Global well-posedness of the one-dimensional cubic nonlinear Schrödinger equation in almost critical spaces

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GLOBAL WELL-POSEDNESS OF THE ONE-DIMENSIONAL CUBIC NONLINEAR SCHRÖDINGER EQUATION IN ALMOST CRITICAL SPACES

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Abstract. In this paper, we first introduce a new function space $MH^{\theta,p}$ whose norm is given by the ℓ^p-sum of modulated H^θ-norms of a given function. In particular, when $\theta < -\frac{1}{2}$, we show that the space $MH^{\theta,p}$ agrees with the modulation space $M^{2,p}(\mathbb{R})$ on the real line and the Fourier-Lebesgue space $\mathcal{F}L^p(\mathbb{T})$ on the circle. We use this equivalence of the norms and the Galilean symmetry to adapt the conserved quantities constructed by Killip-Vişan-Zhang to the modulation space and Fourier-Lebesgue space setting. By applying the scaling symmetry, we then prove global well-posedness of the one-dimensional cubic nonlinear Schrödinger equation (NLS) in almost critical spaces. More precisely, we show that the cubic NLS on $\mathbb{R}$ is globally well-posed in $M^{2,p}(\mathbb{R})$ for any $p < \infty$, while the renormalized cubic NLS on $\mathbb{T}$ is globally well-posed in $\mathcal{F}L^p(\mathbb{T})$ for any $p < \infty$.

In Appendix, we also establish analogous global-in-time bounds for the modified KdV equation (mKdV) in the modulation spaces on the real line and in the Fourier-Lebesgue spaces on the circle. An additional key ingredient of the proof in this case is a Galilean transform which converts the mKdV to the mKdV-NLS equation.

Contents

1. Introduction

1.1. One-dimensional cubic nonlinear Schrödinger equation. In this paper, we study the following Cauchy problem of the one-dimensional cubic nonlinear Schrödinger

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equation (NLS) on $\mathcal{M} = \mathbb{R}$ or $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$:

$$\begin{cases} i\partial_t u = \partial_x^2 u \mp 2|u|^2 u, \\ u|_{t=0} = u_0. \end{cases} \tag{1.1}$$

The equation (1.1) arises in various physical situations for the description of wave propagation in nonlinear optics, fluids, and plasmas; see [53] for a general review. It is also known to be one of the simplest partial differential equations (PDEs) with complete integrability [58, 1, 2, 22]. Our main goal in this paper is to exploit the complete integrable structure of the equation and prove global well-posedness of (1.1) in almost critical spaces.

The Cauchy problem (1.1) has been studied extensively by many mathematicians. Tsutsumi [56] and Bourgain [7] proved global well-posedness of (1.1) in $L^2(\mathcal{M})$ with $\mathcal{M} = \mathbb{R}$ and $\mathbb{T}$, respectively. Before going over the known results for (1.1) below $L^2(\mathcal{M})$, let us first recall two important symmetries that (1.1) enjoys. The scaling symmetry states that if $u(x,t)$ is a solution to (1.1) on $\mathcal{M} = \mathbb{R}$ with initial data u_0, then the λ-scaled function

$$u(x,t) \longmapsto u_\lambda(x,t) = \lambda^{-1} u(\lambda^{-1}x, \lambda^{-2}t) \tag{1.2}$$

is also a solution to (1.1) with the λ-scaled initial data $u_{0,\lambda}(x) = \lambda^{-1}u_0(\lambda^{-1}x)$. Associated to this scaling symmetry, there is a scaling-critical Sobolev regularity s_{crit} such that the homogeneous $\dot{H}^{s_{\text{crit}}}$-norm is invariant under the scaling symmetry. In the case of the one-dimensional cubic NLS (1.1), the scaling-critical Sobolev regularity is $s_{\text{crit}} = -\frac{1}{2}$ and it is known that (1.1) is ill-posed in $H^s(\mathcal{M})$ for $s \le s_{\text{crit}} = -\frac{1}{2}$ with $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$ in the sense of norm inflation [14, 36, 46, 44]; given any $\varepsilon > 0$, there exist a solution u to (1.1) on $\mathcal{M}$ and $t_\varepsilon \in (0, \varepsilon)$ such that

$$\|u(0)\|_{H^s(\mathcal{M})} < \varepsilon \qquad \text{and} \qquad \|u(t_\varepsilon)\|_{H^s(\mathcal{M})} > \varepsilon^{-1}. \tag{1.3}$$

Note that this is a stronger notion of ill-posedness than the failure of continuity of the solution map at $u_0 \equiv 0$. The other symmetry of importance here is the Galilean symmetry; if $u(x,t)$ is a solution to (1.1) on $\mathbb{R}$ with initial condition u_0, then

$$u^\beta(x,t) = \mathcal{G}_\beta(u)(x,t) := e^{-i\beta x} e^{i\beta^2 t} u(x - 2\beta t, t) \tag{1.4}$$

is also a solution to (1.1) with the modulated initial condition $u_0^\beta(x) = e^{-i\beta x}u_0(x)$. On the Fourier side, the Galilean symmetry is expressed as

$$\widehat{u^\beta}(\xi,t) = e^{-i\beta^2 t} e^{-2i\beta\xi t} \widehat{u}(\xi + \beta, t),$$

basically corresponding to a translation in frequencies. In particular, we need to impose $\beta \in \mathbb{Z}$ on the circle $\mathcal{M} = \mathbb{T}$. We point out that the Galilean symmetry induces another critical regularity $s_{\text{crit}}^\infty = 0$ since it preserves the L^2-norm. In fact, there is a dichotomy between the behavior of solutions to (1.1) in $L^2(\mathcal{M})$ and in negative Sobolev spaces. On the one hand, (1.1) is known to be well-posed in $L^2(\mathcal{M})$. On the other hand, it is known to be mildly ill-posed in negative Sobolev spaces in the sense of the failure of local uniform continuity of the solution map: $\Phi(t) : u_0 \in H^s(\mathcal{M}) \mapsto u(t) \in H^s(\mathcal{M})$; see [33, 9, 12]. Moreover, on the circle, (1.1) is known to be ill-posed below $L^2(\mathbb{T})$; see [13, 40, 28]. In particular, in [28], the first author (with Z. Guo) showed non-existence of solutions to (1.1) on $\mathbb{T}$ with initial data outside $L^2(\mathbb{T})$. This last result was proved by first establishing an existence result for the following renormalized cubic NLS on $\mathbb{T}$:

$$\begin{cases} i\partial_t u = \partial_x^2 u \mp 2(|u|^2 - 2 \fint_{\mathbb{T}} |u|^2 dx)u \\ u|_{t=0} = u_0, \end{cases} \qquad (x,t) \in \mathbb{T} \times \mathbb{R}, \tag{1.5}$$

where $\fint f(x)dx := \frac{1}{2\pi}\int_{\mathbb{T}} f(x)dx$. In fact, it is known that, while it is equivalent to the standard cubic NLS (1.1) within $L^2(\mathbb{T})$, this renormalized cubic NLS (1.5) on $\mathbb{T}$ behaves better outside $L^2(\mathbb{T})$ and in fact share many common properties with the cubic NLS on $\mathbb{R}$ outside L^2; see a survey paper [45]. Note that there is no ill-posedness result below $L^2(\mathcal{M})$ for the cubic NLS (1.1) on $\mathbb{R}$ or the renormalized cubic NLS (1.5) on $\mathbb{T}$, contradicting either existence, uniqueness, or continuous dependence, and the well-posedness of (1.1) on $\mathbb{R}$ and (1.5) on $\mathbb{T}$ in negative Sobolev spaces (in particular uniqueness) has been a long-standing challenging open question in the field. In [15, 37, 38], Christ-Colliander-Tao and Koch-Tataru independently proved existence (without uniqueness) of solutions to the cubic NLS (1.1) on $\mathbb{R}$ in negative Sobolev spaces. An analogous existence result for the renormalized cubic NLS (1.5) on $\mathbb{T}$ was established in [28]. More recently, Koch-Tataru [39] and Killip-Vişan-Zhang [34] exploited the complete integrable structure of the equation and proved global-in-time a priori bounds on the H^s-norm of solutions in the scaling-subcritical range: $s > -\frac{1}{2}$. In the following, we combine the result in [34] with the scaling and Galilean symmetries to prove global well-posedness of the cubic NLS (1.1) on $\mathbb{R}$ and the renormalized cubic NLS (1.5) on $\mathbb{T}$ in almost critical spaces with respect to the scaling symmetry.

1.2. **Fourier-Lebesgue spaces and modulation spaces.** In this subsection, we first recall the definitions of the Fourier-Lebesgue spaces and the modulation spaces. Then, we go over the known well-posedness results for the cubic NLS (1.1) on $\mathbb{R}$ and the renormalized cubic NLS (1.5) on $\mathbb{T}$ in these spaces. Lastly, we introduce a new function space $MH^{\theta,p}$ and show that this space coincides with the modulation spaces on $\mathbb{R}$ and the Fourier-Lebesgue spaces on $\mathbb{T}$ in a certain regime (Lemma 1.2). This equivalence of the norms will be a key ingredient for the proof of the main result (Theorem 1.4).

Our conventions for the Fourier transform are as follows:

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-ix\xi}dx \qquad \text{and} \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{f}(\xi)e^{ix\xi}d\xi$$

for functions on the real line $\mathbb{R}$ and

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} f(x)e^{-ix\xi}dx \qquad \text{and} \qquad f(x) = \frac{1}{\sqrt{2\pi}}\sum_{\xi\in\mathbb{Z}} \widehat{f}(\xi)e^{ix\xi}$$

for functions on the circle $\mathbb{T}$ (with $\xi \in \mathbb{Z}$). Given $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$, let $\widehat{\mathcal{M}}$ denote the Pontryagin dual of $\mathcal{M}$, i.e.

$$\widehat{\mathcal{M}} = \begin{cases} \mathbb{R} & \text{if } \mathcal{M} = \mathbb{R}, \\ \mathbb{Z} & \text{if } \mathcal{M} = \mathbb{T}. \end{cases}$$

When $\widehat{\mathcal{M}} = \mathbb{Z}$, we endow it with the counting measure. Given $s \in \mathbb{R}$ and $1 \le p \le \infty$, we define the Fourier-Lebesgue space $\mathcal{F}L^{s,p}(\mathcal{M})$ by the norm:

$$\|f\|_{\mathcal{F}L^{s,p}(\mathcal{M})} = \|\langle\xi\rangle^s \widehat{f}(\xi)\|_{L^p_\xi(\widehat{\mathcal{M}})}$$

with the usual modification when $p=\infty$. Here, $\langle\,\cdot\,\rangle=(1+|\cdot|^2)^{\frac12}$. When $s=0$, we simply set $\mathcal{F}L^p(\mathcal{M})=\mathcal{F}L^{0,p}(\mathcal{M})$. Note that we have $\mathcal{F}L^p(\mathbb{T})\supset L^2(\mathbb{T})$ on the circle for $p\geq 2$.

Next, we recall the definition of the modulation spaces $M^{r,p}_s(\mathbb{R})$ on the real line; see [19, 20]. Let $\psi\in\mathcal{S}(\mathbb{R})$ such that

$$\operatorname{supp}\psi\subset[-1,1]\qquad\text{and}\qquad\sum_{k\in\mathbb{Z}}\psi(\xi-k)\equiv 1.$$

Then, the modulation space $M^{r,p}_s(\mathbb{R})$ is defined as the collection of all tempered distributions $f\in\mathcal{S}'(\mathbb{R})$ such that $\|f\|_{M^{r,p}_s}<\infty$, where the $M^{r,p}_s$-norm is defined by

$$\|f\|_{M^{r,p}_s(\mathbb{R})}=\big\|\langle n\rangle^s\|\psi_n(D)f\|_{L^r_x(\mathbb{R})}\big\|_{\ell^p_n(\mathbb{Z})}. \tag{1.6}$$

Here, $\psi_n(D)$ is the Fourier multiplier operator with the multiplier

$$\psi_n(\xi):=\psi(\xi-n). \tag{1.7}$$

When $s=0$, we simply set $M^{r,p}(\mathbb{R})=M^{r,p}_0(\mathbb{R})$. In the following, we only consider $r=2$. In this case, we have

$$M^{2,p}(\mathbb{R})\supset\mathcal{F}L^p(\mathbb{R})$$

for $p\geq 2$.

Remark 1.1. The modulation spaces $M^{r,p}$ have an equivalent characterization via the short-time (or windowed) Fourier transform (STFT). Given a non-zero window function $\phi\in\mathcal{S}(\mathbb{R})$, we define the STFT $V_\phi f$ of a tempered distribution $f\in\mathcal{S}'(\mathbb{R})$ with respect to ϕ by

$$V_\phi f(x,\xi)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}f(y)\overline{\phi(y-x)}e^{-iy\xi}\,dy.$$

Then, we have the equivalence of norms:

$$\|f\|_{M^{r,p}}\sim_\phi |||f|||_{M^{r,p}}:=\big\|\|V_\phi f\|_{L^r_x}\big\|_{L^p_\xi}, \tag{1.8}$$

where the implicit constants depend on the window function ϕ. In view of the definition (1.8) of the $|||\cdot|||_{M^{r,p}}$-norm, it may be tempting to consider this norm on $\mathbb{T}$. It is, however, known that, for $1\leq r,p\leq\infty$, we have

$$M^{r,p}(\mathbb{T})=\mathcal{F}L^p(\mathbb{T}) \tag{1.9}$$

on the circle; see [50].

Let us now discuss critical regularities for (1.1) and (1.5) in the context of the Fourier-Lebesgue spaces and the modulation spaces. A direct computation shows that the homogeneous Fourier-Lebesgue space $\dot{\mathcal{F}L}^{s,p}(\mathbb{R})$ is invariant under the scaling symmetry (1.2) when $s=s_{\text{crit}}(p)=-\frac1p$ with the understanding $s_{\text{crit}}(\infty)=0$. In particular, when $s=0$, the cubic NLS (1.1) on $\mathbb{R}$ is scaling-critical in $\mathcal{F}L^\infty(\mathbb{R})$. While there is no scaling symmetry on the circle, we say that the renormalized cubic NLS (1.5) on $\mathbb{T}$ is scaling-critical in $\mathcal{F}L^\infty(\mathbb{T})$. On the other hand, the modulation spaces are based on the unit cube decomposition of the frequency space and thus there is no scaling for the modulation spaces.[1] At the same

[1]See [6] for modulation spaces adapted to scaling.

time, a change of variables and interpolating the $r = 2$ and $r = \infty$ cases yield the following bound:

$$\|f_\lambda\|_{\dot{M}^{r,p}_s} \lesssim \lambda^{-s-\frac{1}{p}} \|f\|_{\dot{M}^{r,p}_s} \tag{1.10}$$

any $\lambda \geq 1$, provided that $p \geq r'$ and $r \geq 2$. See also [52] for a further discussion on the scaling properties of the modulation spaces. This shows that $s = s_{\text{crit}}(p) = -\frac{1}{p}$ is (essentially) a scaling-critical regularity for (1.1) in terms of the modulation spaces $M^{r,p}_s(\mathbb{R})$. In particular, when $s = 0$ the cubic NLS (1.1) on $\mathbb{R}$ is (essentially) scaling-critical in $M^{2,\infty}(\mathbb{R})$. We point out that a typical function in these critical spaces $\mathcal{F}L^\infty(\mathcal{M})$ and $M^{2,\infty}(\mathbb{R})$ is the Dirac delta function and that (1.1) on $\mathbb{R}$ and (1.5) on $\mathbb{T}$ are known to be ill-posed with the Dirac delta function as initial data; see [33, 21]. See also Banica-Vega [4, 5] for the work on the cubic NLS (1.1) with the Dirac delta function as initial data.

The Cauchy problems (1.1) on $\mathbb{R}$ and (1.5) on $\mathbb{T}$ have been studied in the context of the Fourier-Lebesgue spaces and the modulation spaces. In particular, local well-posedness in almost critical spaces have been known. In [24], Grünrock studied the cubic NLS (1.1) on $\mathbb{R}$ in the Fourier-Lebesgue spaces and proved local well-posedness in $\mathcal{F}L^p(\mathbb{R})$, $1 < p < \infty$, almost reaching the critical case $p = \infty$. He also proved global well-posedness for $2 \leq p < \frac{5}{2}$. We also mention a precursor to this result by Vargas-Vega [57], establishing well-posedness of (1.1) on $\mathbb{R}$ with infinite L^2-norm initial data. In a recent paper, S. Guo [27] proved local well-posedness of the cubic NLS (1.1) on $\mathbb{R}$ in $M^{2,p}(\mathbb{R})$ for $2 \leq p < \infty$. In the periodic setting, Grünrock-Herr [25] proved local well-posedness of the renormalized cubic NLS (1.5) in $\mathcal{F}L^p(\mathbb{T})$, $1 < p < \infty$. See also Christ [11] for a construction of solutions to (1.5) (without uniqueness) via a power series expansion. In the same paper, Christ also refers to an unpublished work with Erdoğan, claiming small data global well-posedness in $\mathcal{F}L^p(\mathbb{T})$, $p < \infty$. We point out that, as a consequence of the local well-posedness result in [25], the non-existence result in [28] also applies to the cubic NLS (1.1) in the Fourier-Lebesgue setting: $\mathcal{F}L^p(\mathbb{T}) \setminus L^2(\mathbb{T})$, $p > 2$.

While there are some global well-posedness results in the context of the modulation and Fourier-Lebesgue spaces, it is very far from matching the local well-posedness regularities. In the following, we close this gap and prove global well-posedness in almost critical spaces. For this purpose, we first introduce the following modulated Sobolev space $MH^{\theta,p}(\mathbb{R})$ by the norm:

$$\begin{aligned} \|f\|_{MH^{\theta,p}(\mathbb{R})} &= \bigg(\sum_{n\in\mathbb{Z}} \|M_n f\|^p_{H^\theta}\bigg)^{\frac{1}{p}} \\ &= \bigg(\sum_{n\in\mathbb{Z}} \|\langle \xi - n\rangle^\theta \widehat{f}(\xi)\|^p_{L^2_\xi}\bigg)^{\frac{1}{p}}, \end{aligned} \tag{1.11}$$

where M_n denotes the modulation operator defined by

$$M_n f(x) = e^{-inx} f(x). \tag{1.12}$$

On the circle, we define $MH^{\theta,p}(\mathbb{T})$ in an analogous manner. When $\theta \geq 0$, we have $\|f\|_{MH^{\theta,p}} < \infty$ if and only if $f = 0$. Hence, we focus on $\theta < 0$ in the following. In fact, when $\theta < -\frac{1}{2}$, it is easy to see that the $MH^{\theta,p}$-norm is equivalent to the $M^{2,p}$-norm.

Lemma 1.2. (i) *Let $\theta < -\frac{1}{2}$ and $2 \le p \le \infty$. Then, we have*

$$\|f\|_{MH^{\theta,p}} \sim \|f\|_{M^{2,p}}$$

with the understanding that $M^{2,p}(\mathbb{T}) = \mathcal{F}L^p(\mathbb{T})$ on the circle.
(ii) *Let $-\frac{1}{2} \le \theta < 0$ and $2 \le q < p \le \infty$. Then, we have*

$$\|f\|_{M^{2,p}} \lesssim \|f\|_{MH^{\theta,p}} \lesssim \begin{cases} \|f\|_{M^{2,q}}, \\ \|f\|_{M^{2,p}_s}, \end{cases} \tag{1.13}$$

provided that $\frac{1}{q} > \frac{1}{p} + \frac{1}{2} + \theta$ and $s > \frac{1}{2} + \theta$.

The proof of this lemma is elementary and is presented in Section 2. This equivalence of the norms for $\theta < -\frac{1}{2}$ allows us to express the relevant modulation norms on $\mathbb{R}$ and the Fourier-Lebesgue norms on $\mathbb{T}$ in terms of the ℓ^p-sum of the modulated Sobolev norms. On the one hand, we introduce the $MH^{\theta,p}$-norm for a PDE purpose and use it for $\theta = -1$. On the other hand, when $-\frac{1}{2} \le \theta < 0$, it lies between $M^{2,p}$ and $M^{2,q}$ for q satisfying $\frac{1}{q} > \frac{1}{p} + \frac{1}{2} + \theta$. Hence, it may be of interest to study finer properties of $MH^{\theta,p}$. One may also replace the weight $\langle \xi - n\rangle^\theta$ by a general weight function $\phi(\xi - n)$ and define the modulated Sobolev space $MH^{\phi,p}$ adapted to the weight function ϕ via the norm:

$$\|f\|_{MH^{\phi,p}(\mathbb{R})} = \bigg(\sum_{n\in\mathbb{Z}} \|\phi(\xi - n)\widehat{f}(\xi)\|^p_{L^2_\xi}\bigg)^{\frac{1}{p}}.$$

Arguing as in the proof of Lemma 1.2, one can easily prove that

$$\|f\|_{MH^{\phi,p}} \sim \|f\|_{M^{2,p}}$$

for $\phi \in L^2(\mathbb{R})$ which is bounded away from 0 on $[-\frac{1}{2}, \frac{1}{2}]$.

1.3. **Main result.** We briefly go over the main result in the work [34] by Killip-Vişan-Zhang. See [34] for more details. The one-dimensional cubic NLS (1.1) is a completely integrable PDE and it admits the following Lax pair formulation [58, 1]:

$$\frac{d}{dt}L(t;\kappa) = \big[P(t,\kappa), L(t;\kappa)\big],$$

where

$$L(t;\kappa) = \begin{pmatrix} -\partial_x + \kappa & iu \\ \mp i\overline{u} & -\partial_x - \kappa \end{pmatrix}$$

and $P(t;\kappa)$ denotes some operator pencil whose precise form does not play any role in the following. In [34], the authors studied the following perturbation determinant $\alpha(\kappa; u)$:

$$\alpha(\kappa;u) = \operatorname{Re}\sum_{j=1}^\infty \frac{(\mp1)^{j-1}}{j}\operatorname{tr}\Big\{\big[(\kappa - \partial_x)^{-\frac{1}{2}} u(\kappa+\partial_x)^{-1}\overline{u}(\kappa - \partial_x)^{-\frac{1}{2}}\big]^j\Big\}. \tag{1.14}$$

Here, the operators $(\kappa\pm\partial_x)^{-1}$ and $(\kappa\pm\partial_x)^{-\frac{1}{2}}$ are defined as the Fourier multiplier operators. For an operator A on $L^2(\mathcal{M})$ with a continuous integral kernel $K(x,y)$, we define its trace by

$$\operatorname{tr}(A) = \int_{\mathcal{M}} K(x,x)dx.$$

In particular, if A is a Hilbert-Schmidt operator with an integral kernel $K(x,y) \in L^2(\mathcal{M}^2)$, then we have

$$\operatorname{tr}(A^2) = \iint_{\mathcal{M}^2} K(x,y)K(y,x)dxdy.$$

We also set

$$\|A\|_{\mathfrak{I}_2}^2 = \operatorname{tr}(A^*A) = \iint_{\mathcal{M}^2} |K(x,y)|^2 dxdy.$$

Recall from [34, Lemma 1.4] that

$$|\operatorname{tr}(A_1 \cdots A_k)| \le \prod_{j=1}^{k} \|A_j\|_{\mathfrak{I}_2}. \tag{1.15}$$

In the following, we summarize three important properties of $\alpha(\kappa; u)$. Here, we only state the real line case. For the periodic case, the corresponding statements are basically true with a small change in (1.17) for the leading term in the series (1.14); see Lemma 3.3 below.

Lemma 1.3. *The following statements hold:*

(i) [34, Proposition 4.3]*: For a Schwartz class solution u to* (1.1)*, the quantity $\alpha(\kappa; u)$ is conserved, provided that $\kappa > 0$ is sufficiently large such that*

$$\int_{\mathbb{R}} \log\big(4 + \tfrac{\xi^2}{\kappa^2}\big) \frac{|\widehat{u}(\xi)|^2}{\sqrt{4\kappa^2 + \xi^2}} d\xi \le c_0 \tag{1.16}$$

for some absolute constant $c_0 > 0$.

(ii) [34, Lemma 4.2]*: The leading term of the series expansion* (1.14) *is given by*

$$\operatorname{Re}\operatorname{tr}\big\{(\kappa - \partial_x)^{-1} u (\kappa + \partial_x)^{-1} \overline{u}\big\} = \int_{\mathbb{R}} \frac{2\kappa |\widehat{u}(\xi)|^2}{4\kappa^2 + \xi^2} d\xi \tag{1.17}$$

for any $\kappa > 0$ and $u \in \mathcal{S}(\mathbb{R})$.

(iii) [34, Lemma 4.1]*: We have*

$$\big\|(\kappa - \partial_x)^{-\frac{1}{2}} u (\kappa + \partial_x)^{-\frac{1}{2}}\big\|_{\mathfrak{I}_2(\mathbb{R})}^2 \sim \int_{\mathbb{R}} \log\big(4 + \tfrac{\xi^2}{\kappa^2}\big) \frac{|\widehat{u}(\xi)|^2}{\sqrt{4\kappa^2 + \xi^2}} d\xi \tag{1.18}$$

for any $\kappa > 0$ and $u \in \mathcal{S}(\mathbb{R})$.

In view of (1.15) and (1.18), this smallness condition (1.16) guarantees term-by-term differentiation of the series (1.14). Using the properties (i) - (iii), Killip-Vişan-Zhang [34] proved the following global-in-time bound (Theorem 1.3 in [34]) on the H^s-norm of smooth solutions to (1.1) on $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$:

$$\|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s} \Big(1 + \|u_0\|_{H^s}\Big)^{\frac{|s|}{1-|s|}} \tag{1.19}$$

for $-\frac{1}{2} < s < 0$. The main idea of the argument in [34] is to express the H^s-norms (and in fact the Besov norms) as a suitable sum of the right-hand side of (1.17) as κ ranges over dyadic numbers $\kappa_0 \cdot 2^{\mathbb{N}}$ (for some $\kappa_0 > 0$). See Lemma 3.2 and the Z_{κ_0}-norm in the proof of Theorem 4.5 in [34]. The property (iii) above is then to control the error terms (i.e. $j \ge 2$)

in (1.14), which imposes the restriction $s > -\frac{1}{2}$. Note that the restriction $s > -\frac{1}{2}$ is necessary in view of the norm inflation at the critical regularity $s = -\frac{1}{2}$ [36, 46, 44].

We point out that the global-in-time bound (1.19) also holds for smooth solutions u to the renormalized cubic NLS (1.5) on $\mathbb{T}$ since we can convert smooth solutions to the cubic NLS (1.1) and to the renormalized cubic NLS (1.5) by the following invertible gauge transform:

$$\mathcal{J}(u)(t) := e^{\mp 4it \fint |u(t)|^2 dx} u(t), \tag{1.20}$$

while the gauge transform $\mathcal{J}$ preserves the H^s-norm. As mentioned above, uniqueness of solutions to the cubic NLS (1.1) on $\mathbb{R}$ and the renormalized cubic NLS (1.5) on $\mathbb{T}$ in negative Sobolev spaces remains as a very challenging open question. Hence, while the global-in-time bound (1.19) may be used to prove global existence of solutions (without uniqueness) in negative Sobolev spaces, it does not provide global well-posedness at this point.

In the following, we establish global-in-time bound on the $M^{2,p}$-norm of smooth solutions to the cubic NLS (1.1) on $\mathbb{R}$ and the $\mathcal{F}L^p$-norm of smooth solutions to the renormalized cubic NLS (1.5) on $\mathbb{T}$. Then, the local well-posedness in these spaces [27, 25] yields the following global well-posedness result.

Theorem 1.4. *Let $2 \le p < \infty$.*

(i) *There exists $C = C(p) > 0$ such that*

$$\|u(t)\|_{M^{2,p}(\mathbb{R})} \le C(1 + \|u(0)\|_{M^{2,p}(\mathbb{R})})^{\frac{p}{2}-1} \|u(0)\|_{M^{2,p}(\mathbb{R})} \tag{1.21}$$

for any Schwartz class solution u to (1.1) *on $\mathbb{R}$ and any $t \in \mathbb{R}$. In particular, the cubic NLS* (1.1) *on $\mathbb{R}$ is globally well-posed in $M^{2,p}(\mathbb{R})$.*

(ii) *There exists $C = C(p) > 0$ such that*

$$\|u(t)\|_{\mathcal{F}L^p(\mathbb{T})} \le C\big(1 + \|u(0)\|_{\mathcal{F}L^p(\mathbb{T})}\big)^{\frac{p}{2}-1} \|u(0)\|_{\mathcal{F}L^p(\mathbb{T})} \tag{1.22}$$

for any smooth solution u to (1.1) *on $\mathbb{T}$ and any $t \in \mathbb{R}$. In particular, the renormalized cubic NLS* (1.5) *on $\mathbb{T}$ is globally well-posed in $\mathcal{F}L^p(\mathbb{T})$.*

Theorem 1.4 establishes global well-posedness of the cubic NLS (1.1) on $\mathbb{R}$ and the renormalized cubic NLS (1.5)on $\mathbb{T}$ in almost critical spaces, improving significantly the known global well-posedness results [57, 24, 11]. Moreover, the range of $p < \infty$ of Theorem 1.4 is sharp in view of the ill-posedness results with the Dirac delta as initial data which lies in $M^{2,\infty}(\mathbb{R})$ and $\mathcal{F}L^\infty(\mathbb{T})$.[2] See also Remark 1.6 below. In view of the local well-posedness [27, 25], it suffices to establish a priori global-in-time bounds (1.21) on $\mathbb{R}$ and (1.22) on $\mathbb{T}$, controlling the $M^{2,p}$-norms[3] of smooth solutions to the cubic NLS (1.1) on $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$. In the periodic setting, we then obtain the same global-in-time bound (1.22) for smooth solutions to the renormalized cubic NLS (1.5) via the gauge transform (1.20), yielding global well-posedness for the renormalized cubic NLS (1.5) on $\mathbb{T}$. The main idea for proving Theorem 1.4 is to use the equivalence of the norms for the modulation spaces $M^{2,p}$ and

[2] In [21], the ill-posedness result with the Dirac initial data on $\mathbb{T}$ was shown in a topology weaker than $\mathcal{F}L^\infty(\mathbb{T})$. We also point out that $\mathcal{F}L^\infty(\mathbb{T})$ does not admit smooth approximations and hence an a priori bound on the $\mathcal{F}L^\infty(\mathbb{T})$-norm for smooth solutions would not yield the same bound for rough solutions.

[3] In view of (1.9) on the circle, we may use $M^{2,p}(\mathbb{T})$ for $\mathcal{F}L^p(\mathbb{T})$ in the following.

the modulated Sobolev spaces $MH^{-1,p}$ (Lemma 1.2). We then apply the result in [34] to control the growth of the $MH^{-1,p}$-norm. Here, both the scaling and Galilean symmetries play an important role. It follows from (1.11), (1.14), and (1.17) that (the square of) the $MH^{-1,p}$-norm of a solution u is given by the $\ell^{\frac{p}{2}}$-sum of the leading terms for $\alpha(\frac{1}{2}; \mathcal{G}_n(u))$, $n \in \mathbb{Z}$, where $\mathcal{G}_n$ is the Galilean transform defined in (1.4). On the one hand, the Galilean symmetry and Lemma 1.3 imply the conservation of $\alpha(\frac{1}{2}; \mathcal{G}_n(u))$ in the small data case. On the other hand, the scaling symmetry with the subcriticality of the underlying space $M^{2,p}$, $p < \infty$, allows us to reduce the situation to the small data case and handle the error terms in the series (1.14). Compare this with [34], where the main idea in this step is to express the H^s-norm as a suitable sum of the leading terms of $\alpha(\kappa; u)$ as κ ranges over dyadic numbers $\kappa_0 \cdot 2^{\mathbb{N}}$ (for some $\kappa_0 \gg 1$). Here, taking $\kappa_0 \gg 1$ essentially has an effect of scaling, reducing to the small data case (in the subcritical regularity $s > -\frac{1}{2}$). Lastly, we note that, as in [34], we can control the error terms only in the subcritical range, i.e. $p < \infty$ in our setting.

Remark 1.5. (i) In Appendix A, we consider the following complex-valued modified KdV equation on $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$:

$$\partial_t u = -\partial_x^3 u \pm 6|u|^2 \partial_x u$$

and briefly discuss how to derive the same global-in-time bounds (1.21) and (1.22) for Schwartz/smooth solutions to the mKdV. See Theorem A.2.

(ii) The global-in-time bounds (1.21) and (1.22) can be extended to the modulation spaces $M^{2,p}_s(\mathbb{R})$ and the Fourier-Lebesgue spaces $\mathcal{F}L^{s,p}(\mathbb{T})$ of higher regularities. See Appendix B.

Remark 1.6. In [27], S. Guo proved local well-posedness of the cubic NLS (1.1) on $\mathbb{R}$ in a space whose norm is logarithmically stronger than the critical $M^{2,\infty}(\mathbb{R})$-norm. The space is characterized by an Orlicz norm and contains functions whose Fourier transforms decay only logarithmically, i.e. not belonging to $M^{2,p}(\mathbb{R})$ for any finite $p < \infty$. It seems of interest to study the global-in-time behavior of solutions in this logarithmically subcritical space. We also remark that, in [34], Killip-Vişan-Zhang also established global-in-time bounds for solutions to the cubic NLS (1.1) on $\mathbb{R}$ and $\mathbb{T}$ in negative Besov-type spaces which are logarithmically stronger than the critical $H^{-\frac{1}{2}}$, where the norm inflation (1.3) is known.

Remark 1.7. In [18], the first author (with Colliander) studied the renormalized cubic NLS (1.5) on $\mathbb{T}$ with random initial data of the form:

$$u_0^\alpha(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \frac{g_n}{\langle n \rangle^\alpha} e^{inx}, \tag{1.23}$$

where $\{g_n\}_{n\in\mathbb{Z}}$ is a sequence of independent standard complex-valued Gaussian random variables. It is easy to see that such u_0^α almost surely belongs to $H^{\alpha-\frac{1}{2}-\varepsilon}(\mathbb{T}) \setminus H^{\alpha-\frac{1}{2}}(\mathbb{T})$. When $\alpha = 0$, this corresponds to the white noise on $\mathbb{T}$ and is of significant importance to study (1.5) with the white noise initial data. It is also easy to see that u_0^α in (1.23) almost surely belongs to $\mathcal{F}L^p(\mathbb{T})$ for $p > \alpha^{-1}$. Therefore, Theorem 1.4 (ii) yields (a deterministic proof of) almost sure global well-posedness of (1.5) with almost white noise initial data u_0^α, $\alpha > 0$. The $\alpha = 0$ case remains as an important open problem.

2. Equivalence of the norms

In this section, we present a proof of Lemma 1.2. In the following, we only consider the real line case since the proof for the periodic case follows in a similar manner. Obviously, we have

$$\|f\|_{M^{2,p}} \lesssim \|f\|_{MH^{\theta,p}}$$

since $\psi_n(\xi) \lesssim \langle \xi - n\rangle^\theta$, where ψ_n is as in (1.7).

Let $I_k = [k - \frac{1}{2}, k + \frac{1}{2})$, $k \in \mathbb{Z}$. By writing

$$\begin{aligned} \|f\|_{MH^{\theta,p}} &= \Big\| \|\langle \xi - n\rangle^\theta \widehat{f}(\xi)\|_{L^2} \Big\|_{\ell^p_n} \\ &\sim \bigg\| \sum_{k\in\mathbb{Z}} \langle k - n\rangle^{2\theta} \int_{I_k} |\widehat{f}(\xi)|^2 d\xi \bigg\|^{\frac{1}{2}}_{\ell^{\frac{p}{2}}_n}, \end{aligned}$$

it follows from Young's inequality with $p \geq 2$ that

$$\begin{aligned} \|f\|_{MH^{\theta,p}} &\lesssim \bigg(\sum_{n\in\mathbb{Z}} \langle n\rangle^{2\theta} \bigg)^{\frac{1}{2}} \bigg\| \int_{I_n} |\widehat{f}(\xi)|^2 d\xi \bigg\|^{\frac{1}{2}}_{\ell^{\frac{p}{2}}_n} \\ &\lesssim \|f\|_{M^{2,p}}, \end{aligned}$$

provided that $\theta < -\frac{1}{2}$. This proves (i).

Next, we consider the case $-\frac{1}{2} \leq \theta < 0$. In this case, we need to lose either integrability or differentiability. By Young's inequality with $\frac{2}{p} + 1 = \frac{1}{r} + \frac{2}{q}$, we have

$$\begin{aligned} \|f\|_{MH^{\theta,p}} &\lesssim \bigg(\sum_{n\in\mathbb{Z}} \langle n\rangle^{2\theta r} \bigg)^{\frac{1}{2r}} \bigg\| \int_{I_n} |\widehat{f}(\xi)|^2 d\xi \bigg\|^{\frac{1}{2}}_{\ell^{\frac{q}{2}}_n} \\ &\lesssim \|f\|_{M^{2,q}}, \end{aligned} \tag{2.1}$$

provided that $2\theta r < -1$, namely, $\frac{1}{q} > \frac{1}{p} + \frac{1}{2} + \theta$. The second bound in (1.13) follows from applying Hölder's inequality to the right-hand side of (2.1).

3. Control on the modulation and Fourier-Lebesgue norms

In this section, we present the proof of Theorem 1.4. In view of the local well-posedness results [24, 25], it suffices to establish global-in-time bounds for smooth solutions to (1.1) on $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$.

3.1. **On the real line.** We first consider the real line case. Our main goal is to prove the following global-in-time bound.

Proposition 3.1. *Let $2 \leq p < \infty$. Then, there exists $C = C(p) > 0$ such that*

$$\|u(t)\|_{M^{2,p}(\mathbb{R})} \leq C(1 + \|u(0)\|_{M^{2,p}(\mathbb{R})})^{\frac{p}{2}-1} \|u(0)\|_{M^{2,p}(\mathbb{R})}$$

for any Schwartz class solution u to (1.1) *on $\mathbb{R}$ and any $t \in \mathbb{R}$.*

Proof. Let us first consider the small data case. The general case follows from the small data case and the scaling property of the $M^{2,p}$-norm. Fix $2 \leq p < \infty$. Let u be a global-in-time Schwartz class solution to (1.1), satisfying

$$\|u(0)\|_{M^{2,p}} \leq \varepsilon \ll 1 \tag{3.1}$$

for some small $\varepsilon > 0$ (to be chosen later). Given $n \in \mathbb{Z}$, define $\{u_n\}_{n\in\mathbb{Z}}$ by

$$u_n(x,t) = \mathcal{G}_n(u)(x,t) = e^{-inx}e^{in^2t}u(x-2nt,t), \tag{3.2}$$

where $\mathcal{G}_n$ is as in (1.4). Note that we have

$$|\widehat{u}_n(\xi,t)| = |\widehat{u}(\xi+n,t)| \tag{3.3}$$

for any $n \in \mathbb{Z}$ and $\xi, t \in \mathbb{R}$. In view of the Galilean symmetry, u_n is the solution to (1.1) with $u_n|_{t=0} = M_n u(0)$, where M_n is as in (1.12).

In the following, we fix $\kappa = \frac{1}{2}$ and set $\alpha(u) = \alpha(\frac{1}{2}; u)$. From (1.14), (1.15), (1.17), and (1.18) with (3.3), we have

$$\begin{aligned}\bigg|\alpha(u_n(t)) - \int_{\mathbb{R}} \frac{|\widehat{u}_n(\xi,t)|^2}{1+\xi^2}d\xi\bigg| &\le \sum_{j=2}^\infty \frac{1}{j}\Big\|(\tfrac{1}{2}-\partial_x)^{-\frac{1}{2}}\, u_n(t)\, (\tfrac{1}{2}+\partial_x)^{-\frac{1}{2}}\Big\|_{\mathfrak{I}_2(\mathbb{R})}^{2j}\\ &\lesssim \sum_{j=2}^\infty \bigg(\int_{\mathbb{R}} \frac{|\widehat{u}_n(\xi,t)|^2}{(1+\xi^2)^{\frac{1}{2}-\delta}}d\xi\bigg)^j\\ &\lesssim \sum_{j=2}^\infty \bigg(\int_{\mathbb{R}} \frac{|\widehat{u}(\xi,t)|^2}{(1+(\xi-n)^2)^{\frac{1}{2}-\delta}}d\xi\bigg)^j \end{aligned}\tag{3.4}$$

for any $\delta > 0$. By Hölder's inequality, we can choose sufficiently small $\delta = \delta(p) > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}} \frac{|\widehat{u}(\xi,0)|^2}{(1+(\xi-n)^2)^{\frac{1}{2}-\delta}}d\xi &\sim \sum_{k\in\mathbb{Z}} \frac{1}{(1+(k-n)^2)^{\frac{1}{2}-\delta}}\|\widehat{u(0)}\|_{L^2_\xi(I_k)}^2\\ &\lesssim \|u(0)\|_{M^{2,p}}^2\end{aligned}\tag{3.5}$$

uniformly in $n \in \mathbb{Z}$, where $I_k = [k-\frac{1}{2}, k+\frac{1}{2})$, $k \in \mathbb{Z}$, as above. Then, in view of (3.1) and (3.5), we can choose $\varepsilon > 0$ sufficiently small such that the series on the right-hand side of (3.4) is convergent at time $t = 0$. Then, by continuity in time, there exists a small time interval I around $t = 0$ such that the series on the right-hand side of (3.4) is convergent uniformly for any $t \in I$. Moreover, by choosing $\varepsilon > 0$ sufficiently small, we may assume that (1.16) is satisfied for all $t \in I$.

As a consequence, we have

$$\bigg|\alpha(u_n(t)) - \int_{\mathbb{R}} \frac{|\widehat{u}_n(\xi,t))|^2}{1+\xi^2}d\xi\bigg| \lesssim \bigg(\int_{\mathbb{R}} \frac{|\widehat{u}(\xi,t)|^2}{(1+(\xi-n)^2)^{\frac{1}{2}-\delta}}d\xi\bigg)^2$$

for any $t \in I$ and $n \in \mathbb{Z}$. Now, compute the $\ell_n^{\frac{p}{2}}$-norm of both sides. Choose $\delta = \delta(p) > 0$ sufficiently small such that $(\frac{1}{2} - \delta) \cdot \frac{p}{p-1} > \frac{1}{2}$. Then, by Young's inequality, we have

$$\begin{aligned}
&\bigg\| \alpha(u_n(t)) - \int_{\mathbb{R}} \frac{|\widehat{u}_n(\xi,t)|^2}{1+\xi^2} d\xi \bigg\|_{\ell_n^{\frac{p}{2}}} \\
&\quad \lesssim \bigg\| \int_{\mathbb{R}} \frac{|\widehat{u}(\xi,t)|^2}{(1+(\xi-n)^2)^{\frac{1}{2}-\delta}} d\xi \bigg\|_{\ell_n^p}^2 \\
&\quad \sim \bigg\| \sum_{k\in\mathbb{Z}} \frac{1}{(1+(k-n)^2)^{\frac{1}{2}-\delta}} \|\widehat{u}(\xi,t)\|_{L^2_\xi(I_k)}^2 \bigg\|_{\ell_n^p}^2 \\
&\quad \lesssim \Big\| \|\widehat{u}(\xi,t)\|_{L^2_\xi(I_n)}^2 \Big\|_{\ell_n^{\frac{p}{2}}}^2 \\
&\quad \sim \|u(t)\|_{M^{2,p}}^4
\end{aligned} \tag{3.6}$$

for any $t \in I$. Therefore, from Lemma 1.2, (3.6), and the conservation of $\alpha(u_n)$, $n \in \mathbb{Z}$,

$$\begin{aligned}
\|u(t)\|_{M^{2,p}}^2 \sim \|u(t)\|_{MH^{-1,p}}^2 &\le \|\alpha(u_n(t))\|_{\ell_n^{\frac{p}{2}}} + \|u(t)\|_{M^{2,p}}^4 \\
&\lesssim \|u(0)\|_{MH^{-1,p}}^2 + \|u(0)\|_{M^{2,p}}^4 + \|u(t)\|_{M^{2,p}}^4 \\
&\lesssim \|u(0)\|_{M^{2,p}}^2 + \|u(0)\|_{M^{2,p}}^4 + \|u(t)\|_{M^{2,p}}^4
\end{aligned}$$

for all $t \in I$. Namely, we have

$$\|u(t)\|_{M^{2,p}}^2 \le C_0\|u(0)\|_{M^{2,p}}^2 + C_0\|u(0)\|_{M^{2,p}}^4 + C_0\|u(t)\|_{M^{2,p}}^4$$

for all $t \in I$. By choosing $\varepsilon > 0$ sufficiently small, we can apply a continuity argument and conclude that

$$\|u(t)\|_{M^{2,p}} \lesssim \|u(0)\|_{M^{2,p}}$$

for all $t \in \mathbb{R}$. This proves Proposition 3.1 for the small data case.

Next, we consider the general case. Given a global-in-time Schwartz class solution u to (1.1), let u_λ be as in (1.2). Then in view of (1.10), we can choose sufficiently large $\lambda \gg 1$ such that

$$\|u_\lambda(0)\|_{M^{2,p}} \le C\lambda^{-\frac{1}{p}}\|u(0)\|_{M^{2,p}} \le \varepsilon \ll 1 \tag{3.7}$$

as in (3.1). In particular, we may choose

$$\lambda \sim (1 + \|u(0)\|_{M^{2,p}})^p. \tag{3.8}$$

Hence, by the small data case presented above, we obtain

$$\|u_\lambda(t)\|_{M^{2,p}} \lesssim \|u_\lambda(0)\|_{M^{2,p}} \tag{3.9}$$

for all $t \in \mathbb{R}$. Finally, recall that

$$\|u(t)\|_{M^{2,p}} \lesssim \lambda^{\frac{1}{2}}\|u_\lambda(\lambda^2 t)\|_{M^{2,p}}. \tag{3.10}$$

By putting (3.7), (3.9), and (3.10) with (3.8), we conclude that

$$\|u(t)\|_{M^{2,p}} \lesssim (1 + \|u(0)\|_{M^{2,p}})^{\frac{p}{2}-1}\|u(0)\|_{M^{2,p}}$$

for all $t \in \mathbb{R}$. This completes the proof of Proposition 3.1 and hence the proof of Theorem 1.4 (i). □

3.2. **On the circle.** In the remaining part of this paper, we discuss the proof of Theorem 1.4 in the periodic case. While the essential part of the argument remains the same, we need to pay attention to the scaling argument since it modifies the spatial domain. Given $\lambda > 0$, let $\mathbb{T}_\lambda = \mathbb{R}/(2\pi\lambda\mathbb{Z})$ and we use the following convention:

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi\lambda} f(x) e^{-ix\xi} dx \qquad \text{and} \qquad f(x) = \frac{1}{\sqrt{2\pi}\lambda} \sum_{\xi \in \mathbb{Z}_\lambda} \widehat{f}(\xi) e^{ix\xi} \tag{3.11}$$

for functions on the dilated torus $\mathbb{T}_\lambda$, where $\mathbb{Z}_\lambda = \lambda^{-1}\mathbb{Z}$. In this setting, Plancherel's identity is expressed as

$$\|f\|_{L^2(\mathbb{T}_\lambda)} = \|\widehat{f}(\xi)\|_{L^2_\xi(\mathbb{Z}_\lambda, (d\xi)_\lambda)}, \tag{3.12}$$

where $(d\xi)_\lambda$ is the normalized counting measure on $\mathbb{Z}_\lambda$:

$$\int_{\mathbb{Z}_\lambda} f(\xi)(d\xi)_\lambda = \frac{1}{\lambda} \sum_{\xi \in \mathbb{Z}_\lambda} f(\xi).$$

Hence, we define the Fourier-Lebesgue space $\mathcal{F}L^p(\mathbb{T}_\lambda)$ by the norm:

$$\|f\|_{\mathcal{F}L^p(\mathbb{T}_\lambda)} = \|\widehat{f}(\xi)\|_{L^p_\xi(\mathbb{Z}_\lambda, (d\xi)_\lambda)}.$$

For simplicity of the notation, we set $L^p_\xi(\mathbb{Z}_\lambda) = L^p_\xi(\mathbb{Z}_\lambda, (d\xi)_\lambda)$. Under this convention, we have the following scaling property:

$$\|f_\lambda\|_{\mathcal{F}L^p(\mathbb{T}_\lambda)} = \lambda^{-\frac{1}{p}} \|f\|_{\mathcal{F}L^p(\mathbb{T})},$$

where

$$f_\lambda(x) = \lambda^{-1} f(\lambda^{-1}x). \tag{3.13}$$

In particular, note that the $\mathcal{F}L^\infty$-norm is invariant under the scaling symmetry. We also record the following identity:

$$\widehat{fg}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{Z}_\lambda} \widehat{f}(\eta)\widehat{g}(\xi - \eta)(d\eta)_\lambda \tag{3.14}$$

for $\xi \in \mathbb{Z}_\lambda$.

In the following, we restrict our attention to $\lambda \geq 1$. On the standard torus, we have the equivalence of the modulation spaces and the Fourier-Lebesgue spaces (see (1.9)). On a dilated torus $\mathbb{T}_\lambda$, $\lambda \gg 1$, however, these two spaces do not coincide. As in (1.6), we define the modulation spaces on $\mathbb{T}_\lambda$ by the norm:

$$\|f\|_{M^{r,p}(\mathbb{T}_\lambda)} = \big\| \|\psi_n(D) f\|_{L^r_x(\mathbb{T}_\lambda)} \big\|_{\ell^p_n(\mathbb{Z})}.$$

Note that $\psi_n(D)$ is basically a "smooth" projection onto the frequencies $J_n = [n-1, n+1] \cap \mathbb{Z}_\lambda$. On the one hand, when $\lambda = 1$, we have only $O(1)$ many frequencies in J_n, giving rise to the equivalence (1.9). On the other hand, when $\lambda \gg 1$, there are $O(\lambda)$ many frequencies in J_n and hence the $M^{r,p}(\mathbb{T}_\lambda)$- and the $\mathcal{F}L^p(\mathbb{T}_\lambda)$-norms are not equivalent uniformly in period $\lambda \geq 1$. As we see below, for our purpose, it is more convenient to work on the modulation space $M^{2,p}(\mathbb{T}_\lambda)$ on a dilated torus $\mathbb{T}_\lambda$. In view of Plancherel's identity (3.12), we have

$$\begin{aligned}\|f\|_{M^{2,p}(\mathbb{T}_\lambda)} &= \big\|\|\psi_n(\xi)\widehat{f}(\xi)\|_{L^2_\xi(\mathbb{Z}_\lambda)}\big\|_{\ell^p_n(\mathbb{Z})}\\ &= \bigg\|\bigg(\int_{\mathbb{Z}_\lambda}|\psi_n(\xi)\widehat{f}(\xi)|^2(d\xi)_\lambda\bigg)^{\frac{1}{2}}\bigg\|_{\ell^p_n(\mathbb{Z})}.\end{aligned}$$

Given a function f on $\mathbb{T}$, let f_λ be as in (3.13). Then, a straightforward computation shows the following scaling relation:

$$\|f\|_{\mathcal{F}L^p(\mathbb{T})} \lesssim \lambda^{\frac{1}{2}}\|f_\lambda\|_{M^{2,p}(\mathbb{T}_\lambda)}, \tag{3.15}$$

$$\|f_\lambda\|_{M^{2,p}(\mathbb{T}_\lambda)} \lesssim \lambda^{-\frac{1}{p}}\|f\|_{\mathcal{F}L^p(\mathbb{T})} \tag{3.16}$$

for $2 \le p \le \infty$.

Lastly, we define the modulated Sobolev space $MH^{\theta,p}(\mathbb{T}_\lambda)$ on a dilated torus $\mathbb{T}_\lambda$ by the norm:

$$\begin{aligned}\|f\|_{MH^{\theta,p}(\mathbb{T}_\lambda)} &= \big\|\|M_n f\|_{H^\theta(\mathbb{T}_\lambda)}\big\|_{\ell^p_n(\mathbb{Z})}\\ &= \bigg(\sum_{n\in\mathbb{Z}}\|\langle\xi-n\rangle^\theta\widehat{f}(\xi)\|^p_{L^2_\xi(\mathbb{Z}_\lambda)}\bigg)^{\frac{1}{p}},\end{aligned}$$

where M_n is as in (1.12). Note that the outer summation ranges over $n \in \mathbb{Z}$ (and not over $\mathbb{Z}_\lambda$). Then, arguing as in the proof of Lemma 1.2, we obtain the following equivalence when $\theta < -\frac{1}{2}$ and $2 \le p \le \infty$:

$$\|f\|_{MH^{\theta,p}(\mathbb{T}_\lambda)} \sim \|f\|_{M^{2,p}(\mathbb{T}_\lambda)}. \tag{3.17}$$

As in the real line case, the equivalence (3.17) with $\theta = -1$ plays an important role in our analysis.

We first state basic properties of the perturbation determinant $\alpha(\kappa; u)$ in (1.14) on a dilated torus $\mathbb{T}_\lambda$. The following three lemmas are just restatements of Lemma 1.3 on a dilated torus.

Lemma 3.2. *Let $\kappa > \kappa_0$ for some $\kappa_0 > 0$ and $\lambda \ge 1$. Then, we have*

$$\big\|(\kappa-\partial_x)^{-\frac{1}{2}}u(\kappa+\partial_x)^{-\frac{1}{2}}\big\|^2_{\mathfrak{I}_2(\mathbb{T}_\lambda)} \sim \int_{\mathbb{Z}_\lambda}\log\big(4+\tfrac{\xi^2}{\kappa^2}\big)\frac{|\widehat{u}(\xi)|^2}{\sqrt{4\kappa^2+\xi^2}}(d\xi)_\lambda, \tag{3.18}$$

$$\big\|(\kappa+\partial_x)^{-\frac{1}{2}}\overline{u}(\kappa-\partial_x)^{-\frac{1}{2}}\big\|^2_{\mathfrak{I}_2(\mathbb{T}_\lambda)} \sim \int_{\mathbb{Z}_\lambda}\log\big(4+\tfrac{\xi^2}{\kappa^2}\big)\frac{|\widehat{u}(\xi)|^2}{\sqrt{4\kappa^2+\xi^2}}(d\xi)_\lambda, \tag{3.19}$$

for any smooth function u on $\mathbb{T}_\lambda$, where the implicit constant depends on $\kappa_0 > 0$.

Proof. We only consider the first estimate (3.18) as the second estimate (3.19) follows from a similar computation. Let $A = (\kappa-\partial_x)^{-\frac{1}{2}}u(\kappa+\partial_x)^{-\frac{1}{2}}$. Then, with (3.11) and (3.14), we have

$$\begin{aligned} Af(x) &= (\kappa - \partial_x)^{-\frac{1}{2}} u (\kappa + \partial_x)^{-\frac{1}{2}} f \\ &= \frac{1}{2\pi\lambda^2} \sum_{\xi \in \mathbb{Z}_\lambda} e^{i\xi x} (\kappa - i\xi)^{-\frac{1}{2}} \sum_{\eta \in \mathbb{Z}_\lambda} \widehat{u}(\xi - \eta)(\kappa + i\eta)^{-\frac{1}{2}} \widehat{f}(\eta) \\ &= \frac{1}{2\pi\lambda^2} \int_{\mathbb{T}_\lambda} \bigg(\sum_{\xi, \eta \in \mathbb{Z}_\lambda} e^{i\xi x} e^{-i\eta y} \frac{\widehat{u}(\xi - \eta)}{\sqrt{2\pi(\kappa - i\xi)(\kappa + i\eta)}} \bigg) f(y) dy \end{aligned}$$

for $f \in L^2(\mathbb{T}_\lambda)$. Thus, the integral kernel $K(x, y)$ of A is given by

$$K(x, y) = \frac{1}{2\pi\lambda^2} \sum_{\xi, \eta \in \mathbb{Z}_\lambda} e^{i\xi x} e^{-i\eta y} \frac{\widehat{u}(\xi - \eta)}{\sqrt{2\pi(\kappa - i\xi)(\kappa + i\eta)}}.$$

Hence, from the definition of the $\mathfrak{I}_2$-norm and Plancherel's identity (3.12), we have

$$\begin{aligned} \|A\|_{\mathfrak{I}_2}^2 &= \iint_{\mathbb{T}_\lambda^2} |K(x, y)|^2 dx dy \\ &= \frac{1}{2\pi\lambda^2} \sum_{\xi, \eta \in \mathbb{Z}_\lambda} \frac{|\widehat{u}(\xi - \eta)|^2}{\sqrt{(\kappa^2 + \xi^2)(\kappa^2 + \eta^2)}}. \end{aligned} \tag{3.20}$$

On the other hand, with $\xi = \frac{n}{\lambda}$ and $\eta = \frac{m}{\lambda}$, $n, m \in \mathbb{Z}$, we have

$$\begin{aligned} &\frac{1}{\lambda^2} \sum_{\eta \in \mathbb{Z}_\lambda} \frac{1}{\sqrt{(\kappa^2 + (\xi + \eta)^2)(\kappa^2 + \eta^2)}} \\ &\qquad = \sum_{m \in \mathbb{Z}} \frac{1}{\sqrt{((\lambda\kappa)^2 + (n + m)^2)((\lambda\kappa)^2 + m^2)}} \end{aligned}$$

By separately estimating the contributions from (i) $|n + m| \ll |n|$, (ii) $|n + m| \sim |n|$, and (iii) $|n + m| \gg |n|$,

$$\begin{aligned} &\sim \bigg(\log\big(1 + \tfrac{n^2}{(\lambda\kappa)^2}\big) \frac{1}{\sqrt{(\lambda\kappa)^2 + n^2}} + \frac{|n|}{(\lambda\kappa)^2 + n^2} \bigg) \\ &\sim \frac{1}{\lambda} \log\big(4 + \tfrac{\xi^2}{\kappa^2}\big) \frac{1}{\sqrt{4\kappa^2 + \xi^2}}. \end{aligned} \tag{3.21}$$

Therefore, the first estimate (3.18) follows from (3.20) and (3.21). □

Next, we estimate the leading term in (1.14).

Lemma 3.3. *Let $\kappa > 0$ and $\lambda \geq 1$. Then, we have*

$$\operatorname{Re} \operatorname{tr} \big\{ (\kappa - \partial_x)^{-1} u (\kappa + \partial_x)^{-1} \overline{u} \big\} = \frac{1 + e^{-2\pi\lambda\kappa}}{1 - e^{-2\pi\lambda\kappa}} \cdot \int_{\mathbb{Z}_\lambda} \frac{2\kappa |\widehat{u}(\xi)|^2}{4\kappa^2 + \xi^2} (d\xi)_\lambda \tag{3.22}$$

for any smooth function u on $\mathbb{T}_\lambda$.

Proof. Recall from [34] that for $\kappa > 0$, the operators $(\kappa - \partial_x)^{-1}$ and $(\kappa + \partial_x)^{-1}$, admit the convolution kernels $k_-^\kappa(x) = \mathbf{1}_{(-\infty, 0]}(x) \cdot e^{\kappa x}$ and $k_+^\kappa(x) = k_-^\kappa(-x)$. Define $K_\mp^\kappa(x)$ to be the periodization of $k_\mp^\kappa(x)$ with period $2\pi\lambda$:

$$K_\mp^\kappa(x) = \sum_{n \in \mathbb{Z}} k_\mp^\kappa(x - 2\pi\lambda n).$$

Then, the Fourier coefficients are given by

$$\begin{aligned}\widehat{K^\kappa_\mp}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_0^{2\pi\lambda} K^\kappa_\mp(x)e^{-ix\xi}dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} k^\kappa_\mp(x)e^{-ix\xi}dx\\ &= \frac{1}{\kappa\mp i\xi}\end{aligned}$$

for $\xi\in\mathbb{Z}_\lambda$. Namely, $K^\kappa_\mp(x)$ represents the convolution kernels for $(\kappa-\partial_x)^{-1}$ and $(\kappa+\partial_x)^{-1}$ on $\mathbb{T}_\lambda$, respectively. Also, note that $K^\kappa_+(-x) = K^\kappa_-(x)$ and

$$K^\kappa_-(x) = \frac{e^{\kappa(x-2\pi\lambda\lceil\frac{x}{2\pi\lambda}\rceil)}}{1-e^{-2\pi\lambda\kappa}},$$

where $\lceil\cdot\rceil$ denotes the ceiling function, i.e. $\lceil x\rceil$ denotes the smallest integer n with $n\geq x$. In particular, we have

$$\begin{aligned}\big(K^\kappa_-\big)^2(x) &= \frac{1-e^{-4\pi\lambda\kappa}}{(1-e^{-2\pi\lambda\kappa})^2}\cdot\frac{e^{2\kappa(x-2\pi\lambda\lceil\frac{x}{2\pi\lambda}\rceil)}}{1-e^{-4\pi\lambda\kappa}}\\ &= \frac{1+e^{-2\pi\lambda\kappa}}{1-e^{-2\pi\lambda\kappa}}\cdot K^{2\kappa}_-(x).\end{aligned}$$

From these observations and Parseval's identity, we have

$$\begin{aligned}\text{LHS of (3.22)} &= \operatorname{Re}\iint_{\mathbb{T}^2_\lambda} K_-(x-y)u(y)K_+(y-x)\overline{u(x)}dxdy\\ &= \operatorname{Re}\iint_{\mathbb{T}^2_\lambda}\big(K_-(x-y)\big)^2u(y)\overline{u(x)}dxdy\\ &= \frac{1+e^{-2\pi\lambda\kappa}}{1-e^{-2\pi\lambda\kappa}}\cdot\operatorname{Re}\int_{\mathbb{Z}_\lambda}\frac{|\widehat{u}(\xi)|^2}{2\kappa-i\xi}(d\xi)_\lambda\\ &= \frac{1+e^{-2\pi\lambda\kappa}}{1-e^{-2\pi\lambda\kappa}}\cdot\int_{\mathbb{Z}_\lambda}\frac{2\kappa|\widehat{u}(\xi)|^2}{4\kappa^2+\xi^2}(d\xi)_\lambda.\end{aligned}$$

This proves the identity (3.22). □

The conservation of $\alpha(\kappa;u)$ follows as in the real line case.

Lemma 3.4. *Let $\kappa>\kappa_0$ for some $\kappa_0>0$ and $\lambda\geq 1$. For a smooth solution u to* (1.1) *on a dilated torus, the quantity $\alpha(\kappa;u)$ defined in* (1.14) *is conserved, provided that*

$$\int_{\mathbb{Z}_\lambda}\log\big(4+\tfrac{\xi^2}{\kappa^2}\big)\frac{|\widehat{u}(\xi)|^2}{\sqrt{4\kappa^2+\xi^2}}(d\xi)_\lambda\leq c_0 \tag{3.23}$$

for some absolute constant $c_0>0$, depending on $\kappa_0>0$.

The proof of Lemma 3.4 is mostly based on algebraic computations and thus we omit details. See the proof of Proposition 4.3 in [34]. As in Lemma 1.3, the smallness condition (3.23) guarantees term-by-term differentiation of the series (1.14). Note that the dependence of c_0 on κ_0 comes from the dependence of the implicit constant on κ_0 in Lemma 3.2.

As in the real line case, Theorem 1.4 (ii) on $\mathbb{T}$ follows once we prove the following proposition.

Proposition 3.5. *Let $2 \le p < \infty$. Then, there exists $C = C(p) > 0$ such that*

$$\|u(t)\|_{\mathcal{F}L^p(\mathbb{T})} \le C\big(1 + \|u(0)\|_{\mathcal{F}L^p(\mathbb{T})}\big)^{\frac{p}{2}-1}\|u(0)\|_{\mathcal{F}L^p(\mathbb{T})}$$

for any smooth solution u to (1.1) *on $\mathbb{T}$ and any $t \in \mathbb{R}$.*

Proof. Fix $2 \le p < \infty$. Let u be a global-in-time smooth solution to (1.1) on $\mathbb{T}$. For $\lambda \in \mathbb{N}$, let u_λ denote the scaled solution to (1.1) on the dilated torus $\mathbb{T}_\lambda$ defined by (1.2). Given small $\varepsilon > 0$ (to be chosen later), it follows from (3.16) that we can choose sufficiently large $\lambda = \lambda(\|u(0)\|_{\mathcal{F}L^p(\mathbb{T})}) \gg 1$ such that

$$\|u_\lambda(0)\|_{M^{2,p}(\mathbb{T}_\lambda)} \le C\lambda^{-\frac{1}{p}}\|u(0)\|_{\mathcal{F}L^p(\mathbb{T})} \le \varepsilon \ll 1. \tag{3.24}$$

In particular, we may choose

$$\lambda \sim (1 + \|u(0)\|_{\mathcal{F}L^p(\mathbb{T})})^p. \tag{3.25}$$

Given $n \in \mathbb{Z}$, define $\{u_n\}_{n\in\mathbb{Z}}$ by

$$u_{\lambda,n}(x,t) = \mathcal{G}_n(u_\lambda)(x,t) = e^{-inx}e^{in^2t}u_\lambda(x - 2nt, t).$$

In view of the Galilean symmetry, $u_{\lambda,n}$ is the solution to (1.1) on $\mathbb{T}_\lambda$ with $u_{\lambda,n}|_{t=0} = M_n u_\lambda(0)$. Moreover, we have

$$|\widehat{u}_{\lambda,n}(\xi,t)| = |\widehat{u}_\lambda(\xi + n, t)|. \tag{3.26}$$

for any $n \in \mathbb{Z}$, $\xi \in \mathbb{Z}_\lambda$, and $t \in \mathbb{R}$. Here, we used the fact that λ is an integer such that $\mathbb{Z} \subset \mathbb{Z}_\lambda$.

In the following, we fix $\kappa = \frac{1}{2}$ and set $\alpha(u) = \alpha(\frac{1}{2}; u)$ and

$$C_\lambda = \frac{1 + e^{-\pi\lambda}}{1 - e^{-\pi\lambda}}.$$

Note that $C_\lambda \sim 1$ for $\lambda \ge 1$. From (1.14) with Lemmas 3.2 and 3.3 and (3.26), we have

$$\begin{aligned}\bigg|\alpha(u_{\lambda,n}(t)) - C_\lambda \int_{\mathbb{Z}_\lambda} \frac{|\widehat{u}_{\lambda,n}(\xi,t)|^2}{1+\xi^2}(d\xi)_\lambda\bigg| &\le \sum_{j=2}^\infty \frac{1}{j}\Big\|(\tfrac{1}{2} - \partial_x)^{-\frac{1}{2}}\, u_{\lambda,n}(t)\, (\tfrac{1}{2} + \partial_x)^{-\frac{1}{2}}\Big\|_{\mathfrak{I}_2(\mathbb{T}_\lambda)}^{2j} \\ &\lesssim \sum_{j=2}^\infty \bigg(\int_{\mathbb{Z}_\lambda} \frac{|\widehat{u}_\lambda(\xi,t)|^2}{(1+(\xi-n)^2)^{\frac{1}{2}-\delta}}(d\xi)_\lambda\bigg)^j\end{aligned}$$

for any $\delta > 0$. By Hölder's inequality, we can choose sufficiently small $\delta = \delta(p) > 0$ such that

$$\begin{aligned}\int_{\mathbb{Z}_\lambda} \frac{|\widehat{u}_\lambda(\xi,t)|^2}{(1+(\xi-n)^2)^{\frac{1}{2}-\delta}}(d\xi)_\lambda &\sim \sum_{k\in\mathbb{Z}} \frac{1}{(1+(k-n)^2)^{\frac{1}{2}-\delta}}\|\widehat{u}_\lambda(\xi,0)\|^2_{L^2_\xi(I_k\cap\mathbb{Z}_\lambda,(d\xi)_\lambda)} \\ &\lesssim \|u_\lambda(0)\|^2_{M^{2,p}(\mathbb{T}_\lambda)}\end{aligned} \tag{3.27}$$

uniformly in $n \in \mathbb{Z}$, where $I_k = [k - \frac{1}{2}, k + \frac{1}{2})$, $k \in \mathbb{Z}$, as above. Then, in view of (3.24), (3.27), and continuity in time, we can argue as in the real line case and conclude that there exists a time interval I around $t = 0$ such that

$$\bigg|\alpha(u_{\lambda,n}(t)) - C_\lambda \int_{\mathbb{Z}_\lambda} \frac{|\widehat{u}_{\lambda,n}(\xi,t))|^2}{1+\xi^2}(d\xi)_\lambda\bigg| \lesssim \bigg(\int_{\mathbb{Z}_\lambda} \frac{|\widehat{u}_\lambda(\xi,t)|^2}{(1+(\xi-n)^2)^{\frac{1}{2}-\delta}}(d\xi)_\lambda\bigg)^2$$

for any $t \in I$ and $n \in \mathbb{Z}$, by choosing $\varepsilon > 0$ sufficiently small. Moreover, we may assume that (3.23) is satisfied for all $t \in I$ so that Lemma 3.4 is applicable.

Now, compute the $\ell_n^{\frac{p}{2}}$-norm of both sides. By choosing $\delta = \delta(p) > 0$ sufficiently small, Young's inequality yields

$$\begin{aligned}
\bigg\| \alpha(u_{\lambda,n}(t)) - C_\lambda \int_{\mathbb{Z}_\lambda} \frac{|\widehat{u}_{\lambda,n}(\xi,t))|^2}{1+\xi^2}(d\xi)_\lambda \bigg\|_{\ell_n^{\frac{p}{2}}} \\
&\lesssim \bigg\| \sum_{k\in\mathbb{Z}} \frac{1}{(1+(k-n)^2)^{\frac{1}{2}-\delta}} \|\widehat{u}_\lambda(\xi,t)\|^2_{L^2_\xi(I_k\cap\mathbb{Z}_\lambda,(d\xi)_\lambda)} \bigg\|^2_{\ell_n^p} \\
&\lesssim \Big\| \|\widehat{u}_\lambda(\xi,t)\|^2_{L^2_\xi(I_k\cap\mathbb{Z}_\lambda,(d\xi)_\lambda)} \Big\|^2_{\ell_n^{\frac{p}{2}}} \\
&\sim \|u_\lambda(t)\|^4_{M^{2,p}(\mathbb{T}_\lambda)}
\end{aligned} \tag{3.28}$$

for any $t \in I$. Therefore, from (3.17), (3.28), and the conservation of $\alpha(u_{\lambda,n})$, $n \in \mathbb{Z}$,

$$\begin{aligned}
\|u_\lambda(t)\|^2_{M^{2,p}(\mathbb{T}_\lambda)} &\sim \|u_\lambda(t)\|^2_{MH^{-1,p}(\mathbb{T}_\lambda)} \\
&\lesssim \|u_\lambda(0)\|^2_{MH^{-1,p}(\mathbb{T}_\lambda)} + \|u_\lambda(0)\|^4_{M^{2,p}(\mathbb{T}_\lambda)} + \|u_\lambda(t)\|^4_{M^{2,p}(\mathbb{T}_\lambda)} \\
&\lesssim \|u_\lambda(0)\|^2_{M^{2,p}(\mathbb{T}_\lambda)} + \|u_\lambda(0)\|^4_{M^{2,p}(\mathbb{T}_\lambda)} + \|u_\lambda(t)\|^4_{M^{2,p}(\mathbb{T}_\lambda)}
\end{aligned}$$

for all $t \in I$. Hence, by choosing $\varepsilon > 0$ sufficiently small, we conclude from the continuity argument that

$$\|u_\lambda(t)\|_{M^{2,p}(\mathbb{T}_\lambda)} \lesssim \|u_\lambda(0)\|_{M^{2,p}(\mathbb{T}_\lambda)}$$

for all $t \in \mathbb{R}$. Combining this with (3.15) and (3.24), we obtain

$$\|u(t)\|_{\mathcal{F}L^p(\mathbb{T})} \lesssim \lambda^{\frac{1}{2}-\frac{1}{p}} \|u(0)\|_{\mathcal{F}L^p(\mathbb{T})} \tag{3.29}$$

for all $t \in \mathbb{R}$. Finally, Proposition 3.5 follows from (3.29) with (3.25). □

Appendix A. On the modified KdV equation

In this appendix, we consider the complex-valued modified KdV equation (mKdV) on $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$:

$$\begin{cases} \partial_t u = -\partial_x^3 u \pm 6|u|^2\partial_x u, \\ u|_{t=0} = u_0 \end{cases} \tag{A.1}$$

and discuss how to derive the global-in-time bounds (1.21) and (1.22) for Schwartz solutions to the mKdV (A.1). The equation (A.1) is known to be completely integrable and is closely related to the cubic NLS (1.1) [29, 51, 30, 34]. When the initial data u_0 is real-valued, the corresponding solution u to (A.1) remains real-valued, thus solving the following real-valued mKdV:

$$\partial_t u = -\partial_x^3 u \pm 6u^2\partial_x u. \tag{A.2}$$

The mKdV enjoys the following scaling symmetry

$$u(x,t) \longmapsto u_\lambda(x,t) = \lambda^{-1} u(\lambda^{-1}x, \lambda^{-3}t),$$

inducing the same scaling-critical regularity as the cubic NLS (1.1). In terms of the homogeneous Fourier-Lebesgue space $\dot{\mathcal{F}L}^{s,p}(\mathbb{R})$, the critical regularity is given by $s_{\text{crit}}(p) = -\frac{1}{p}$.

The Cauchy problem (A.1) has been extensively studied; the complex-valued mKdV (A.1) is known to be locally well-posed in $H^s(\mathcal{M})$ for $s \geq \frac{1}{4}$ on the real line [32, 55] and for $s \geq \frac{1}{3}$ on the circle [8, 54, 43, 42]. In the real-valued setting, the I-method has been applied to prove global well-posedness of the mKdV (A.2) in $H^s(\mathcal{M})$ for $s \geq \frac{1}{4}$ on the real line and for $s \geq \frac{1}{2}$ on the circle [17, 35]. See also [16, 41] for global existence results in $H^s(\mathbb{R})$, $s > -\frac{1}{8}$, and $L^2(\mathbb{T})$ in the real-valued setting. We point out that, in the periodic setting, the well-posedness studies stated above have been performed on the following renormalized mKdV on $\mathbb{T}$:

$$\partial_t u = -\partial_x^3 u \pm 6(|u|^2 - \textstyle\fint_{\mathbb{T}} |u|^2 dx)\partial_x u. \tag{A.3}$$

As in the case of the cubic NLS (1.1) and the renormalized cubic NLS (1.5), it is easy to see, via the transform:

$$u(x,t) \longmapsto u(x \pm 6\mu t, t)$$

with $\mu = \fint_{\mathbb{T}} |u|^2 dx$, that the mKdV (A.1) and the renormalized mKdV (A.3) are equivalent in $L^2(\mathbb{T})$. It is, however, known that the renormalized mKdV (A.3) behaves better outside $L^2(\mathbb{T})$. Indeed, the (unrenormalized) mKdV (A.1) is known to be ill-posed in negative Sobolev spaces $H^s(\mathbb{T})$, $s < 0$, [41] and Fourier-Lebesgue spaces $\mathcal{F}L^p(\mathbb{T})$, $p > 2$ [30].

In [34], Killip-Vişan-Zhang showed that Lemma 1.3 also holds for Schwartz solutions u to the complex-valued mKdV (A.1). Therefore, while it is not explicitly stated in [34], their result yields global well-posedness of the complex-valued mKdV (A.1) in $H^s(\mathcal{M})$ for $s \geq \frac{1}{4}$ on the real line and for $s \geq \frac{1}{3}$ on the circle, thus matching the known local well-posedness results.

In the real-valued and defocusing case (i.e. with the + sign in (A.1), (A.2), and (A.3)), there are several further results exploiting the completely integrable structure of the equation. Before proceeding further, let us define the notion of *sensible weak solutions*. See also [21, 47].

Definition A.1. Let $B(\mathbb{T})$ be a Banach space of functions on $\mathbb{T}$. Given $u_0 \in B(\mathbb{T})$, we say that $u \in C((t_0, t_1); B(\mathbb{T}))$ is a sensible weak solution to an equation on (t_0, t_1), $-\infty \leq t_0 < 0 < t_1 \leq \infty$, if, for any sequence $\{u_{0,n}\}_{n\in\mathbb{N}}$ of smooth functions tending to u_0 in $B(\mathbb{T})$, the corresponding (classical) solutions $u_n(t)$ with $u_n|_{t=0} = u_{0,n}$ converge to $u(t)$ in $B(\mathbb{T})$ for each $t \in (t_0, t_1)$.

We point that this notion of sensible weak solutions is rather weak. In particular, sensible weak solutions do not have to satisfy the equation even in the distributional sense. In [31], Kappeler-Topalov proved global well-posedness (in the sense of sensible weak solutions) of the periodic real-valued defocusing mKdV (A.2) in $L^2(\mathbb{T})$. In a recent paper [30], Kappeler-Molnar studied the periodic real-valued defocusing renormalized mKdV (A.3) in the Fourier-Lebesgue spaces and proved, in the sense of sensible weak solutions, local well-posedness and small data global well-posedness in $\mathcal{F}L^p(\mathbb{T})$, $2 \leq p < \infty$.[4] On the one hand, Molinet [41] applied the short-time Fourier restriction norm method and proved that the solutions constructed in [31] are indeed distributional solutions. On the other hand, the solutions outside $L^2(\mathbb{T})$ constructed in [30] are not yet known to be distributional solutions at this point.

[4]The small data global well-posedness also applies to the focusing case. See Remark on p. 2217 in [30].

We now state our result for the mKdV.

Theorem A.2. *Let $2 \leq p < \infty$.*

(i) *The global-in-time bound* (1.21) *in the modulation spaces $M^{2,p}(\mathbb{R})$ holds for any Schwartz class solution u to the complex-valued mKdV* (A.1) *on $\mathbb{R}$.*

(ii) *The global-in-time bound* (1.22) *in the Fourier-Lebesgue spaces $\mathcal{F}L^p(\mathbb{T})$ holds for any smooth solution u to the complex-valued mKdV* (A.1) *on $\mathbb{T}$. In particular, the same global-in-time bound* (1.22) *also holds any smooth solution u to the complex-valued renormalized mKdV* (A.3) *on $\mathbb{T}$.*

On the circle, in view of the aforementioned local well-posedness (in the sense of sensible weak solutions) in the Fourier-Lebesgue spaces $\mathcal{F}L^p(\mathbb{T})$, $2 \leq p < \infty$, in [30] and the global-in-time bound (1.22) in Theorem A.2 (ii), one may be tempted to conclude global well-posedness (in the sense of sensible weak solutions) of the real-valued defocusing renormalized mKdV (A.3), extending globally in time the sensible weak solutions constructed in [30]. We, however, point out that the construction of the sensible weak solutions in [30] is carried out through the Birkhoff coordinates and the local existence time is characterized by the openness of the range of the Birkhoff map. In particular, the local existence time in [30] is not given in terms of the size of initial data in an explicit manner and hence we do not know how to conclude such global well-posedness (in the sense of sensible weak solutions) of the real-valued defocusing renormalized mKdV (A.3) in $\mathcal{F}L^p(\mathbb{T})$, $2 \leq p < \infty$. On the real line, there is no known local well-posedness in the modulation spaces $M^{2,p}(\mathbb{R})$, $p > 2$, and hence the global-in-time bound does not lead to any global well-posedness at this point. See Remark A.4 for a further discussion on this issue.

Theorem A.2 follows from a consideration analogous to the proof of Theorem 1.4 for the cubic NLS but with one important difference. On the one hand, the Galilean symmetry (1.4) played an important role in the proof of Theorem 1.4 for the cubic NLS. On the other hand, it is known that the mKdV (A.1) does not enjoy the Galilean symmetry. Nonetheless, if we define a Galilean transform $\mathfrak{G}_\beta$, $\beta \in \mathbb{R}$, by

$$u^\beta(x,t) = \mathfrak{G}_\beta(u)(x,t) := e^{-i\beta x} e^{2i\beta^3 t} u(x - 3\beta^2 t, t), \tag{A.4}$$

then a direct computation shows that if u is a solution to the mKdV (A.1), then $v = \mathfrak{G}_\beta(u)$ satisfies the following mKdV-NLS equation:

$$\partial_t v = (-\partial_x^3 v \pm 6|v|^2 \partial_x v) + 3\beta(-i\partial_x^2 v \pm 2i|v|^2 v). \tag{A.5}$$

Then, from the conservation of $\alpha(\kappa; u)$ in (1.14) for the mKdV flow and the cubic NLS flow ([34, Propositions 4.3 and 4.4]), we conclude that $\alpha(\kappa; u)$ is also conserved under the mKdV-NLS equation (A.5).

Lemma A.3. *Let $\beta \in \mathbb{R}$. For a Schwartz class solution u to the mKdV-NLS equation* (A.5)*, the quantity $\alpha(\kappa; u)$ defined in* (1.14) *is conserved, provided that $\kappa > 0$ is sufficiently large such that the smallness condition* (1.16) *holds.*

Once we have Lemma A.3 and observe that $u^\beta = \mathfrak{G}_\beta(u)$ in (A.4) satisfies

$$|\widehat{u^\beta}(\xi, t)| = |\widehat{u}(\xi + \beta, t)|$$

(compare this with (3.3)), the global-in-time bounds (1.21) and (1.22) for the mKdV (A.1) follow exactly in the same manner as in the proof of Theorem 1.4. Hence, we omit details.

Remark A.4. There are local well-posedness results by Grünrock [23] and Grünrock-Vega [26] on the modified KdV (A.1) in the Fourier-Lebesgue spaces on the real line. In particular, it was shown in [26] that the mKdV (A.1) is locally well-posed in $\mathcal{F}L^{s,p}(\mathbb{R})$ for $2 \le p < \infty$ and $s \ge \frac{1}{2p}$. By taking $p \to \infty$, we see that this yields local well-posedness in almost critical Fourier-Lebesgue spaces. Unfortunately, our result does not allow us to control the Fourier-Lebesgue norms on the real line and thus we do not know how to extend the local-in-time solutions in [23, 26] globally in time at this point.

In terms of the modulation spaces $M^{2,p}_s(\mathbb{R})$, we recently proved local well-posedness of the mKdV (A.1) for $s \ge \frac{1}{4}$ and $2 \le p < \infty$ [48], thus extending the local well-posedness in [32, 55].[5] On the one hand, $\dot{\mathcal{F}L}^{\frac{1}{4},\infty}(\mathbb{R})$ scales like $\dot{H}^{-\frac{1}{4}}(\mathbb{R})$ and thus we may say that $M^{2,\infty}_{\frac{1}{4}}(\mathbb{R})$ "scales like" $\dot{H}^{-\frac{1}{4}}(\mathbb{R})$ in view of the embedding:

$$M^{2,p}_s(\mathbb{R}) \supset \mathcal{F}L^{s,p}(\mathbb{R})$$

for $p \ge 2$. On the other hand, the $M^{2,p}_s(\mathbb{R})$-norm is weaker than the $\mathcal{F}L^{s,p}$-norm and the solution map to the mKdV (A.1) on $\mathbb{R}$ fails to be locally uniformly continuous in $M^{2,p}_s(\mathbb{R})$ as soon as $s < \frac{1}{4}$. This is sharp contrast with the Fourier-Lebesgue case, where local well-posedness was proved via a contraction argument even for some $s < \frac{1}{4}$ [26]. Lastly, note that a slight modification of the proof of Theorem A.2 then yields a global-in-time bound in $M^{2,p}_s(\mathbb{R})$ for $2 \le p < \infty$ and $\frac{1}{4} \le s < 1 - \frac{1}{p}$, which yields global well-posedness of the mKdV (A.1) in the same range. See Theorem B.1 below. Combining this global-in-time bound and a persistence-of-regularity argument, we proved global well-posedness of the mKdV (A.1) in $M^{2,p}_s(\mathbb{R})$ for $s \ge \frac{1}{4}$ and $2 \le p < \infty$. See [48] for details.

Remark A.5. As in the case of the cubic NLS, the Dirac delta function plays a special role for the mKdV. On the one hand, there is an existence result for the mKdV (A.1) on $\mathbb{R}$ with the Dirac delta function (with a small multiplicative constant) as initial data [49]. On the other hand, via a scaling analysis, one can easily see that continuous dependence must fail at the Dirac delta function in the $\mathcal{F}L^{s,p}(\mathbb{R})$- and $M^{2,p}_s(\mathbb{R})$-topologies for $sp < -1$. See also [3] for an analogous ill-posedness at the Dirac delta function in the periodic case (in the focusing case).[6]

Remark A.6. In the following, we briefly discuss an alternative proof of the global-in-time bounds (1.21) and (1.22) in Theorems 1.4 and A.2. This alternative approach has been brought to our attention by R. Killip. The main idea is to consider $\alpha(\kappa; u)$ in (1.14) with a complex number $\kappa \in \mathbb{C}$. Then, we have the following statements.

[5] See also a recent preprint [10] for analogous local well-posedness of (A.1), including $p = \infty$.

[6] While the argument in [3] is carried out in $H^s(\mathbb{T})$, it can be easily adapted to the Fourier-Lebesgue setting $\mathcal{F}L^{s,p}(\mathbb{T})$, $sp < -1$. We also point out that their result in the defocusing case (Theorem 6.3) seems to be incorrect. In particular, the proof of Theorem 6.2 contains an error; the sech function in the proof needs to be replaced by the csch function, which causes a breakdown of the proofs of Theorems 6.2 and 6.3.

Lemma A.7. *For $\kappa \in \mathbb{C}$ with $\operatorname{Re}\kappa > 0$ and $u \in \mathcal{S}(\mathbb{R})$, we have*

$$\operatorname{Re}\operatorname{tr}\big\{(\kappa-\partial_x)^{-1}u(\kappa+\partial_x)^{-1}\overline{u}\big\} = \int \frac{2(\operatorname{Re}\kappa)|\widehat{u}(\xi+2\operatorname{Im}\kappa)|^2\, d\xi}{4(\operatorname{Re}\kappa)^2+\xi^2},$$
$$\big\|(\kappa-\partial_x)^{-\frac12}u(\kappa+\partial_x)^{-\frac12}\big\|_{\mathfrak{I}_2(\mathbb{R})}^2 \sim \int \log\big(4+\tfrac{\xi^2}{(\operatorname{Re}\kappa)^2}\big)\frac{|\widehat{u}(\xi+2\operatorname{Im}\kappa)|^2\, d\xi}{\sqrt{4(\operatorname{Re}\kappa)^2+\xi^2}}.$$

Moreover, for a Schwartz class solution u to the cubic NLS (1.1) *or the mKdV* (A.1)*, the quantity $\alpha(\kappa;u)$ is conserved, provided that $\operatorname{Re}\kappa > 0$ is sufficiently large such that*

$$\int_{\mathbb{R}} \log\big(4+\tfrac{\xi^2}{(\operatorname{Re}\kappa)^2}\big)\frac{|\widehat{u}(\xi+2\operatorname{Im}\kappa)|^2}{\sqrt{4(\operatorname{Re}\kappa)^2+\xi^2}}d\xi \le c_0$$

for some absolute constant $c_0 > 0$.

Here, we stated the results only on the real line but similar statements hold on the circle. Once we have Lemma A.7, we may use $\alpha(\frac12 + i\frac{n}{2};u)$ instead of $\alpha(\frac12;u_n)$ in (3.4), where u_n is defined in (3.2). In particular, this allows us to proceed and establish the global-in-time bounds (1.21) and (1.22) without using the Galilean symmetry (1.4) for the cubic NLS (1.1) and the Galilean transform (A.4) for the mKdV (A.1).

Appendix B. Controlling the modulation and Fourier-Lebesgue norms of higher regularities

In this appendix, we briefly discuss how to derive the following global-in-time bounds on the modulation and Fourier-Lebesgue norms of higher regularities.

Theorem B.1. *Let $2 \le p < \infty$ and $0 \le s < 1 - \frac{1}{p}$.*

(i) *There exists $C = C(p) > 0$ such that*

$$\|u(t)\|_{M^{2,p}_s(\mathbb{R})} \le C(1+\|u(0)\|_{M^{2,p}_s(\mathbb{R})})^{\frac{p}{2}-1}\|u(0)\|_{M^{2,p}_s(\mathbb{R})}$$

for any Schwartz class solution u to the cubic NLS (1.1) *or the mKdV* (A.1) *on $\mathbb{R}$ and any $t\in\mathbb{R}$.*

(ii) *There exists $C = C(p) > 0$ such that*

$$\|u(t)\|_{\mathcal{F}L^{s,p}(\mathbb{T})} \le C\big(1+\|u(0)\|_{\mathcal{F}L^{s,p}(\mathbb{T})}\big)^{\frac{p}{2}-1}\|u(0)\|_{\mathcal{F}L^{s,p}(\mathbb{T})}$$

for any smooth solution u to the cubic NLS (1.1) *or the mKdV* (A.1) *on $\mathbb{T}$ and any $t \in \mathbb{R}$.*

One may use a differencing technique as in Section 3 of [34] to establish global-in-time bounds for higher values of s but we do not pursue it in this paper. It is worthwhile to note that when $p = 2$, Theorem B.1 yields global-in-time control on the H^s-norm of a solution for $0 \le s < \frac12$ *without* using a differencing technique. Compare this with Section 4 of [34], where their argument (without a differencing technique) yields global-in-time control for $-\frac12 \le s < 0$.

In order to prove Theorem B.1, we first introduce the following modulated Sobolev space $MH^{\theta,p}_s(\mathbb{R})$ with a weight by the norm:

$$\begin{aligned}\|f\|_{MH^{\theta,p}_s(\mathbb{R})} &= \bigg(\sum_{n\in\mathbb{Z}} \langle n\rangle^{sp} \|M_n f\|^p_{H^\theta}\bigg)^{\frac{1}{p}}\\ &= \bigg(\sum_{n\in\mathbb{Z}} \langle n\rangle^{sp} \|\langle \xi - n\rangle^\theta \widehat{f}(\xi)\|^p_{L^2_\xi}\bigg)^{\frac{1}{p}},\end{aligned}$$

where M_n is the modulation operator defined in (1.12). On the circle, we define $MH^{\theta,p}_s(\mathbb{T})$ in an analogous manner. Then, we have the following equivalence of the norms.

Lemma B.2. *Let $s \geq 0$ and $\theta + s < -\frac{1}{2}$. Then, we have*

$$\|f\|_{MH^{\theta,p}_s} \sim \|f\|_{M^{2,p}_s}$$

with the understanding that $M^{2,p}_s(\mathbb{T}) = \mathcal{F}L^{s,p}(\mathbb{T})$ on the circle.

The proof of Lemma B.2 is analogous to that of Lemma 1.2 and thus we omit details. In the following, we only consider the cubic NLS (1.1) on the real line case and indicate where a modification appears in proving Theorem B.1 (i). Since the $s = 0$ case is already contained in Theorems 1.4 and A.2, we restrict our attention to $s > 0$.

In view of Lemma B.2 with $\theta = -1$, we only need to control the following quantity:

$$\bigg\|\langle n\rangle^{2s}\alpha(u_n(t)) - \langle n\rangle^{2s}\int_{\mathbb{R}} \frac{|\widehat{u}_n(\xi,t))|^2}{1+\xi^2}d\xi\bigg\|_{\ell^{\frac{p}{2}}_n}$$

for $t \in I$, where I is the time interval used in the proof of Proposition 3.1. Proceeding as in (3.6), we have

$$\begin{aligned}\bigg\|\langle n\rangle^{2s}\alpha(u_n(t)) &- \langle n\rangle^{2s}\int_{\mathbb{R}} \frac{|\widehat{u}_n(\xi,t))|^2}{1+\xi^2}d\xi\bigg\|_{\ell^{\frac{p}{2}}_n}\\ &\lesssim \bigg\|\langle n\rangle^{s}\int_{\mathbb{R}} \frac{|\widehat{u}(\xi,t)|^2}{\langle \xi - n\rangle^{1-2\delta}}d\xi\bigg\|^2_{\ell^p_n}\\ &\sim \bigg\|\sum_{k\in\mathbb{Z}} \frac{\langle n\rangle^s}{\langle k-n\rangle^{1-2\delta}\langle k\rangle^{2s}} \cdot \langle k\rangle^{2s}\|\widehat{u}(\xi,t)\|^2_{L^2_\xi(I_k)}\bigg\|^2_{\ell^p_n}\end{aligned} \tag{B.1}$$

for any $t \in I$. When $\langle k\rangle \gtrsim \langle n\rangle$, we apply Young's inequality as in (3.6) and obtain

$$\begin{aligned}\text{LHS of (B.1)} &\lesssim \bigg\|\sum_{k\in\mathbb{Z}} \frac{1}{\langle k-n\rangle^{1-2\delta}} \cdot \langle k\rangle^{2s}\|\widehat{u}(\xi,t)\|^2_{L^2_\xi(I_k)}\bigg\|^2_{\ell^p_n}\\ &\lesssim \|u(t)\|^4_{M^{2,p}_s}\end{aligned} \tag{B.2}$$

for $s > 0$, by choosing $\delta = \delta(p) > 0$ sufficiently small. When $\langle k\rangle \ll \langle n\rangle$, it follows from Young's and Hölder's inequalities: $\frac{1}{p} + 1 = \frac{1}{q} + (\frac{1}{r} + \frac{2}{p})$ that

$$\begin{aligned}\text{LHS of (B.1)} &\lesssim \bigg\| \sum_{k\in\mathbb{Z}} \frac{1}{\langle k-n\rangle^{1-2\delta-s}} \cdot \frac{1}{\langle k\rangle^{2s}} \cdot \langle k\rangle^{2s} \|\widehat{u}(\xi,t)\|^2_{L^2_\xi(I_k)} \bigg\|^2_{\ell^p_n} \\ &\le \bigg\| \frac{1}{\langle n\rangle^{1-2\delta-s}} \bigg\|^2_{\ell^q_n} \bigg\| \frac{1}{\langle n\rangle^{2s}} \bigg\|^2_{\ell^r_n} \|u(t)\|^4_{M^{2,p}_s} \\ &\lesssim \|u(t)\|^4_{M^{2,p}_s}, \end{aligned} \tag{B.3}$$

provided that (i) $1-2\delta-s > 1-\frac{1}{p}-\frac{1}{r}$, (ii) $\frac{1}{r}+\frac{2}{p} \le 1$, and (iii) $r > \frac{1}{2s}$. When $s \le \frac{1}{2} - \frac{1}{p}$, by choosing $\frac{1}{r} = 2s-$ and $\delta > 0$ sufficiently small, we see that all the conditions (i) - (iii) are satisfied. When $s > \frac{1}{2} - \frac{1}{p}$, by choosing $\frac{1}{r} = 1 - \frac{2}{p}$ and $\delta > 0$ sufficiently small, the conditions (i) - (iii) are satisfied for $2 \le p < \infty$ and $\frac{1}{2} - \frac{1}{p} < s < 1 - \frac{1}{p}$. Therefore, putting the two cases together, we conclude that the estimate (B.3) holds for $2 \le p < \infty$ and $0 < s < 1 - \frac{1}{p}$.

Once we have (B.2) and (B.3), we can proceed as in the proof of Proposition 3.1, with Lemma B.2 in place of Lemma 1.2. The proof for the periodic case follows in a similar manner.

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