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# LIOUVILLE-TYPE THEOREMS FOR THE STATIONARY MHD EQUATIONS IN 2D

WENDONG WANG AND YUZHAO WANG

**ABSTRACT.** This paper is devoted to investigating Liouville type properties of the two-dimensional stationary incompressible Magnetohydrodynamics equations. More precisely, we show that there are no non-trivial solutions to MHD equations either the Dirichlet integral or some  $L^p$  norm of the velocity-magnetic fields is suitably bounded, which generalize the well-known results for the 2D Navier-Stokes equations by Gilbarg-Weinberger [13] and Koch-Nadirashvili-Seregin-Sverak [19]. Compared to the Navier-Stokes equations, there is no maximum principle for solutions to the MHD equation. To overcome this difficulty, we develop a different approach, which does not appeal to the special structure of the vorticity equation as [13] did.

**Keywords:** Liouville type theorems, MHD equations, Navier-Stokes equations

## 1. INTRODUCTION

In this note, the main concern is the two-dimensional (2D) stationary incompressible Magnetohydrodynamics (MHD) equations on the whole plane  $\mathbb{R}^2$ :

$$\begin{cases} -\mu\Delta u + u \cdot \nabla u + \nabla\pi = b \cdot \nabla b, \\ -\nu\Delta b + u \cdot \nabla b = b \cdot \nabla u, \\ \operatorname{div} u = 0, \quad \operatorname{div} b = 0, \end{cases} \quad (1)$$

where  $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and  $b : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  denote the velocity field and the magnetic field respectively;  $\mu > 0$  is the viscosity coefficient and  $\nu > 0$  is the resistivity coefficient. Magnetohydrodynamics is the study of the magnetic properties of electrically conducting fluids, which appear naturally in various branches of physics and engineering such as plasmas, liquid metals, salt water, and electrolytes, etc. It models the phenomena that the magnetic fields induce currents in a moving conductive fluid, which in turn affects the magnetic field itself. As a consequence, the mathematical model of MHD is a coupled system of the Navier-Stokes equations of fluid dynamics governing the velocity field and the Maxwell's equation of electromagnetism describing the magnetic field. Due to its fundamental importance, the field of MHD has attracted considerable attention. In particular, Hannes Alfvén received the Nobel Prize in Physics in 1970 for his pioneering contribution in this field. We focus on the stationary equations (1), which plays a particularly important role in understanding the long time behaviour of its dynamical model. For further physical background and mathematical theory, we refer to Schnack [26] and the references therein.

It is obvious to observe that (1) admits trivial solutions  $u = b = C$ . A natural question to ask is whether there exists non-trivial solutions to (1) with the finite Dirichlet energy:

$$D(u, b) = \int_{\mathbb{R}^2} |\nabla u|^2 + |\nabla b|^2 dx. \quad (2)$$

This question is known as the Liouville-type problem concerning MHD equations (1). Liouville theory for MHD, if holds, claims that the stationary Magnetohydrodynamics flow owns the rigidity in the sense that the Dirichlet energy of the flow is either null or infinite. Liouville theory has been established for various fluid models, such as stationary Navier-Stokes equations.

When  $b = 0$  and  $\mu = 1$ , the MHD equation (1) reduces to the standard 2D Navier-Stokes (NS) equations,

$$\begin{cases} -\Delta u + u \cdot \nabla u + \nabla \pi = 0, \\ \operatorname{div} u = 0, \end{cases} \quad (3)$$

for which Liouville properties are well understood. For instance, Gilbarg-Weinberger [13] proved that there are only constant solutions to (3) provided the Dirichlet energy is finite, that is

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx < \infty.$$

Their proof relies on the fact that the vorticity of the 2D NS equations (3) satisfies a nice elliptic equation, to which the maximum principle applies. To be more precise, for a solution  $u$  to (3), define  $w = \partial_2 u_1 - \partial_1 u_2$  to be its vorticity. Then  $w$  solves the following elliptic equation

$$\Delta w - u \cdot \nabla w = 0,$$

which satisfies the maximal principle. The assumption on boundedness of the Dirichlet energy can be relaxed to  $\nabla u \in L^p(\mathbb{R}^2)$  with some  $p \in (\frac{6}{5}, 3]$ , see [2]. As a different type of Liouville property for the 2D NS, Koch-Nadirashvili-Seregin-Sverak [19] showed that any bounded solution to (3) is trivial solution, say  $u \equiv C$ , as a byproduct of their results on the non-stationary case. In [19] they exploited the maximum principle of a parabolic type, see also a note of Koch [17]. Recently, it was extended to the case of generalized Newtonian fluids, where the viscosity is a function depending on the shear rate in [7, 9]. See also [29] for a similar result for  $u \in L^p(\mathbb{R}^2)$  with  $p > 1$  on the generalized Newtonian fluid. Other types of Liouville properties for the stationary NS on the plane were also extensively studied, such as under the growth condition  $\limsup |x|^{-\alpha} |u(x)| < \infty$  as  $|x| \rightarrow \infty$  for some  $\alpha > 0$ , see [10, 2]; existence and asymptotic behavior of solutions in an exterior domain, see [14, 24, 25, 15, 22, 18, 5]. For more references on Liouville theorems of (3), we refer to [11, 6, 28, 16, 8] and the references therein.

If we take the magnetic effect into account, i.e.  $b$  is not necessarily vanishing, the understanding of the Liouville-type properties of MHD on the plane is far from satisfactory. The involvement of the magnetic field makes the problem much more complicated. As pointed out by Oleinik-Samokhin in [21], the magnetic field applied to a conducting viscous flow may affect the process of separation of the boundary layer as well as the speed. However, some numerical experiments in [23] seem to indicate that the velocity field should play a more important role than the magnetic field in the regularity theory. See also [27]. One may wonder whether the velocity fields also play the leading role in a Liouville theory for MHD equations. Partial progress has been made for the three-dimensional case, where Chae-Weng [4] recently proved the axially symmetric Dirichlet solution  $(u, b)$  is trivial if  $u \in L^3(\mathbb{R}^3)$ . It is still a challenging open problem to remove the  $L^3(\mathbb{R}^3)$  boundedness. In this note, we will give an affirmative answer to the two-dimensional case. In particular, we show that by adding the magnetic effect the Liouville

theorem still holds, provided that the magnetic fields are suitably controlled by the Reynold number.

Before proceeding with our main result, we define the weak solution to the MHD system (1).

**Definition 1.1.** *We say that  $(u, b)$  is a weak solution to the 2D MHD equations (1) in a domain  $\Omega \subset \mathbb{R}^2$  provided that:*

- (i).  $u, b \in L^s_{loc}(\Omega)$  for some  $s > 2$ ;
- (ii).  $\operatorname{div} u = 0$  and  $\operatorname{div} b = 0$ , in the weak sense;
- (iii).  $(u, b)$  satisfies the following system

$$\mu \int_{\Omega} u \cdot \Delta \phi \, dx + \int_{\Omega} (u \cdot \nabla \phi) \cdot u \, dx = \int_{\Omega} (b \cdot \nabla \phi) \cdot b \, dx$$

and

$$\nu \int_{\Omega} b \cdot \Delta \phi \, dx + \int_{\Omega} (u \cdot \nabla \phi) \cdot b \, dx = \int_{\Omega} (b \cdot \nabla \phi) \cdot u \, dx$$

for all  $\phi \in C_0^\infty(\Omega)$  with  $\phi = (\phi_1, \phi_2)$  and  $\operatorname{div} \phi = 0$ .

In what follows, we shall take  $\Omega = \mathbb{R}^2$  unless otherwise specified.

To establish the Liouville theory for the 2D stationary MHD equations (1), one may want to mimic the arguments in [13] or [19] for Navier Stokes equations (3). However, this is not the case. For instance, due to the presence of the magnetic fields, the maximum principle doesn't hold for the vorticity of the MHD equations. Therefore, Gilbarg-Weinberger's argument fails to apply to the 2D MHD equations. Nevertheless, we step forward in this direction and provide positive answers to this question by assuming that the magnetic fields are suitably bounded.

Our first main result is as follows,

**Theorem 1.1.** *Let  $(u, b)$  be a weak solution of the 2D MHD equations (1) defined over the entire plane. Assume that  $D(u, b) \leq D_0 < \infty$  and there exists an absolute constant  $C_*$  such that*

$$\|b\|_{L^1(\mathbb{R}^2)} D_0^{\frac{1}{2}} \leq C_* \min\{\mu\nu, \mu^{\frac{1}{2}}\nu^{\frac{3}{2}}\}.$$

*Then  $u$  and  $b$  are constants.*

**Remark 1.** Similar analysis as Galdi in [11], for any weak solutions  $(u, b)$  to the stationary MHD equations (1), if  $u, b \in L^s_{loc}(\Omega; \mathbb{R}^2)$  with some  $s > 2$ , then  $u, b \in W^{1,2}_{loc}(\Omega; \mathbb{R}^2)$  and  $u, b$  are smooth as a consequence of the regularity property of Stokes equations. For more details, we refer readers to [11, Chapter IX]. Therefore, the weak solutions to (1) are indeed smooth under the conditions of Theorem 1.1.

**Remark 2.** We stress that smallness conditions only apply to the magnetic field  $b$ . Note that if  $(u, b)$  be a solution of (1), then

$$u^\lambda(x) \doteq \lambda u(\lambda x), \quad b^\lambda(x) \doteq \lambda b(\lambda x)$$

is also a solution of (1). The quantities  $\|b\|_{L^1(\mathbb{R}^2)}\|\nabla u\|_{L^2(\mathbb{R}^2)}$  and  $\|b\|_{L^1(\mathbb{R}^2)}\|\nabla b\|_{L^2(\mathbb{R}^2)}$  are invariant under the natural scaling. Furthermore, our proof doesn't appeal to the special structure of the vorticity equation of the 2D NS equations as [13] did, and so it is more robust in extending to more general settings.

**Remark 3.** The suitable smallness of the magnetic field plays a vital role in our proof of Theorem 1.1. When  $\mu \approx \nu \approx (Re)^{-1}$  and  $\|u\|_{\dot{H}^1} \lesssim 1$ , Theorem 1.1 shows that  $\|b\|_{L^1 \cap \dot{H}^1} \lesssim (Re)^{-2}$  implies the trivial solution are stable in this space. The upper bound  $(Re)^{-2}$  also appeared in the study of in the time-dependent Navier-Stokes equations as the stability threshold of shear flow. See Bedrossian-Germain-Masmoudi [1]. However, it is still not clear whether this bound is essential to the Liouville theorem Theorem 1.1.

Motivated by [19] and [29], our second result is concerned with the Liouville property for  $L^p$  solutions,

**Theorem 1.2.** *Let  $(u, b)$  be a weak solution of the 2D MHD equations (1) defined over the entire plane. Then  $u, b$  are constants if one of the following conditions holds:*

- (1)  $u, b \in L^p(\mathbb{R}^2, \mathbb{R}^2)$  for some  $p \in [2, 6]$ ;
- (2)  $\|u\|_{L^p(\mathbb{R}^2)} + \|b\|_{L^p(\mathbb{R}^2)} \leq L < \infty$  for some  $p \in (6, \infty]$ , and there exists an absolute constant  $C_*$  such that  $\|b\|_{L^1(\mathbb{R}^2)} L^{\frac{p}{p-2}} \leq C_* \min\{\mu\nu, \mu^{\frac{1}{2}}\nu^{\frac{3}{2}}\}$ .

**Remark 4.** Note that for  $p \in [2, 6]$ , no smallness conditions are needed, that is to say there are no non-trivial  $L^p$  solutions to (1). However, it is different when  $p > 6$ . The main difference comes from a simple fact: the estimate of the nonlinear term  $u \cdot \nabla u$  or  $b \cdot \nabla b$ , and  $R^{-1} \int_{B_R \setminus B_{R/2}} |b|^3 dx = o(R)$  as  $R \rightarrow \infty$  if  $b \in L^p(\mathbb{R}^2)$  satisfying  $p \leq 6$  (see Section 4.1, 4.2). When  $p \in (6, \infty]$ , we need to assume that the scaling invariant norms  $\|b\|_{L^1(\mathbb{R}^2)}\|u\|_{L^p(\mathbb{R}^2)}^{\frac{p}{p-2}}$  and  $\|b\|_{L^1(\mathbb{R}^2)}\|b\|_{L^p(\mathbb{R}^2)}^{\frac{p}{p-2}}$  are sufficiently small. Moreover, the above result generalizes the corresponding theorems for the Navier-Stokes equation (3) in [19] or [29] to the setting of MHD equations.

## 2. PRELIMINARIES

In this section, we prepare some preliminary lemmas that we shall rely on. Throughout this article,  $C(a_1, \dots, a_k)$  denotes a constant depending on  $a_1, \dots, a_k$ , which may be different from line to line. We denote the ball with centre  $x_0$  of radius  $R$  by  $B_R(x_0)$ . If  $x_0 = 0$ , we simply write  $B_R = B_R(0)$ . Let a radially decaying smooth  $\eta(x)$  be a test function such that

$$\eta(x) = \begin{cases} 1, & x \in B_1, \\ 0, & x \in B_2^c. \end{cases}$$

and let

$$\eta_R(x) = \eta\left(\frac{x}{R}\right) \tag{4}$$

for  $R > 0$ . One notices that  $|\nabla^k \eta_R| \leq \frac{C}{R^k}$ .

Let us recall a result of Gilbarg-Weinberger in [13] about the decay of functions with finite Dirichlet integrals.

**Lemma 2.1** (Lemma 2.1, 2.2, [13]). *Let a  $C^1$  vector-valued function  $f(x) = (f_1, f_2)(x) = f(r, \theta)$  with  $r = |x|$  and  $x_1 = r \cos \theta$ . There holds finite Dirichlet integral in the range  $r > r_0$ , that is*

$$\int_{r>r_0} |\nabla f|^2 dx dy < \infty.$$

Then, we have

$$\lim_{r \rightarrow \infty} \frac{1}{\ln r} \int_0^{2\pi} |f(r, \theta)|^2 d\theta = 0.$$

and furthermore, there is an increasing sequence  $\{r_n\}$  with  $r_n \in (2^n, 2^{n+1})$ , such that

$$\lim_{n \rightarrow \infty} \frac{|f(r_n, \theta)|^2}{\ln r_n} = 0,$$

uniformly in  $\theta$ .

If, furthermore, we assume  $\nabla f \in L^p(\mathbb{R}^2)$  for some  $2 < p < \infty$ , then the above decay property can be improved to be point-wise uniformly. More precisely, we have

**Lemma 2.2** (Theorem II.9.1 [11]). *Let  $\Omega \subset \mathbb{R}^2$  be an exterior domain and let*

$$\nabla f \in L^2 \cap L^p(\Omega),$$

for some  $2 < p < \infty$ . Then

$$\lim_{|x| \rightarrow \infty} \frac{|f(x)|}{\sqrt{\ln(|x|)}} = 0,$$

uniformly.

We also need a Giaquinta's iteration lemma [12, Lemma 3.1], also see a proof in [3, Lemma 8].

**Lemma 2.3** (Lemma 3.1 [12]). *Let  $f(r)$  be a non-negative bounded function on  $[R_0, R_1] \subset \mathbb{R}_+$ . If there are negative constants  $A, B, D$  and positive exponents  $b < a$  and a parameter  $\theta \in (0, 1)$  such that for all  $R_0 \leq \rho < \tau \leq R_1$*

$$f(\rho) \leq \theta f(\tau) + \frac{A}{(\tau - \rho)^a} + \frac{B}{(\tau - \rho)^b} + D,$$

then for all  $R_0 \leq \rho < \tau \leq R_1$

$$f(\rho) \leq C(a, \theta) \left[ \frac{A}{(\tau - \rho)^a} + \frac{B}{(\tau - \rho)^b} + D \right].$$

### 3. PROOF OF THEOREM 1.1

For a solution of  $(u, b)$  to (1), consider the vorticity  $w = \partial_2 u_1 - \partial_1 u_2$  and the current density  $h = \partial_2 b_1 - \partial_1 b_2$ . It is easy to check that  $(w, h)$  satisfies

$$\begin{cases} -\mu \Delta w + u \cdot \nabla w = b \cdot \nabla h, \\ -\nu \Delta h + u \cdot \nabla h = b \cdot \nabla w + H, \end{cases} \quad (5)$$

where

$$H = 2\partial_2 b_2 (\partial_2 u_1 + \partial_1 u_2) + 2\partial_1 u_1 (\partial_2 b_1 + \partial_1 b_2). \quad (6)$$

One crucial step of the proof is to get the higher regularity estimates of the solutions of (1). Different from the argument in [13], we have to exploit something new to overcome the obstacle due to the lack of maximum principle for the 2D MHD equations. Before proceeding with the proof of Theorem 1.1, we prove the following smoothing property for the solution of (5).

**Lemma 3.1.** *Let the vorticity  $w$  and the current  $h$  as in the MHD equations (5) with finite Dirichlet integral, i.e.  $D(u, b) < \infty$ . Then, we have*

$$\int_{\mathbb{R}^2} |\nabla w|^2 + |\nabla h|^2 dx < \infty; \quad (7)$$

and furthermore, under the polar coordinate  $x = r \cos \theta$  and  $y = r \sin \theta$ , we have

$$\lim_{r \rightarrow \infty} \frac{|u(r, \theta)|^2}{\ln r} + \frac{|b(r, \theta)|^2}{\ln r} = 0 \quad (8)$$

uniformly in  $\theta$ .

*Proof.* We assume  $\mu = \nu = 1$  without loss of generality. Choose a cut-off function  $\phi(x) \in C_0^\infty(B_R)$  with  $0 \leq \phi \leq 1$  satisfying the following two properties:

i).  $\phi$  is radially decreasing and satisfies

$$\phi(x) = \phi(|x|) = \begin{cases} 1, & |x| \leq \rho, \\ 0, & |x| \geq \tau, \end{cases}$$

where  $0 < \frac{R}{2} \leq \rho < \tau \leq R$ ;

ii).  $|\nabla \phi|(x) \leq \frac{C}{\tau - \rho}$  for all  $x \in \mathbb{R}^2$ .

Multiplying both sides of (5) by  $\phi^2 w$  and  $\phi^2 h$  respectively and then integrating over  $\mathbb{R}^2$  to get

$$\int_{\mathbb{R}^2} \phi^2 |\nabla w|^2 dx = - \int_{\mathbb{R}^2} \nabla w \cdot \nabla(\phi^2) w dx - \int_{\mathbb{R}^2} u \cdot \nabla w \phi^2 w dx + \int_{\mathbb{R}^2} b \cdot \nabla h \phi^2 w dx$$

and

$$\int_{\mathbb{R}^2} \phi^2 |\nabla h|^2 dx = - \int_{\mathbb{R}^2} \nabla h \cdot \nabla(\phi^2) h dx - \int_{\mathbb{R}^2} u \cdot \nabla h \phi^2 h dx + \int_{\mathbb{R}^2} b \cdot \nabla w \phi^2 h dx + \int_{\mathbb{R}^2} H \phi^2 h dx.$$

By noticing the cancelation

$$\int_{\mathbb{R}^2} b \cdot \nabla h \phi^2 w dx + \int_{\mathbb{R}^2} b \cdot \nabla w \phi^2 h dx = - \int_{\mathbb{R}^2} b \cdot \nabla(\phi^2) h w dx,$$

and then applying integration by parts, we arrive at

$$\begin{aligned} & \int_{\mathbb{R}^2} \phi^2 |\nabla w|^2 dx + \int_{\mathbb{R}^2} \phi^2 |\nabla h|^2 dx \\ &= - \int_{\mathbb{R}^2} \nabla w \cdot \nabla(\phi^2) w dx - \int_{\mathbb{R}^2} \nabla h \cdot \nabla(\phi^2) h dx + \frac{1}{2} \int_{\mathbb{R}^2} u \cdot \nabla(\phi^2) w^2 dx \\ & \quad + \frac{1}{2} \int_{\mathbb{R}^2} u \cdot \nabla(\phi^2) h^2 dx - \int_{\mathbb{R}^2} b \cdot \nabla(\phi^2) h w dx + \int_{\mathbb{R}^2} H h \phi^2 dx \\ & \doteq I_1 + \cdots + I_6. \end{aligned} \quad (9)$$

In what follows we shall estimate  $I_j$  for  $j = 1, 2, \dots, 6$  one by one.

For the term  $I_1$ , by Hölder's inequality and (2) we have

$$\begin{aligned} I_1 &\leq \frac{C}{\tau - \rho} \|\nabla w\|_{L^2(B_\tau)} \|w\|_{L^2(B_\tau)} \\ &\leq \frac{1}{8} \int_{B_\tau} |\nabla w|^2 dx + \frac{C}{(\tau - \rho)^2}, \end{aligned}$$

and similarly

$$I_2 \leq \frac{1}{8} \int_{B_\tau} |\nabla h|^2 dx + \frac{C}{(\tau - \rho)^2}.$$

For the terms  $I_3, \dots, I_5$ , it only needs to consider  $I_5$  since other terms can be treated similarly. Let

$$\bar{f}(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r, \theta) d\theta,$$

then by Wirtinger's inequality (for example, see Ch II.5 [11]) we have

$$\int_0^{2\pi} |f - \bar{f}|^2 d\theta \leq \int_0^{2\pi} |\partial_\theta f|^2 d\theta. \quad (10)$$

By Hölder inequality, (10) and Lemma 2.1 we have

$$\begin{aligned} I_5 &\leq \left| \int_{\mathbb{R}^2} h w b \cdot \nabla \phi dx \right| \\ &\leq \left| \int_{\mathbb{R}^2} h w (b - \bar{b}) \cdot \nabla \phi dx \right| + \left| \int_{\mathbb{R}^2} h w \bar{b} \cdot \nabla \phi dx \right| \\ &\leq \frac{C}{\tau - \rho} \left( \int_{B_\tau} w^4 \right)^{\frac{1}{4}} \left( \int_{B_\tau} h^4 \right)^{\frac{1}{4}} \left( \int_{\frac{R}{2} < |r| < R} \int_0^{2\pi} |b(r, \theta) - \bar{b}|^2 d\theta r dr \right)^{\frac{1}{2}} \\ &\quad + \frac{C}{\tau - \rho} \int_{B_\tau \setminus B_{\frac{R}{2}}} |wh| \left( \int_0^{2\pi} |b(r, \theta)|^2 d\theta \right)^{\frac{1}{2}} dx \\ &\leq \frac{CR}{\tau - \rho} \left( \int_{B_\tau} w^4 \right)^{\frac{1}{4}} \left( \int_{B_\tau} h^4 \right)^{\frac{1}{4}} \left( \int_{\frac{R}{2} < |r| < R} \frac{1}{r^2} \int_0^{2\pi} |\partial_\theta b|^2 d\theta r dr \right)^{\frac{1}{2}} \\ &\quad + C \frac{(\ln R)^{\frac{1}{2}}}{\tau - \rho} \int_{B_\tau} (w^2 + h^2) dx. \end{aligned}$$

Using the following Poincaré-Sobolev inequality (see, for example, Theorem 8.11 and 8.12 [20])

$$\|w\|_{L^4(B_\tau)} \leq C \|\nabla w\|_{L^2(B_\tau)}^{\frac{1}{2}} \|w\|_{L^2(B_\tau)}^{\frac{1}{2}} + C\tau^{-\frac{1}{2}} \|w\|_{L^2(B_\tau)}, \quad (11)$$

we obtain

$$I_5 \leq \frac{CR}{\tau - \rho} \left( \int_{B_\tau} |\nabla w|^2 + |\nabla h|^2 \right)^{\frac{1}{2}} + \frac{CR\tau^{-1}}{\tau - \rho} + \frac{C\sqrt{\ln R}}{\tau - \rho},$$

where we used the boundedness of Dirichlet integral. Thus

$$I_3 + I_4 + I_5 \leq \frac{1}{8} \int_{B_\tau} |\nabla h|^2 + |\nabla w|^2 dx + \frac{CR\tau^{-1}}{\tau - \rho} + \frac{C\sqrt{\ln R}}{\tau - \rho} + \frac{CR^2}{(\tau - \rho)^2}.$$



For the term  $I_6$ , using (11) again we get

$$\begin{aligned} I_6 &= \int_{\mathbb{R}^2} \phi h H \, dx \leq C \|\nabla b\|_{L^4(B_\tau)}^2 \|w\|_{L^2(B_\tau)} \\ &\leq \frac{1}{8} \int_{B_\tau} |\nabla^2 b|^2 \, dx + C(1 + \tau^{-1}). \end{aligned}$$

Moreover, due to  $\nabla^\perp = (\partial_2, -\partial_1)^\top$  and  $\operatorname{div} u = 0$ , there holds

$$\Delta u = \nabla^\perp (\partial_2 u_1 - \partial_1 u_2) = \nabla^\perp w.$$

Thus by integration by parts we have

$$\begin{aligned} &\int_{\mathbb{R}^2} \phi^2 |\nabla^2 u|^2 \, dx \\ &\leq \frac{4}{3} \int_{\mathbb{R}^2} \phi^2 |\Delta u|^2 \, dx + \frac{C}{(\tau - \rho)^2} \int_{B_\tau} |\nabla u|^2 \, dx \\ &\leq \frac{4}{3} \int_{B_\tau} \phi^2 |\nabla w|^2 \, dx + \frac{C}{(\tau - \rho)^2}. \end{aligned} \tag{12}$$

Collecting the estimates  $I_1, \dots, I_6$ , by (12) we get

$$\begin{aligned} &\frac{3}{4} \int_{B_\rho} |\nabla^2 u|^2 + |\nabla^2 b|^2 \, dx \\ &\leq \frac{1}{2} \int_{B_\tau} |\nabla^2 b|^2 + |\nabla^2 u|^2 \, dx \\ &\quad + C(1 + \tau^{-1}) + \frac{CR\tau^{-1}}{\tau - \rho} + \frac{C\sqrt{\ln R}}{\tau - \rho} + \frac{CR^2}{(\tau - \rho)^2} + \frac{C}{(\tau - \rho)^2}. \end{aligned}$$

Then by applying Lemma 2.3, we obtain

$$\int_{B_{R/2}} |\nabla^2 u|^2 + |\nabla^2 b|^2 \, dx \leq CR^{-2} + C \frac{\sqrt{\ln R}}{R} + C.$$

Finally, by taking  $R \rightarrow \infty$ , we arrive at (7).

Now we turn to the proof of (8). By Gagliardo-Nirenberg inequality, one notices that

$$\|\nabla u\|_{L^4(\mathbb{R}^2)} \leq C \|\nabla u\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla^2 u\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}.$$

Then (8) follows from (7) and Lemma 2.2. Therefore, the proof is complete.  $\square$

**Remark 5.** Lemma 3.1 roughly says that by assuming the boundedness of the Dirichlet integral (2), i.e. the  $L^2$  norm of the gradient, one can bound the second order derivatives, (7). This is a manifestation of the smoothing effect, which will be used as a substitution of the maximal principle in [13]. Please also note that the assumptions on the magnetic field  $b$  in Lemma 3.1 holds automatically for the Navier-Stokes equation since then  $b = 0$ . Therefore, in this perspective, the smoothing effect exploited by Lemma 3.1 is more robust than the maximal principle used in [13].

Now we are ready to demonstrate the proof of Theorem 1.1.

*Proof of Theorem 1.1.* Making the inner product with  $\eta_R^2 w$  on both sides of the equation (5)<sub>1</sub>, and  $\eta_R^2 h$  on both sides of the equation (5)<sub>2</sub>, we have

$$\begin{aligned}
& \mu \int_{B_R} |\nabla w|^2 + \nu \int_{B_R} |\nabla h|^2 dx \\
& \leq \frac{C}{R^2} \left( \int_{B_{2R} \setminus B_R} |w|^2 + |h|^2 dx \right) + \frac{C}{R} \left( \int_{B_{2R} \setminus B_R} |u||w|^2 + |h|^2|u| + |b||h||w| dx \right) \\
& \quad + \left| \int_{B_{2R}} H h \eta_R^2 dx \right| \\
& \doteq I_1 + I_2 + I_3,
\end{aligned} \tag{13}$$

where  $\eta_R$  is as in (4) and  $H$  is as in (6).

Terms  $I_1$  and  $I_2$  are easy to estimate. By  $D(u, b) \leq D_0$  and (8) we have

$$|I_1| + |I_2| \leq CR^{-2} + \frac{C}{R} \sqrt{\ln R}. \tag{14}$$

It remains to bound  $I_3$ . In what follows, we may assume that  $I_3 = 2\partial_2 b_2 \partial_2 u_1$  since the treatments for other terms are similar.

$$\begin{aligned}
|I_3| &= \left| \int_{B_{2R}} \eta_R^2 h \partial_2 b_2 \partial_2 u_1 dx \right| \\
&\leq \left| \int_{B_{2R}} b_2 \partial_2 \partial_2 u_1 h \eta_R^2 dx \right| + \left| \int_{B_{2R}} b_2 \partial_2 u_1 \partial_2 h \eta_R^2 dx \right| + \left| \int_{B_{2R}} b_2 \partial_2 u_1 h \partial_2 \eta_R^2 dx \right| \\
&\leq \left( \int_{B_{2R}} |b_2 h \eta_R|^2 dx \right)^{1/2} \left( \int_{B_{2R}} |\partial_2 \partial_2 u_1 \eta_R|^2 dx \right)^{1/2} \\
&\quad + \left( \int_{B_{2R}} |\partial_2 h \eta_R|^2 dx \right)^{1/2} \left( \int_{B_{2R}} |b_2 \partial_2 u_1 \eta_R|^2 dx \right)^{1/2} + \frac{C}{R} \sqrt{\ln R}.
\end{aligned}$$

where we used  $D(u, b) \leq D_0$  and (8). For the first factor of the first term, due to the Gagliardo-Nirenberg inequality we have

$$\begin{aligned}
& \left( \int_{B_{2R}} |b_2 h \eta_R|^2 dx \right)^{1/2} \\
& \leq \|b\|_{L^4(\mathbb{R}^2)} \|h\|_{L^4(\mathbb{R}^2)} \\
& \leq C \|b\|_{L^1(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla^2 b\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|h\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla h\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \\
& \leq C \left( \|b\|_{L^1(\mathbb{R}^2)} D_0^{\frac{1}{2}} \right)^{\frac{1}{2}} \|\nabla h\|_{L^2(\mathbb{R}^2)},
\end{aligned}$$

where we used (7),  $D(u, b) \leq D_0$ , and (12). Similarly, we have

$$\left( \int_{B_{2R}} |b_2 \partial_2 u_1|^2 dx \right)^{1/2} \leq C \left( \|b\|_{L^1(\mathbb{R}^2)} D_0^{\frac{1}{2}} \right)^{\frac{1}{2}} \|\nabla h\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla w\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}.$$

Hence, by letting  $R \rightarrow \infty$ , we conclude

$$I_3 \leq C \left( \|b\|_{L^1(\mathbb{R}^2)} D_0^{\frac{1}{2}} \right)^{\frac{1}{2}} \left( \|\nabla h\|_{L^2(\mathbb{R}^2)} \|\nabla w\|_{L^2(\mathbb{R}^2)} + \|\nabla h\|_{L^2(\mathbb{R}^2)}^{\frac{3}{2}} \|\nabla w\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \right).$$

By choosing  $\|b\|_{L^1(\mathbb{R}^2)} D_0^{\frac{1}{2}}$  small enough, we arrive at

$$I_3 \leq \frac{\mu}{16} \|\nabla h\|_2^2 + \frac{\nu}{16} \|\nabla w\|_2^2. \quad (15)$$

For instance, one may choose  $\|b\|_{L^1(\mathbb{R}^2)} D_0^{\frac{1}{2}} \leq C_* \min\{\mu\nu, \mu^{\frac{1}{2}}\nu^{\frac{3}{2}}\}$ , where  $C_*$  is an absolute constant.

By collecting (13), (14), and (15), we finally get

$$\mu \int_{B_R} |\nabla w|^2 + \nu \int_{B_R} |\nabla h|^2 dx \leq CR^{-2} + \frac{C}{R} \ln R + \frac{\mu}{16} \|\nabla w\|_2^2 + \frac{\nu}{16} \|\nabla h\|_2^2.$$

Consequently, letting  $R \rightarrow \infty$ , we conclude that

$$\nabla w = \nabla h = 0.$$

It follows that both  $w$  and  $h$  are constants. Due to  $D(u, b) \leq D_0$ , we conclude that  $w = 0$  and  $h = 0$ . Finally, since  $\operatorname{div} u = 0$  and  $\operatorname{div} b = 0$ , it follows that  $u$  and  $b$  are constants. Furthermore, one notices that  $b = 0$  since  $b \in L^1$ . Thus the proof is finished.  $\square$

#### 4. PROOF OF THEOREM 1.2

In this section, the proof relies on a Giaquinta's iteration lemma [12, Lemma3.1]. We assume that  $\mu = \nu = 1$  for simplicity. The proof is split into four cases:  $3 \leq p \leq 6$ ,  $2 \leq p < 4$ ,  $6 < p < \infty$ , and  $p = \infty$ . The arguments for the former two cases are similar, the main point of which is to establish a gradient estimate; while the later two cases appeal to estimates involving second order derivatives. We shall give full detailed proofs for the first and third cases, and indicate where modification is needed to treat the second and fourth cases.

Let us start with the first case.

**4.1. Case  $3 \leq p \leq 6$ .** At first, we fix a  $R \in \mathbb{R}_+$  and the cut-off function  $\phi(x) \in C_0^\infty(B_R)$  as in the previous section. But here the choice of the parameters  $\rho, \tau$  satisfies

$$0 < \frac{R}{2} < \frac{2}{3}\tau < \frac{3}{4}R \leq \rho < \tau \leq R.$$

Due to Theorem III 3.1 in [11], there exists a constant  $C(s)$  and a vector-valued function  $\bar{w} : B_\tau \setminus B_{\frac{2}{3}\tau} \rightarrow \mathbb{R}^2$  such that  $\bar{w} \in W_0^{1,s}(B_\tau \setminus B_{\frac{2}{3}\tau})$ , and  $\nabla \cdot \bar{w}(x) = \nabla_x \cdot [\phi(x)u(x)]$ . Moreover, we get

$$\int_{B_\tau \setminus B_{\frac{2}{3}\tau}} |\nabla \bar{w}(x)|^s dx \leq C(s) \int_{B_\tau \setminus B_{\frac{2}{3}\tau}} |\nabla \phi \cdot u|^s dx. \quad (16)$$

We thus can extend  $\bar{w}$  to the whole space  $\mathbb{R}^2$ , which vanishes outside of the domain  $B_\tau$ .

*Proof of Theorem 1.2: case  $3 \leq p \leq 6$ .* Making the inner products  $(\phi u - \bar{w})$  and  $\phi b$  on both sides of the equation (1), by  $\nabla \cdot \bar{w} = \nabla \cdot [\phi u]$  we have

$$\begin{aligned} & \int_{B_\tau} \phi |\nabla u|^2 dx \\ &= - \int_{B_\tau} \nabla \phi \cdot \nabla u \cdot u dx + \int_{B_\tau \setminus B_{\frac{2}{3}\tau}} \nabla \bar{w} : \nabla u dx - \int_{B_\tau} u \cdot \nabla u \cdot \phi u dx \end{aligned}$$

$$\begin{aligned}
& + \int_{B_\tau \setminus B_{\frac{2}{3}\tau}} u \cdot \nabla u \cdot \bar{w} \, dx + \int_{B_\tau} b \cdot \nabla b \cdot \phi u \, dx - \int_{B_\tau \setminus B_{\frac{2}{3}\tau}} b \cdot \nabla b \cdot \bar{w} \, dx \\
& = I_1 + \cdots + I_6,
\end{aligned}$$

and

$$\begin{aligned}
& \int_{B_\tau} \phi |\nabla b|^2 \, dx \\
& = - \int_{B_\tau} \nabla \phi \cdot \nabla b \cdot b \, dx - \int_{B_\tau} u \cdot \nabla b \cdot \phi b \, dx + \int_{B_\tau} b \cdot \nabla u \cdot \phi b \, dx \\
& = I'_1 + I'_2 + I'_3.
\end{aligned}$$

For the term  $I_1$ , by Hölder inequality we have

$$|I_1| \leq \frac{C}{\tau - \rho} \left( \int_{B_\tau} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B_R \setminus B_{R/2}} |u|^2 \, dx \right)^{\frac{1}{2}}.$$

For the term  $I_2$ , Hölder inequality and (16) imply that

$$\begin{aligned}
|I_2| & \leq C \left( \int_{B_\tau} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \|\nabla \bar{w}\|_{L^2(B_\tau \setminus B_{\frac{2}{3}\tau})} \\
& \leq \frac{C}{\tau - \rho} \|\nabla u\|_{L^2(B_\tau)} \|u\|_{L^2(B_R \setminus B_{R/2})}.
\end{aligned}$$

By integration by parts, it is easy to find that

$$|I_3| \leq \frac{C}{\tau - \rho} \|u\|_{L^3(B_R \setminus B_{R/2})}^3.$$

For the term  $I_4$ , integration by parts leads to

$$I_4 = - \int_{B_\tau \setminus B_{\frac{2}{3}\tau}} u \cdot \nabla \bar{w} \cdot u \, dx.$$

Then in view of (16) we find

$$\begin{aligned}
|I_4| & \leq \|u\|_{L^3(B_R \setminus B_{R/2})}^2 \|\nabla \bar{w}\|_{L^3} \\
& \leq \frac{C}{\tau - \rho} \|u\|_{L^3(B_R \setminus B_{R/2})}^3.
\end{aligned}$$

For the term  $I_5$ , we need a cancellation with  $I'_3$ . More precisely,

$$I_5 + I'_3 = - \int_{B_\tau} (b \otimes b) : (\nabla \phi \otimes u),$$

and it follows that

$$|I_5 + I'_3| \leq \frac{C}{\tau - \rho} \left( \|u\|_{L^3(B_R \setminus B_{R/2})}^3 + \|b\|_{L^3(B_R \setminus B_{R/2})}^3 \right).$$

The treatment for  $I_6$  is similar to  $I_4$  and

$$|I_6| \leq \frac{C}{\tau - \rho} \|b\|_{L^3(B_R \setminus B_{R/2})}^2 \|u\|_{L^3(B_R \setminus B_{R/2})}.$$

For the terms  $I'_1$  and  $I'_2$ , similar as  $I_1$  and  $I_3$  respectively, we find

$$|I'_1| + |I'_2| \leq \frac{C}{\tau - \rho} \|\nabla b\|_{L^2(B_\tau)} \|b\|_{L^2(B_R \setminus B_{R/2})} + \frac{C}{\tau - \rho} \left( \|u\|_{L^3(B_R \setminus B_{R/2})}^3 + \|b\|_{L^3(B_R \setminus B_{R/2})}^3 \right).$$

By setting

$$f(r) = \int_{B_r} |\nabla u|^2 + |\nabla b|^2 dx, \quad (17)$$

collecting the above estimates we have

$$\begin{aligned} f(\rho) &\leq \frac{1}{2} f(\tau) + \frac{C}{\tau - \rho} \left( \|u\|_{L^3(B_R \setminus B_{R/2})}^3 + \|b\|_{L^3(B_R \setminus B_{R/2})}^3 \right) \\ &\quad + \frac{C}{(\tau - \rho)^2} \left( \|u\|_{L^2(B_R \setminus B_{R/2})}^2 + \|b\|_{L^2(B_R \setminus B_{R/2})}^2 \right). \end{aligned}$$

Now we apply Lemma 2.3 with  $R_0 = \frac{3R}{4}$  and  $R_1 = R$  to obtain

$$\begin{aligned} &\int_{B_{R/2}} |\nabla u|^2 + |\nabla b|^2 dx \\ &\leq \frac{C}{R^2} \left( \|u\|_{L^2(B_R \setminus B_{R/2})}^2 + \|b\|_{L^2(B_R \setminus B_{R/2})}^2 \right) \\ &\quad + \frac{C}{R} \left( \|u\|_{L^3(B_R \setminus B_{R/2})}^3 + \|b\|_{L^3(B_R \setminus B_{R/2})}^3 \right) \\ &\leq CR^{-\frac{4}{p}} \left( \|u\|_{L^p(B_R \setminus B_{R/2})}^2 + \|b\|_{L^p(B_R \setminus B_{R/2})}^2 \right) \\ &\quad + CR^{1-\frac{6}{p}} \left( \|u\|_{L^p(B_R \setminus B_{R/2})}^3 + \|b\|_{L^p(B_R \setminus B_{R/2})}^3 \right). \end{aligned} \quad (18)$$

for all  $p \geq 3$ .

Hence, for  $p \in [3, 6]$ , we get

$$\lim_{R \rightarrow \infty} \int_{B_{R/2}} |\nabla u|^2 + |\nabla b|^2 dx = 0,$$

provided  $u, b \in L^p(\mathbb{R}^2)$ . It follows that  $u$  and  $b$  are constants, thus  $u \equiv b \equiv 0$ . Therefore we finish the proof.  $\square$

By incorporating with the translation, the estimate (18) implies the following uniform local estimate

**Corollary 4.1.** *For smooth solutions  $u, b$  to the MHD equations (1), we have*

$$\begin{aligned} &\int_{B_{R/2}(x_0)} |\nabla u|^2 + |\nabla b|^2 dx \\ &\leq CR^{-\frac{4}{p}} \left( \|u\|_{L^p(B_R(x_0) \setminus B_{R/2}(x_0))}^2 + \|b\|_{L^p(B_R(x_0) \setminus B_{R/2}(x_0))}^2 \right) \\ &\quad + CR^{1-\frac{6}{p}} \left( \|u\|_{L^p(B_R(x_0) \setminus B_{R/2}(x_0))}^3 + \|b\|_{L^p(B_R(x_0) \setminus B_{R/2}(x_0))}^3 \right), \end{aligned} \quad (19)$$

provided  $u, b \in L^p(\mathbb{R}^2)$  with  $3 \leq p \leq \infty$ .

In particular, the above lemma says that  $\nabla u$  and  $\nabla b$  are uniformly locally in  $L^2(\mathbb{R}^2)$ , which will be denoted by  $u, b \in \dot{H}_{\text{uloc}}^1$ , by assuming  $u, b \in L^p(\mathbb{R}^2)$  for some  $p \geq 3$ . From Corollary 4.1 one easily obtains the following estimate on the growth of the Dirichlet integral,

**Corollary 4.2.** *For smooth solutions  $u, b$  to the MHD equations (1), we have*

$$\int_{B_R(x_0)} |\nabla u|^2 + |\nabla b|^2 dx \lesssim 1 + R^{1-\frac{6}{p}},$$

provided  $u, b \in L^p(\mathbb{R}^2)$  with  $3 \leq p \leq \infty$ .

These two properties are of particular importance in what follows.

#### 4.2. Case $2 \leq p < 4$ .

*Proof of Theorem 1.2: case  $2 \leq p < 4$ .* The argument for this case is similar to that of the previous one. While different treatments are needed to deal with the nonlinear terms  $I_3, \dots, I_6$ , and  $I'_2, I'_3$ . However, the methods to estimate each of these terms are similar and thus we only compute one term, say  $I_4$ , to illustrate the idea.

With the help of (16) and (11), we have

$$\begin{aligned} |I_4| &= \left| \int_{B_\tau \setminus B_{\frac{2}{3}\tau}} u \cdot \nabla \bar{w} \cdot u dx \right| \\ &\leq \|u\|_{L^4(B_\tau \setminus B_{\frac{2}{3}\tau})}^2 \|\nabla \bar{w}\|_{L^2(B_\tau \setminus B_{\frac{2}{3}\tau})} \\ &\leq \frac{C}{\tau - \rho} \|u\|_{L^2(B_R \setminus B_{R/2})} \left[ \|u\|_{L^2(B_\tau)} \|\nabla u\|_{L^2(B_\tau)} + \tau^{-1} \|u\|_{L^2(B_\tau)}^2 \right] \\ &\leq \frac{1}{8} \|\nabla u\|_{L^2(B_\tau)}^2 + \frac{C}{(\tau - \rho)^2} \|u\|_{L^2(B_R \setminus B_{R/2})}^2 \|u\|_{L^2(B_\tau)}^2 \\ &\quad + \frac{C}{\tau(\tau - \rho)} \|u\|_{L^2(B_R \setminus B_{R/2})} \|u\|_{L^2(B_\tau)}^2. \end{aligned}$$

Similar arguments for all other terms finally lead to

$$\begin{aligned} f(\rho) &\leq \frac{1}{2} f(\tau) + \frac{C}{(\tau - \rho)^2} \left( \|u\|_{L^2(B_R)}^2 + \|b\|_{L^2(B_R)}^2 \right) + \frac{C}{(\tau - \rho)^2} \left( \|u\|_{L^2(B_R)}^4 + \|b\|_{L^2(B_R)}^4 \right) \\ &\quad + \frac{C}{\tau(\tau - \rho)} \left( \|u\|_{L^2(B_R)}^3 + \|b\|_{L^2(B_R)}^3 \right), \end{aligned}$$

where  $f(\rho)$  was defined in (17) and  $\frac{3R}{4} \leq \rho < \tau \leq R$ . Then we apply Lemma 2.3 to obtain

$$\begin{aligned} &\int_{B_{R/2}} |\nabla u|^2 + |\nabla b|^2 dx \\ &\leq \frac{C}{R^2} \left( \|u\|_{L^2(B_R)}^2 + \|b\|_{L^2(B_R)}^2 \right) + \frac{C}{R^2} \left( \|u\|_{L^2(B_R)}^3 + \|b\|_{L^2(B_R)}^3 \right) \\ &\quad + \frac{C}{R^2} \left( \|u\|_{L^2(B_R)}^4 + \|b\|_{L^2(B_R)}^4 \right) \\ &\leq CR^{-\frac{4}{p}} \left( \|u\|_{L^p(B_R)}^2 + \|b\|_{L^2(B_R)}^2 \right) + CR^{1-\frac{6}{p}} \left( \|u\|_{L^p(B_R)}^3 + \|b\|_{L^p(B_R)}^3 \right) \\ &\quad + CR^{2-\frac{8}{p}} \left( \|u\|_{L^p(B_R)}^4 + \|b\|_{L^p(B_R)}^4 \right), \end{aligned}$$

which implies the triviality of  $u, b$  when  $2 \leq p < 4$ . Therefore we complete the proof for this case.  $\square$

**4.3. Case  $6 < p < \infty$ .** We'll reduce this case to the case of  $p = \infty$ , i.e. we show that the  $L^p$  boundedness implies the  $L^\infty$  boundedness. See Corollary 4.4. Then the completed proof for this case will then be presented in the next subsection when considering the case of  $p = \infty$ . Under the natural scaling, we can assume that  $\|u\|_{L^p(\mathbb{R}^2)} + \|b\|_{L^p(\mathbb{R}^2)} \leq 1$ .

Now we turn to the vorticity and current-density equation (5). As we have seen in the previous subsection, when  $p \leq 6$ , from (19) one has a decay estimate on the gradients

$$\int_{B_R} |\nabla u|^2 + |\nabla b|^2 dx = o(1),$$

as  $R \rightarrow \infty$ , from which the theorem follows. However, this argument fails when  $p > 6$  as the left hand side of (19) may fail to decay to zero as  $R \rightarrow \infty$ . To circumvent this difficulty, we exploit the local properties of the solution instead. To be more precise, by choosing  $R = 2$ , (19) becomes

$$\int_{B_1(x_0)} |\nabla u|^2 + |\nabla b|^2 dx \leq C(p) \left[ 1 + \|u\|_{L^p(B_2(x_0))}^3 + \|b\|_{L^p(B_2(x_0))}^3 \right] \leq C(p),$$

from which we shall show  $u, b$  are bounded globally. To this purpose, we shall first prove that  $\nabla^2 u$  and  $\nabla^2 b$  are uniformly locally  $L^2$  bounded

**Lemma 4.3.** *For  $p > 6$ , assume  $u, b \in L^p(\mathbb{R}^2)$  are smooth solutions to (1) and*

$$\|u\|_{L^p(\mathbb{R}^2)} + \|b\|_{L^p(\mathbb{R}^2)} \leq 1.$$

*Then, we have*

$$\sup_{x_0} \int_{B_1(x_0)} |\nabla^2 u|^2 + |\nabla^2 b|^2 dx < \infty. \quad (20)$$

*Proof.* The idea of the proof is similar to that of Lemma 3.1. In view of Corollary 4.1, we have the local boundedness (19). Then, there holds

$$\int_{B_1(x_0)} |\nabla u|^2 + |\nabla b|^2 dx \leq C. \quad (21)$$

Without loss of any generality, we may assume  $x_0 = 0$  for simplicity.

Let  $\phi$  be defined as in Subsection 4.1 with  $R = 2$ . Multiplying both sides of the vorticity and current-density equation (5) by  $\phi^2 w$  and  $\phi^2 h$  respectively and then integrating over  $\mathbb{R}^2$  to get

$$\begin{aligned} & \int \phi^2 |\nabla w|^2 dx + \int \phi^2 |\nabla h|^2 dx \\ &= - \int \nabla w \cdot \nabla(\phi^2) w dx - \int \nabla h \cdot \nabla(\phi^2) h dx + \frac{1}{2} \int u \cdot \nabla(\phi^2) w^2 dx \\ & \quad + \frac{1}{2} \int u \cdot \nabla(\phi^2) h^2 dx - \int b \cdot \nabla(\phi^2) h w dx + \int H h \phi^2 dx \\ & \doteq I_1 + \cdots + I_6. \end{aligned} \quad (22)$$

In what follows we shall estimate  $I_j$  for  $j = 1, 2, \dots, 6$  one by one.

For the term  $I_1$ , by Hölder's inequality and (21) we have

$$\begin{aligned} I_1 &\leq \frac{C}{\tau - \rho} \|\nabla w\|_{L^2(B_\tau)} \|w\|_{L^2(B_2)} \\ &\leq \frac{1}{8} \int_{B_\tau} |\nabla w|^2 dx + \frac{C}{(\tau - \rho)^2}, \end{aligned}$$

where  $B_\tau \subset B_2$ . For the term  $I_2$ , similar argument as the case  $I_1$  gives

$$I_2 \leq \frac{C}{\tau - \rho} \|\nabla h\|_{L^2(B_\tau)} \|h\|_{L^2(B_2)} \leq \frac{1}{8} \int_{B_\tau} |\nabla h|^2 dx + \frac{C}{(\tau - \rho)^2}.$$

To estimate the term  $I_3$ , we set

$$(w^2)_{B_\tau} = \frac{1}{|B_\tau|} \int_{B_\tau} w^2 dx \quad \text{and} \quad u_{B_\tau} = \frac{1}{|B_\tau|} \int_{B_\tau} u dx,$$

be the means of  $w^2$  and  $u$  over the ball  $B_\tau$ , we have

$$\begin{aligned} I_3 &\leq \left| \int u \cdot \nabla w \phi^2 w dx \right| = \frac{1}{2} \left| \int u \cdot \nabla (w^2 - (w^2)_{B_\tau}) \phi^2 dx \right| \\ &\leq \frac{C}{\tau - \rho} \int_{B_\tau} |w^2 - (w^2)_{B_\tau}| |u - u_{B_\tau}| dx + C \frac{|u_{B_\tau}|}{\tau - \rho} \int_{B_\tau} w^2 dx \\ &\leq \varepsilon \int_{B_\tau} |w^2 - (w^2)_{B_\tau}|^2 dx + \frac{C_\varepsilon}{(\tau - \rho)^2} \int_{B_\tau} |u - u_{B_\tau}|^2 dx + C \frac{|u_{B_\tau}|}{\tau - \rho} \int_{B_\tau} w^2 dx \end{aligned}$$

where  $\varepsilon > 0$  will be determined later. The last two terms can be easily bounded by

$$\frac{C_\varepsilon}{(\tau - \rho)^2} \int_{B_2} |\nabla u|^2 dx + \frac{C}{\tau - \rho} \int_{B_2} w^2 dx \int_{B_2} |u| dx,$$

where we used Poincaré's inequality. For the first term, by Poincaré inequality, Hölder's inequality, and Young's inequality, we arrive at

$$\begin{aligned} &\int_{B_\tau} |w^2 - (w^2)_{B_\tau}|^2 dx \\ &\leq C \left( \int_{B_\tau} |\nabla(w^2)| dx \right)^2 \\ &\leq C \int_{B_\tau} |\nabla w|^2 dx \int_{B_2} |w|^2 dx. \end{aligned}$$

To summary, by choosing  $\varepsilon$  small enough and applying (21) we have

$$I_3 \leq \frac{1}{8} \int_{B_\tau} |\nabla w|^2 dx + \frac{C}{\tau - \rho} + \frac{C}{(\tau - \rho)^2}. \quad (23)$$

For the term  $I_4$ , the proof is similar to that of  $I_3$ , we have

$$I_4 \leq \frac{1}{8} \int_{B_\tau} |\nabla h|^2 dx + \frac{C}{\tau - \rho} + \frac{C}{(\tau - \rho)^2}.$$

To bound  $I_5$ , we use a similar argument as that of  $I_3$  but with the following application of Poincaré inequality instead

$$\int_{B_\tau} |wh - (wh)_{B_\tau}|^2 dx \leq C \int_{B_\tau} |\nabla h|^2 dx \int_{B_2} |w|^2 dx + C \int_{B_\tau} |\nabla w|^2 dx \int_{B_2} |h|^2 dx.$$



One then gets

$$I_5 \leq \frac{1}{8} \int_{B_\tau} |\nabla h|^2 dx + \frac{1}{8} \int_{B_\tau} |\nabla w|^2 dx + \frac{C}{\tau - \rho} + \frac{C}{(\tau - \rho)^2}.$$

Lastly, for the term  $I_6$ , by using Hölder inequality and Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} \int \phi^2 h H dx &\leq \|\phi \nabla b\|_{L^4(B_\tau)}^2 \|\nabla u\|_{L^2(B_2)} \\ &\leq C \|\phi \nabla b\|_{L^2(B_\tau)} \|\nabla(\phi \nabla b)\|_{L^2(B_\tau)} \|w\|_{L^2(B_2)} \\ &\leq \frac{1}{8} \|\nabla h\|_{L^2(B_\tau)}^2 + C + \frac{C}{\tau - \rho}, \end{aligned}$$

where we also used (12).

By denoting

$$g(r) = \int_{B_r} |\nabla h|^2 dx + \int_{B_r} |\nabla w|^2 dx,$$

we finally arrive at

$$g(\rho) \leq \frac{1}{2} g(\tau) + C + \frac{C}{\tau - \rho} + \frac{C}{(\tau - \rho)^2}$$

for all  $\frac{3}{4} \leq \rho < \tau \leq 2$ . Then an application of Lemma 2.3 yields

$$\int_{B_1} |\nabla h|^2 dx + \int_{B_1} |\nabla w|^2 dx \leq C.$$

Then the desired bound (20) follows.  $\square$

One direct consequence of Lemma 4.3 is the boundedness of  $u$  and  $b$ .

**Corollary 4.4.** *With the same assumptions as Lemma 4.3, we have*

$$\|u\|_{L^\infty(\mathbb{R}^2; \mathbb{R}^2)} + \|b\|_{L^\infty(\mathbb{R}^2; \mathbb{R}^2)} \leq C(p) < \infty.$$

**4.4. Case  $p = \infty$ .** In Subsection 4.3, it is showed that  $u, b \in L^\infty$  provided  $u, b \in L^p(\mathbb{R}^2; \mathbb{R}^2)$  for some  $p \in (6, \infty)$ . Next we shall start with  $u, b \in L^\infty$ . At first, we strengthen the local estimate (20) into a global one. More precisely, we have

**Lemma 4.5.** *Let smooth solutions  $u, b$  to the MHD equations (1) satisfying*

$$\|u\|_{L^\infty(\mathbb{R}^2; \mathbb{R}^2)} + \|b\|_{L^\infty(\mathbb{R}^2; \mathbb{R}^2)} \leq 1.$$

*Then, there exists an absolute constant  $C_*$  such that if*

$$\|b\|_{L^1(\mathbb{R}^2)} \leq C_* \min\{\mu\nu, \mu^{\frac{1}{2}}\nu^{\frac{3}{2}}\},$$

*there holds*

$$\int_{\mathbb{R}^2} |\nabla^2 u|^2 dx + \int_{\mathbb{R}^2} |\nabla^2 b|^2 dx \leq C.$$

*Proof.* In view of (12), it suffices to show

$$\int_{\mathbb{R}^2} |\nabla w|^2 dx + \int_{\mathbb{R}^2} |\nabla h|^2 dx \lesssim 1.$$

The proof is a modification of the proof of Lemma 4.3. As in the proof of Lemma 4.3, we get

$$\begin{aligned}
& \mu \int \phi^2 |\nabla w|^2 dx + \nu \int \phi^2 |\nabla h|^2 dx \\
&= - \int \nabla w \cdot \nabla \phi^2 w dx - \int \nabla h \cdot \nabla \phi^2 h dx + \frac{1}{2} \int u \cdot \nabla \phi^2 w^2 dx \\
&\quad + \frac{1}{2} \int u \cdot \nabla \phi^2 h^2 dx - \int b \cdot \nabla \phi^2 h w dx + \int H \phi^2 h dx \\
&= I_1 + \cdots + I_6.
\end{aligned} \tag{24}$$

where  $\phi$  is a test function as in the proof of Lemma 4.1 with  $|\nabla \phi| \leq \frac{C}{\tau - \rho}$  and  $|\nabla^2 \phi| \leq \frac{C}{(\tau - \rho)^2}$ . We shall show all the above terms are bounded uniformly in  $R$ .

For the term  $I_1$ , by Corollary 4.2 we have

$$|I_1| = \frac{1}{2} \int w^2 |\Delta \phi^2| dx \leq \frac{C}{(\tau - \rho)^2} \int_{T_\tau} w^2 dx \leq \frac{C\tau}{(\tau - \rho)^2},$$

where  $T_\tau = B_\tau \setminus B_{\frac{2}{3}\tau}$ . For the term  $I_3$ , since  $u \in L^\infty$  and then we have

$$|I_3| = C \int w^2 |\nabla \phi^2| dx \leq \frac{C}{\tau - \rho} \int_{T_\tau} w^2 dx \leq \frac{C\tau}{\tau - \rho}.$$

For the term  $I_2, I_4, I_5$ , similar as  $I_1$  and  $I_3$  we have

$$|I_2| + |I_4| + |I_5| \leq \frac{C\tau}{\tau - \rho} + \frac{C\tau}{(\tau - \rho)^2}.$$

Now we turn to the term  $I_6$ , which is the main difficulty. Without loss of any generality, we may assume  $H = \partial_2 b_2 \partial_2 u_1$ . Applying integration by parts we obtain

$$\begin{aligned}
|I_6| &= \left| \int \phi^2 h \partial_2 b_2 \partial_2 u_1 dx \right| \\
&\leq \left| \int \partial_2 \phi^2 h \partial_2 u_1 b_2 dx \right| + \left| \int \phi^2 \partial_2 h \partial_2 u_1 b_2 dx \right| + \left| \int \phi^2 h \partial_2^2 u_1 b_2 dx \right| \\
&= I_{61} + I_{62} + I_{63}.
\end{aligned}$$

The first term can be bounded easily by using  $b \in L^\infty$  and Corollary 4.2,

$$I_{61} \leq \frac{C}{\tau - \rho} \int_{B_\tau} |h \partial_2 u| dx \leq \frac{C\tau}{\tau - \rho}.$$

The terms  $I_{62}$  and  $I_{63}$  can be treated in a similar way. Therefore we only estimate the former, for which we have

$$\begin{aligned}
& \int \phi^2 \partial_2 h \partial_2 u_1 b_2 dx \\
&= \int \partial_2 h \partial_2 (u_1 \phi) \phi b_2 dx - \int \partial_2 h u_1 \partial_2 \phi b_2 \phi dx \\
&= I_{621} + I_{622}.
\end{aligned}$$

We notice that the term  $I_{622}$  is easy to control,

$$I_{622} \leq \frac{C}{\tau - \rho} \|\nabla h\|_{L^2(B_\tau)} \|\phi u\|_{L^4} \|b\|_{L^4}$$

$$\begin{aligned}
&\leq \frac{C}{\tau - \rho} \|\nabla h\|_{L^2(B_\tau)} \|u\|_{L^4(B_\tau)} \|b\|_{L^1}^{\frac{1}{4}} \|b\|_{L^\infty}^{\frac{3}{4}} \\
&\leq \frac{C\tau^{\frac{1}{2}}}{\tau - \rho} \|\nabla h\|_{L^2(B_\tau)} \|u\|_{L^\infty(B_\tau)} \|b\|_{L^1}^{\frac{1}{4}} \|b\|_{L^\infty}^{\frac{3}{4}},
\end{aligned}$$

which is sufficient for our purpose. Now we turn to the term  $I_{621}$ ,

$$\begin{aligned}
I_{621} &\leq \|\nabla h\|_{L^2(B_\tau)} \|\partial_2(u_1\phi)\phi b_2\|_{L^2} \\
&\leq \|\nabla h\|_{L^2(B_\tau)} \|\nabla(u\phi)\|_{L^4(\mathbb{R}^2)} \|\phi b\|_{L^4(\mathbb{R}^2)}.
\end{aligned}$$

Then we apply the following two Gagliardo-Nirenberg inequalities,

$$\|\nabla f\|_{L^4(\mathbb{R}^2)} \leq \|\nabla^2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|f\|_{L^\infty(\mathbb{R}^2)}^{\frac{1}{2}},$$

and

$$\|f\|_{L^4(\mathbb{R}^2)} \leq \|\nabla^2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|f\|_{L^1(\mathbb{R}^2)}^{\frac{1}{2}},$$

to obtain

$$\begin{aligned}
I_{621} &\leq C \|\nabla h\|_{L^2(B_\tau)} \left( \|\nabla^2(u\phi)\|_{L^2(\mathbb{R}^2)} \|\nabla^2(b\phi)\|_{L^2(\mathbb{R}^2)} \right)^{\frac{1}{2}} \|\phi b\|_{L^1(\mathbb{R}^2)}^{\frac{1}{2}} \|\phi u\|_{L^\infty(\mathbb{R}^2)}^{\frac{1}{2}} \\
&\leq \frac{1}{4} \left( \int_{B_\tau} \mu |\nabla h|^2 + \nu |\nabla w|^2 dx \right) + C + \frac{CR}{(\tau - \rho)^2} + \frac{C\tau^2}{(\tau - \rho)^4},
\end{aligned}$$

provided  $\|b\|_{L^1}$  is small enough and we used (12) and Corollary 4.2.

Finally, by setting

$$g(r) = \int_{B_r} |\nabla h|^2 + |\nabla w|^2 dx,$$

we arrive at

$$g(\rho) \leq \frac{1}{2}g(\tau) + C + \frac{CR}{\tau - \rho} + \frac{CR^2}{(\tau - \rho)^4},$$

where  $\frac{3R}{4} \leq \rho < \tau \leq R$ .

Hence, by Lemma 2.3, we have

$$\int_{B_{R/2}} |\nabla w|^2 + |\nabla h|^2 dx \leq C + \frac{C}{R^2}.$$

Letting  $R \rightarrow \infty$ , the proof is complete.  $\square$

In fact, by assuming  $b$  is small enough in  $L^1$  space, we can conclude that  $\nabla^2 u$  and  $\nabla^2 b$  are both trivial. More precisely, we have

**Lemma 4.6.** *Let  $(u, b)$  be a weak solution of the 2D MHD equations (1) defined over the entire plane. Assume that*

$$\|u\|_{L^\infty(\mathbb{R}^2)} + \|b\|_{L^\infty(\mathbb{R}^2)} \leq 1.$$

*There exists a constant  $C_*$  such that if*

$$\|b\|_{L^1(\mathbb{R}^2)} \leq C_* \min\{\mu\nu, \mu^{\frac{1}{2}}\nu^{\frac{3}{2}}\},$$

*then*

$$\nabla^2 u \equiv 0, \quad \nabla^2 b \equiv 0.$$

*Proof.* In view of Lemma 4.5, we may assume  $\nabla^2 u, \nabla^2 b \in L^2(\mathbb{R}^2)$  in what follows. We shall revisit the proof of Lemma 4.5 and show the terms  $I_1$  to  $I_5$  in (24) vanishes and  $I_6$  becomes small as  $R$  goes to infinity.

Let  $\phi$  be replaced by  $\eta_R$  in Lemma 4.5, then one notices that the terms  $I_1$  and  $I_2$  tend to zero as  $R \rightarrow \infty$ . The treatments for terms  $I_2, I_4, I_5$  are similar, thus we only focus on the term  $I_3$ . Let  $\chi(x)$  be a test function such that

$$\chi_R(x) = \begin{cases} 1, & x \in B_R \setminus B_{R/2}, \\ 0, & x \in B_{2R}^c \cup B_{R/4}. \end{cases}$$

and  $|\nabla^k \chi| \leq \frac{C}{R^k}$ . Then

$$\begin{aligned} I_3 &\leq \frac{C}{R} \int_{B_R \setminus B_{R/2}} |u| w^2 dx \leq \frac{C}{R} \int_{B_R \setminus B_{R/2}} w^2 dx \leq \frac{C}{R} \|\nabla(u\chi_R)\|_{L^2(T_R)}^2 \\ &\leq \frac{C}{R} \|u\|_{L^2(T_R)} [\|\chi_R \nabla^2 u\|_{L^2(T_R)} + \|\nabla u |\nabla \chi_R|\|_{L^2(T_R)} + \|u |\nabla^2 \chi_R|\|_{L^2(T_R)}] \\ &\leq C \|\nabla^2 u\|_{L^2(T_R)} + \frac{C}{\sqrt{R}}, \end{aligned}$$

where  $T_R = B_{2R} \setminus B_{R/4}$ , and we used Corollary 4.2. Obviously,  $I_3$  tends to zero as  $R \rightarrow \infty$  by Lemma 4.5.

Now we turn to the term  $I_6$ . Unlike other terms, we will not show  $I_6$  goes to zero as  $R \rightarrow \infty$ , instead we shall show  $I_6$  tends to something smaller than the left hand side of (24), which implies the desired result.

Without loss of any generality, assume  $H = \partial_2 b_2 \partial_2 u_1$ . Therefore,

$$\begin{aligned} I_6 &= \int_{\mathbb{R}^2} \phi h \partial_2 b_2 \partial_2 u_1 dx \\ &= - \int_{\mathbb{R}^2} \phi h \partial_2^2 u_1 b_2 dx - \int_{\mathbb{R}^2} \phi \partial_2 h \partial_2 u_1 b_2 dx - \int_{\mathbb{R}^2} \partial_2 \phi h \partial_2 u_1 b_2 dx \\ &= J_1 + J_2 + J_3. \end{aligned}$$

Then we shall estimate  $J_1, J_2$ , and  $J_3$  one by one. For the term  $J_1$ ,

$$\begin{aligned} J_1 &\leq \|\partial_2^2 u_1\|_{L^2(\mathbb{R}^2)} \|h\|_{L^4(\mathbb{R}^2)} \|b\|_{L^4(\mathbb{R}^2)} \\ &\leq C \|\partial_2^2 u\|_{L^2(\mathbb{R}^2)} \|b\|_{L^\infty(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla^2 b\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|b\|_{L^1(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla^2 b\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \\ &\leq \frac{\mu}{4} \|\nabla^2 u\|_{L^2(\mathbb{R}^2)} + \frac{\nu}{4} \|\nabla^2 b\|_{L^2(\mathbb{R}^2)}, \end{aligned}$$

provided  $\|b\|_{L^1}$  sufficiently small, that is  $\|b\|_{L^1} \leq C_* \mu \nu$  holds for a small  $C_*$ . For the term  $J_2$ , it can be estimated in the same way

$$\begin{aligned} J_2 &\leq \|\partial_2 h\|_{L^2(\mathbb{R}^2)} \|\partial_2 u\|_{L^4(\mathbb{R}^2)} \|b\|_{L^4(\mathbb{R}^2)} \\ &\leq C \|\partial_2^2 b\|_{L^2(\mathbb{R}^2)} \|u\|_{L^\infty(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla^2 u\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|b\|_{L^1(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla^2 b\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \\ &\leq \frac{\mu}{4} \|\nabla^2 u\|_{L^2(\mathbb{R}^2)} + \frac{\nu}{4} \|\nabla^2 b\|_{L^2(\mathbb{R}^2)}, \end{aligned}$$

provided  $\|b\|_{L^1} \leq C_* \mu^{\frac{1}{2}} \nu^{\frac{3}{2}}$ . The term  $J_3$  can be dealt with similarly as  $I_3$ , which also vanishes as  $R \rightarrow \infty$ . Thus the proof is finished.  $\square$

Now we are ready to finish the remaining part  $6 < p \leq \infty$  of Theorem 1.2.

*Proof of Theorem 1.2: case  $6 < p \leq \infty$ .* For  $6 < p \leq \infty$ , assume that  $(u, b)$  are nontrivial and

$$\|u\|_{L^p(\mathbb{R}^2)} + \|b\|_{L^p(\mathbb{R}^2)} = L > 0 \quad (25)$$

then consider the scaling solution  $(u^\lambda(x), b^\lambda(x))$ , where

$$u^\lambda(x) = \lambda u(\lambda x), \quad b^\lambda(x) = \lambda b(\lambda x)$$

Then by scaling property we get

$$\|u^\lambda\|_{L^p(\mathbb{R}^2)} + \|b^\lambda\|_{L^p(\mathbb{R}^2)} \leq 1$$

if  $\lambda^{-\frac{p-2}{p}} = L$ . By the assumption of Theorem 1.2, we get there exists an absolute constant  $C_*$  such that

$$L^{\frac{p}{p-2}} \|b\|_{L^1(\mathbb{R}^2)} \leq C_* \min\{\mu\nu, \mu^{\frac{1}{2}}\nu^{\frac{3}{2}}\}$$

and hence

$$\|b^\lambda\|_{L^1(\mathbb{R}^2)} = \lambda^{-1} \|b\|_{L^1(\mathbb{R}^2)} \leq C_* \min\{\mu\nu, \mu^{\frac{1}{2}}\nu^{\frac{3}{2}}\} \quad (26)$$

Then it follows from Corollary 4.4 and Lemma 4.6 that

$$\nabla^2(u^\lambda) \equiv 0, \quad \nabla^2(b^\lambda) \equiv 0,$$

which implies  $u, b$  are constants, since  $u, b \in L^p(\mathbb{R}^2)$ . This is a contradiction with (25). Hence  $(u, b)$  are trivial solutions.

The proof is complete.  $\square$

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