Adopting the notation in this paper, we can define for any integer  $k \geq 0$ :

$$R_k(H) := \left\{ z \in \mathbb{C} : \exists k > 0, \exists (a_n) \in H^{k+1}, \right. \\ \left. P(w) = \sum_{n=0}^k a_n w^n, z \text{ is a } k\text{-th order root of } P(w) \right\}.$$

$$R_k^*(H) := \left\{ z \in \mathbb{C} : \exists (a_n) \in H^{\mathbb{N}}, \right. \\ \left. P(w) = \sum_{n=0}^{\infty} a_n w^n, z \text{ is a } k\text{-th order root of } P(w) \right\}.$$

Not so much has been studied about the above multiple root set, some partial results can be found in [8]. We can, for example, consider the following questions:

- Are  $R_k(H)$ ,  $R_k^*(H)$  dense in any non-degenerate annulus?
- What about the connectedness and path-connectedness of  $R_k(H)$ ,  $R_k^*(H)$ ?
- What can we say about the boundary of  $R_k(H)$ ,  $R_k^*(H)$ ?

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