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CLOSURE PROPERTIES OF SOLUTIONS TO HEAT INEQUALITIES

JONATHAN BENNETT AND NEAL BEZ

Abstract. We prove that if $u_1, u_2 : (0,\infty) \times \mathbb{R}^d \to (0,\infty)$ are sufficiently well-behaved solutions to certain heat inequalities on $\mathbb{R}^d$ then the function $u : (0,\infty) \times \mathbb{R}^d \to (0,\infty)$ given by $u^{1/p} = u_1^{1/p_1} * u_2^{1/p_2}$ also satisfies a heat inequality of a similar type provided $\frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p}$. On iterating, this result leads to an analogous statement concerning n-fold convolutions. As a corollary, we give a direct heat-flow proof of the sharp n-fold Young convolution inequality and its reverse form.

1. Introduction

It is known that if $d \in \mathbb{N}$ and $u_1, u_2 : (0,\infty) \times \mathbb{R}^d \to (0,\infty)$ satisfy the heat inequality

$$\partial_t u \geq \frac{1}{4\pi} \Delta u, \tag{1.1}$$

then any geometric mean of u_1 and u_2 satisfies the same heat inequality; i.e. for $1 \leq p_1, p_2 \leq \infty$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} = 1$, the function

$$u := u_1^{1/p_1} u_2^{1/p_2}$$

also satisfies the heat inequality (1.1). As a corollary to this, provided that u_1 and u_2 are sufficiently well-behaved, by the divergence theorem it follows that the quantity

$$Q(t) := \int_{\mathbb{R}^d} u_1(t,x)^{1/p_1} u_2(t,x)^{1/p_2} dx$$

is nondecreasing for all $t > 0$. Furthermore, on insisting that, for $j = 1, 2$, u_j satisfies (1.1) with equality and sufficiently well-behaved initial data $f_j^{p_j}$, it follows from this monotonicity that

$$\int_{\mathbb{R}^d} f_1(x) f_2(x) dx = \lim_{t \to 0} Q(t) \leq \lim_{t \to \infty} Q(t) = \|f_1\|_{p_1} \|f_2\|_{p_2};$$

that is, we recover the classical Hölder inequality.

This closure property of solutions to heat inequalities may be generalised considerably. Let $m, d \in \mathbb{N}$, $d_1, \ldots, d_m \in \mathbb{N}$, $1 \leq p_1, \ldots, p_m \leq \infty$ and for each $1 \leq j \leq m$ let $B_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$

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be such that $B_j^*B_j$ is a projection and

$$\sum_{j=1}^m \tfrac{1}{p_j} B_j^* B_j = I_d,$$

where I_d denotes the identity on $\mathbb{R}^d$. If $u_j : (0,\infty) \times \mathbb{R}^{d_j} \to (0,\infty)$ satisfies the heat inequality (1.1) for each j, then one may show that the same is true of the "geometric mean"

$$u(t,x) := \prod_{j=1}^m u_j(t, B_j x)^{1/p_j}.$$

(Here Δ acts in the number of variables dictated by context.) Very much as before, an immediate consequence of this is that $\int u(t,\cdot)$ is nondecreasing for all $t > 0$, and that from this monotonicity one may deduce the inequality

$$\int_{\mathbb{R}^d} \prod_{j=1}^m f_j(B_j x) dx \leq \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^{d_j})}.$$

This is the celebrated geometric Brascamp–Lieb inequality due to Ball [1] for rank-one projections and Barthe [3] in the general rank case. Such a heat-flow approach to proving inequalities, by its nature, generates sharp constants and guarantees the existence of centred gaussian extremisers. All of these observations were first made by Carlen, Lieb and Loss [13] for rank-one projections and by Bennett, Carbery, Christ and Tao [11] in the general rank case. Recently, Barthe and Huet [7] have given a different heat-flow proof of the geometric Brascamp–Lieb inequality and, moreover, the same line of argument also led them to a heat-flow proof of Barthe's reverse form of the geometric Brascamp–Lieb inequality. See [2], [3] for a statement of this reverse inequality. The reader is referred to [5] and the references therein for further discussion of heat-flow methods in the context of such geometric inequalities.

Aside from the geometric means above, and the trivial operation of ordinary addition, it is not difficult to verify that harmonic addition also preserves the set of solutions of (1.1); i.e. if $u_1, u_2 : (0,\infty) \times \mathbb{R}^d \to (0,\infty)$ satisfy (1.1), then the function u given by

$$\frac{1}{u} = \frac{1}{u_1} + \frac{1}{u_2}$$

also satisfies (1.1). By the divergence theorem, this closure property is easily seen to imply a variant of the triangle inequality for harmonic addition.

All of the above closure properties involve pointwise operations. The main purpose of this article is to establish closure properties under rather different operations involving convolution. As a consequence we provide heat-flow proofs of sharp convolution inequalities that do not proceed via duality.

1.1. **Main Results.**

Let $d \in \mathbb{N}$ and suppose $0 < p_1, p_2, p < \infty$ satisfy

$$\frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p}. \tag{1.2}$$

Let $0 \leq \sigma_1, \sigma_2, \sigma < \infty$ satisfy

$$\frac{1}{p_1}\left(1 - \frac{1}{p_1}\right)\sigma_2 = \frac{1}{p_2}\left(1 - \frac{1}{p_2}\right)\sigma_1 \tag{1.3}$$

and

$$\sigma p = \sigma_1 p_1 + \sigma_2 p_2.$$

The main contribution in this paper is captured by the following. We clarify that the operation $*$ will denote spatial convolution.

Theorem 1. *For $j = 1, 2$ suppose that $u_j : (0, \infty) \times \mathbb{R}^d \to (0, \infty)$ is such that $u_j(t, \cdot)^{1/p_j}$, $\partial_t(u_j(t, \cdot)^{1/p_j})$, $\nabla(u_j(t, \cdot)^{1/p_j})$, $u_j(t, \cdot)^{1/p_j}|\nabla \log u_j(t, \cdot)|^2$ and $\Delta(u_j(t, \cdot)^{1/p_j})$ are rapidly decreasing in space locally uniformly in time $t > 0$. Let $u : (0, \infty) \times \mathbb{R}^d \to (0, \infty)$ be given by*

$$u^{1/p} := u_1^{1/p_1} * u_2^{1/p_2}.$$

Then $u(t, \cdot)^{1/p}$, $\partial_t(u(t, \cdot)^{1/p})$, $\nabla(u(t, \cdot)^{1/p})$, $u(t, \cdot)^{1/p}|\nabla \log u(t, \cdot)|^2$ and $\Delta(u(t, \cdot)^{1/p})$ are also rapidly decreasing in space locally uniformly in time $t > 0$. Furthermore,

(1) *if $p_j \geq 1$ and*

$$\partial_t u_j \geq \frac{\sigma_j}{4\pi} \Delta u_j,$$

for $j = 1, 2$, then

$$\partial_t u \geq \frac{\sigma}{4\pi} \Delta u; \tag{1.4}$$

(2) *if $p_j \leq 1$ and*

$$\partial_t u_j \leq \frac{\sigma_j}{4\pi} \Delta u_j,$$

for $j = 1, 2$, then

$$\partial_t u \leq \frac{\sigma}{4\pi} \Delta u. \tag{1.5}$$

An important feature of this closure property is that the (technical) regularity ingredients are all satisfied when u_1 and u_2 are solutions to heat *equations* with sufficiently well-behaved initial data. Indeed, we shall see that Theorem 1 implies the following.

Corollary 2. *For $j = 1, 2$ let u_j satisfy the heat equation*

$$\partial_t u_j = \frac{\sigma_j}{4\pi} \Delta u_j$$

with initial data a compactly supported positive finite Borel measure. Let $Q : (0, \infty) \to (0, \infty)$ be given by

$$Q(t) := \|u_1(t, \cdot)^{1/p_1} * u_2(t, \cdot)^{1/p_2}\|_{L^p(\mathbb{R}^d)}.$$

If $p_1, p_2 \geq 1$ then $Q(t)$ is nondecreasing for each $t > 0$ and if $p_1, p_2 \leq 1$ then $Q(t)$ is nonincreasing for each $t > 0$.

The proofs of Theorem 1 and Corollary 2 appear in Section 2.

Remark 3. Under the hypotheses of Corollary 2, it follows from our proof in Section 2 that

$$\begin{aligned} Q'(t) = &\frac{\varepsilon}{8\pi Q(t)^{p-1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u_1(t, \cdot)^{1/p_1} * u_2(t, \cdot)^{1/p_2})(x)^{p-2} u_1(t, x-y)^{1/p_1} u_2(t, y)^{1/p_2} \times \\ & u_1(t, x-z)^{1/p_1} u_2(t, z)^{1/p_2} \left| \left(\tfrac{\sigma_1}{p_1}|\tfrac{1}{p_1} - 1|\right)^{1/2} \tfrac{\nabla u_1}{u_1}(t, x-y) + \left(\tfrac{\sigma_2}{p_2}|\tfrac{1}{p_2} - 1|\right)^{1/2} \tfrac{\nabla u_2}{u_2}(t, y) \right. \\ & \left. - \left(\tfrac{\sigma_1}{p_1}|\tfrac{1}{p_1} - 1|\right)^{1/2} \tfrac{\nabla u_1}{u_1}(t, x-z) - \left(\tfrac{\sigma_2}{p_2}|\tfrac{1}{p_2} - 1|\right)^{1/2} \tfrac{\nabla u_2}{u_2}(t, z) \right|^2 \, dxdydz \end{aligned}$$

for each $t > 0$. Here ε is defined to be 1 if $p_1, p_2 \geq 1$ and -1 if $p_1, p_2 \leq 1$. Consequently, if exactly one of p_1 and p_2 is equal to 1 then the corresponding monotonicity in Corollary 2 is strict. In this case and if p_j is equal to 1, then it is amusing to note that σ_j is zero by (1.3); that is, the heat-flow u_j is constant in time.

We now describe the sharp convolution inequalities that follow from these results. Recall that the sharp form of the Young convolution inequality on $\mathbb{R}^d$ states that if $p_1, p_2 \geq 1$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p}$ then

$$\|f_1 * f_2\|_{L^p(\mathbb{R}^d)} \leq \left(\frac{C_{p_1} C_{p_2}}{C_p}\right)^d \|f_1\|_{L^{p_1}(\mathbb{R}^d)} \|f_2\|_{L^{p_2}(\mathbb{R}^d)} \tag{1.6}$$

for any nonnegative functions f_j in $L^{p_j}(\mathbb{R}^d)$, where $C_r := \left(r^{1/r}/r'^{1/r'}\right)^{1/2}$. The sharp constant in (1.6) is due to Beckner [8], [9] and Brascamp and Lieb [12]. The sharp reverse form of (1.6) states that if $p_1, p_2 \leq 1$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p}$ then

$$\|f_1 * f_2\|_{L^p(\mathbb{R}^d)} \geq \left(\frac{C_{p_1} C_{p_2}}{C_p}\right)^d \|f_1\|_{L^{p_1}(\mathbb{R}^d)} \|f_2\|_{L^{p_2}(\mathbb{R}^d)} \tag{1.7}$$

for any nonnegative functions f_j in $L^{p_j}(\mathbb{R}^d)$. Leindler [15] proved (1.7) with a nonsharp constant and Brascamp and Lieb found the sharp constant in [12] (see also Barthe's simpler argument in [4] which proves both forms with sharp constants). It is easy to see that from Corollary 2 one may recover both (1.6) and (1.7). To see this, let $0 < p_1, p_2, p < \infty$ satisfy (1.2) and note that it suffices to verify both inequalities when the $f_j^{p_j}$ are bounded, integrable and compactly supported functions. For $j = 1, 2$ let u_j satisfy the heat equation

$$\partial_t u_j = \frac{\sigma_j}{4\pi} \Delta u_j \tag{1.8}$$

with initial data $u_j(0, x) := f_j(x)^{p_j}$. By the dominated convergence theorem, one can show that,

$$\lim_{t \to 0} Q(t) = \|f_1 * f_2\|_{L^p(\mathbb{R}^d)}$$

and, combined with a simple change of variables,

$$\lim_{t \to \infty} Q(t) = \left\|H_{\sigma_1}^{1/p_1} * H_{\sigma_2}^{1/p_2}\right\|_{L^p(\mathbb{R}^d)} \|f_1\|_{L^{p_1}(\mathbb{R}^d)} \|f_2\|_{L^{p_2}(\mathbb{R}^d)}.$$

Here,

$$H_t(x) := (1/t)^{d/2} e^{-\pi|x|^2/t}$$

is the appropriate heat kernel at time t and σ_1, σ_2 satisfy (1.3). A direct computation shows that

$$\left\|H_{\sigma_1}^{1/p_1} * H_{\sigma_2}^{1/p_2}\right\|_{L^p(\mathbb{R}^d)} = \left(\frac{C_{p_1} C_{p_2}}{C_p}\right)^d$$

and hence, Corollary 2 immediately implies both (1.6) and (1.7). We also remark that if the initial data for u_1 and u_2 are extremal then $Q(t)$ is constant in time. It is possible to recover the complete characterisation of the extremals in the Young convolution inequality and its reverse form from the expression for $Q'(t)$ in Remark 3. We omit the details of this.

An alternative perspective on the sharp Young convolution inequality on $\mathbb{R}^d$ is to consider the following dual formulation. Suppose $1 \leq p_1, p_2, p_3 < \infty$ satisfy $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 2$. The inequality

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_1(x)^{1/p_1} f_2(y)^{1/p_2} f_3(x-y)^{1/p_3}\,dxdy \leq \prod_{j'=1}^{3} C^d_{p_{j'}} \prod_{j=1}^{3} \|f_j\|^{1/p_j}_{L^1(\mathbb{R}^d)} \tag{1.9}$$

for all nonnegative integrable functions f_j is equivalent to the Young convolution inequality in (1.6). It is known that if each f_j evolves under an appropriate heat-flow u_j then the quantity

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u_1(t,x)^{1/p_1} u_2(t,y)^{1/p_2} u_3(t,x-y)^{1/p_3}\,dxdy$$

is nondecreasing for each $t > 0$ from which the inequality in (1.9) follows. This type of dualised heat-flow approach to the Young convolution inequality on $\mathbb{R}^d$ can be found in [13] and [11]. Carlen, Lieb and Loss have also shown that this type of heat-flow approach can be used to prove certain analogues of the Young convolution inequality in other settings. See [13] and [14] for analogues on the euclidean sphere and the permutation group, respectively (see also [6]).

It is also worth noting that in our undualised setup when the exponent p is a natural number and $1 \leq p_1, p_2 < \infty$, by multiplying out the pth power of the integral one may deduce the monotonicity of Q directly from [11] (see also [13]).

1.2. **Iterated convolutions.** Naturally Theorem 1 self-improves to a result involving iterated convolutions, which we now state. Suppose $0 < p_1, \ldots, p_n, p < \infty$ satisfy

$$\sum_{j=1}^{n} \frac{1}{p_j} = n - 1 + \frac{1}{p}. \tag{1.10}$$

Let $0 \leq \sigma_1, \ldots, \sigma_n, \sigma < \infty$ satisfy

$$\frac{1}{p_j}\left(1 - \frac{1}{p_j}\right)\sigma_k = \frac{1}{p_k}\left(1 - \frac{1}{p_k}\right)\sigma_j$$

for each $j, k = 1, \ldots, n$ and

$$\sigma p = \sum_{j=1}^{n} \sigma_j p_j.$$

As before these relations uniquely define $\sigma_1, \ldots, \sigma_n, \sigma$ up to a common scale factor.

Corollary 4. *For $j = 1, \ldots, n$ suppose that $u_j : (0,\infty) \times \mathbb{R}^d \to (0,\infty)$ is such that $u_j(t,\cdot)^{1/p_j}$, $\partial_t(u_j(t,\cdot)^{1/p_j})$, $\nabla(u_j(t,\cdot)^{1/p_j})$, $u_j(t,\cdot)^{1/p_j}|\nabla \log u_j(t,\cdot)|^2$ and $\Delta(u_j(t,\cdot)^{1/p_j})$ are rapidly decreasing in space locally uniformly in time for $t > 0$. Let $u : (0,\infty) \times \mathbb{R}^d \to (0,\infty)$ be given by*

$$u^{1/p} := u_1^{1/p_1} * \cdots * u_n^{1/p_n}.$$

Then $u(t,\cdot)^{1/p}$, $\partial_t(u(t,\cdot)^{1/p})$, $\nabla(u(t,\cdot)^{1/p})$, $u(t,\cdot)^{1/p}\,|\nabla \log u(t,\cdot)|^2$ and $\Delta(u(t,\cdot)^{1/p})$ are also rapidly decreasing in space locally uniformly in time for $t > 0$. Furthermore,

(1) *if $p_j \geq 1$ and*

$$\partial_t u_j \geq \frac{\sigma_j}{4\pi}\Delta u_j,$$

for $j = 1, \ldots, n$, then

$$\partial_t u \geq \frac{\sigma}{4\pi}\Delta u;$$

(2) if $p_j \leq 1$ and

$$\partial_t u_j \leq \frac{\sigma_j}{4\pi}\Delta u_j,$$

for $j = 1, \ldots, n$, then

$$\partial_t u \leq \frac{\sigma}{4\pi}\Delta u.$$

It is a simple exercise to verify that Corollary 4 follows from Theorem 1.

Corollary 5. *For $j = 1, \ldots, n$ let u_j satisfy the heat equation*

$$\partial_t u_j = \frac{\sigma_j}{4\pi}\Delta u_j$$

with initial data a compactly supported positive finite Borel measure. Let $Q : (0, \infty) \to (0, \infty)$ be given by

$$Q(t) := \|u_1(t, \cdot)^{1/p_1} * \cdots * u_n(t, \cdot)^{1/p_n}\|_{L^p(\mathbb{R}^d)}.$$

If $p_1, \ldots, p_n \geq 1$ then $Q(t)$ is nondecreasing for each $t > 0$ and if $p_1, \ldots, p_n \leq 1$ then $Q(t)$ is nonincreasing for each $t > 0$.

We remark that Corollary 5 follows from Corollary 4 in the same way that Corollary 2 follows from Theorem 1. As one may expect, from Corollary 5 (and its proof) we recover the sharp n-fold Young convolution inequality, its reverse form and a complete characterisation of extremals. We omit the details of this.

When p is an even integer, this n-fold Young convolution inequality is of course related to the Hausdorff–Young inequality via Plancherel's theorem. In particular,

$$\|\widehat{u(t, \cdot)^{1/p'}}\|_{L^p(\mathbb{R}^d)} = \|u(t, \cdot)^{1/p'} * \cdots * u(t, \cdot)^{1/p'}\|_{L^2(\mathbb{R}^d)}^{2/p} \tag{1.11}$$

where $\widehat{\ }$ denotes the Fourier transform and the iterated convolution is $p/2$-fold. By Corollary 5 it follows that the above quantity is nondecreasing for $t > 0$ if u satisfies the heat equation $\partial_t u = \frac{1}{4\pi}\Delta u$ with initial data a compactly supported finite positive Borel measure. We remark that the nondecreasingness of the quantity in (1.11) also follows from [11]. We refer the interested reader to [10] for an explicit verification of how this fact follows from [11] and for counterexamples to the monotonicity of

$$\|\widehat{u(t, \cdot)^{1/p'}}\|_{L^p(\mathbb{R}^d)}$$

whenever p is not an even integer.

1.3. **Further results.** We now describe some extensions of our results when the scaling condition (1.2) (or more generally (1.10)) is relaxed. Let $1 \leq p_1, p_2, p < \infty$ be such that

$$\frac{1}{p_1} + \frac{1}{p_2} \geq 1 + \frac{1}{p}$$

and suppose that $0 \leq \alpha_1, \alpha_2 \leq 1$ satisfy

$$\frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} = 1 + \frac{1}{p}. \tag{1.12}$$

Let $0 \leq \sigma_1, \sigma_2, \sigma < \infty$ satisfy

$$\frac{1}{p_1}\left(1 - \frac{\alpha_1}{p_1}\right)\sigma_2 = \frac{1}{p_2}\left(1 - \frac{\alpha_2}{p_2}\right)\sigma_1 \tag{1.13}$$

and

$$\sigma p = \sigma_1 p_1 + \sigma_2 p_2.$$

Theorem 6. *For $j = 1,2$ suppose that $u_j : (0,\infty) \times \mathbb{R}^d \to (0,\infty)$ is such that $u_j(t,\cdot)^{1/p_j}$, $\partial_t(u_j(t,\cdot)^{1/p_j})$, $\nabla(u_j(t,\cdot)^{1/p_j})$, $u_j(t,\cdot)^{1/p_j}|\nabla \log u_j(t,\cdot)|^2$ and $\Delta(u_j(t,\cdot)^{1/p_j})$ are rapidly decreasing in space locally uniformly in time for $t > 0$. Let $u : (0,\infty) \times \mathbb{R}^d \to (0,\infty)$ be given by*

$$u(t,x)^{1/p} := t^{d(1/p_1+1/p_2-1-1/p)/2}(u_1(t,\cdot)^{1/p_1} * u_2(t,\cdot)^{1/p_2})(x).$$

Then $u(t,\cdot)^{1/p}$, $\partial_t(u(t,\cdot)^{1/p})$, $\nabla(u(t,\cdot)^{1/p})$, $u(t,\cdot)^{1/p}|\nabla \log u(t,\cdot)|^2$ and $\Delta(u(t,\cdot)^{1/p})$ are also rapidly decreasing in space locally uniformly in time for $t > 0$. Furthermore, if

$$\partial_t u_j \geq \frac{\sigma_j}{4\pi}\Delta u_j \tag{1.14}$$

and, for each $t > 0$,

$$\sigma_j \mathrm{div}\left(\frac{\nabla u_j}{u_j}\right)(t,\cdot) \geq -\frac{2d\pi}{t} \tag{1.15}$$

for $j = 1,2$ then

$$\partial_t u \geq \frac{\sigma}{4\pi}\Delta u \tag{1.16}$$

and, for each $t > 0$,

$$\sigma \mathrm{div}\left(\frac{\nabla u}{u}\right)(t,\cdot) \geq -\frac{2d\pi}{t}. \tag{1.17}$$

Corollary 7. *For $j = 1,2$ let u_j satisfy the heat equation*

$$\partial_t u_j = \frac{\sigma_j}{4\pi}\Delta u_j$$

with initial data a compactly supported positive finite Borel measure. Let $Q : (0,\infty) \to (0,\infty)$ be given by

$$Q(t) := t^{d(1/p_1+1/p_2-1-1/p)/2}\left\|u_1(t,\cdot)^{1/p_1} * u_2(t,\cdot)^{1/p_2}\right\|_{L^p(\mathbb{R}^d)}.$$

Then $Q(t)$ is nondecreasing for each $t > 0$.

We remark that the idea behind this extension of Theorem 1(1) lies in [11] and involves a certain log-convexity property for solutions to heat equations. In particular, if there is equality in (1.14) and the initial data for u_j is some finite positive Borel measure then (1.15) is automatic by Corollary 8.7 of [11].

Theorem 6 self-improves to a result involving iterated convolutions, as was the case with Theorem 1. Again, we leave the details of this to the interested reader.

Finally we remark that all of our results also hold in the setting of the torus. This will be clear from our proofs.

2. Proof of Theorem 1, Corollary 2 and Theorem 6

An elementary but crucial component of the proof of Theorem 1 and Theorem 6 is the following.

Lemma 8. *Let $0 < p_1, p_2, \Lambda_1, \Lambda_2 < \infty$. For $j = 1,2$ suppose that $u_j : (0,\infty) \times \mathbb{R}^d \to (0,\infty)$ is such that $u_j(t,\cdot)^{1/p_j}$, $\partial_t(u_j(t,\cdot)^{1/p_j})$, $\nabla(u_j(t,\cdot)^{1/p_j})$, $u_j(t,\cdot)^{1/p_j}|\nabla \log u_j(t,\cdot)|^2$ and $\Delta(u_j(t,\cdot)^{1/p_j})$ are rapidly decreasing in space locally uniformly in time for $t > 0$. Then*

$$\Lambda_1(u_1^{1/p_1} * u_2^{1/p_2})(u_1^{1/p_1}|\tfrac{\nabla u_1}{u_1}|^2 * u_2^{1/p_2}) + \Lambda_2(u_1^{1/p_1} * u_2^{1/p_2})(u_1^{1/p_1} * u_2^{1/p_2}|\tfrac{\nabla u_2}{u_2}|^2)$$
$$+ 2\Lambda_1^{1/2}\Lambda_2^{1/2}(u_1^{1/p_1} * u_2^{1/p_2})(u_1^{1/p_1}\tfrac{\nabla u_1}{u_1} * u_2^{1/p_2}\tfrac{\nabla u_2}{u_2}) - (p_1\Lambda_1^{1/2} + p_2\Lambda_2^{1/2})^2|\nabla(u_1^{1/p_1} * u_2^{1/p_2})|^2$$

evaluated at $(t,x) \in (0,\infty) \times \mathbb{R}^d$ coincides with

$$\tfrac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u_1(t,x-y)^{1/p_1}u_2(t,y)^{1/p_2}u_1(t,x-z)^{1/p_1}u_2(t,z)^{1/p_2} \times$$
$$|\Lambda_1^{1/2}\tfrac{\nabla u_1}{u_1}(t,x-y) + \Lambda_2^{1/2}\tfrac{\nabla u_2}{u_2}(t,y) - \Lambda_1^{1/2}\tfrac{\nabla u_1}{u_1}(t,x-z) - \Lambda_2^{1/2}\tfrac{\nabla u_2}{u_2}(t,z)|^2\, dydz.$$

Proof. We remark that each convolution term is well-defined by the regularity hypotheses on u_1 and u_2. Moreover, since $\nabla(u_j^{1/p_j}) = \frac{1}{p_j}u_j^{1/p_j}\frac{\nabla u_j}{u_j}$ is rapidly decreasing in space for $j = 1,2$ it follows that $\nabla(u_1^{1/p_1} * u_2^{1/p_2})$ coincides with $\frac{1}{p_1}(u_1^{1/p_1}\frac{\nabla u_1}{u_1} * u_2^{1/p_2})$ and $\frac{1}{p_2}(u_1^{1/p_1} * u_2^{1/p_2}\frac{\nabla u_2}{u_2})$, depending on whether one applies the gradient to the left or right of the convolution. Thus, Lemma 8 follows upon expanding the square in the integrand and collecting like terms. □

2.1. **Proof of Theorem 1.** We begin by justifying the closure of the technical regularity ingredients in Theorem 1. For $j = 1,2$ let v_j be the time dependent vector field on $\mathbb{R}^d$ given by $v_j := \frac{\nabla u_j}{u_j}$.

Since the convolution of two rapidly decreasing functions on $\mathbb{R}^d$ is also rapidly decreasing, it is straightforward to check that $u^{1/p}$ is rapidly decreasing locally uniformly in time. For the time derivative, we note that

$$\partial_t(u^{1/p}) = \partial_t(u_1^{1/p_1}) * u_2^{1/p_2} + u_1^{1/p_1} * \partial_t(u_2^{1/p_2}), \tag{2.1}$$

where the interchange of differentiation and integration is justified since u_j^{1/p_j} and $\partial_t(u_j^{1/p_j})$ are rapidly decreasing in space locally uniformly in time for $j = 1,2$. Hence $\partial_t(u^{1/p})$ is also rapidly decreasing in space locally uniformly in time. Similarly, it follows that $\nabla(u^{1/p})$ is rapidly decreasing in space locally uniformly in time and, moreover, we may write

$$\nabla(u^{1/p}) = \nabla(u_1^{1/p_1}) * u_2^{1/p_2} = \tfrac{1}{p_1}u_1^{1/p_1}v_1 * u_2^{1/p_2} \tag{2.2}$$

or, by symmetry, $\nabla(u^{1/p}) = \frac{1}{p_2}u_1^{1/p_1} * u_2^{1/p_2}v_2$. Next, observe that

$$u^{1/p}|\nabla \log u|^2 = p^2\frac{|\nabla(u^{1/p})|^2}{u^{1/p}} = \frac{p^2}{p_1^2}\frac{|u_1^{1/p_1}v_1 * u_2^{1/p_2}|^2}{u^{1/p}} \leq \frac{p^2}{p_1^2}u_1^{1/p_1}|v_1|^2 * u_2^{1/p_2}$$

by (2.2) and the Cauchy-Schwarz inequality. Since $u_1^{1/p_1}|v_1|^2$ and u_2^{1/p_2} are rapidly decreasing in space locally uniformly in time by assumption, it follows that $u^{1/p}|\nabla \log u|^2$ is also rapidly decreasing locally uniformly in time.

Finally, we note that $\Delta(u^{1/p})$ is rapidly decreasing in space locally uniformly in time since (2.2) and our hypotheses on u_1 and u_2 imply that

$$\Delta(u^{1/p}) = \Delta(u_1^{1/p_1}) * u_2^{1/p_2}.$$

This concludes our justification of the closure of the regularity ingredients in Theorem 1. It is, however, a convenient opportunity to note that we may also write

$$\Delta(u^{1/p}) = \operatorname{div}(\nabla(u_1^{1/p_1}) * u_2^{1/p_2}) = \begin{cases} \frac{1}{p_1^2} u_1^{1/p_1}|v_1|^2 * u_2^{1/p_2} + \frac{1}{p_1} u_1^{1/p_1}\operatorname{div}(v_1) * u_2^{1/p_2} \\ \frac{1}{p_1 p_2} u_1^{1/p_1} v_1 * u_2^{1/p_2} v_2 \end{cases} \tag{2.3}$$

depending on whether we apply the divergence to the term on the left or right of the convolution. Since

$$\Delta(u_1^{1/p_1}) = \tfrac{1}{p_1}\operatorname{div}(u_1^{1/p_1} v_1) = \tfrac{1}{p_1^2} u_1^{1/p_1}|v_1|^2 + \tfrac{1}{p_1} u_1^{1/p_1}\operatorname{div}(v_1)$$

it follows from the regularity of u_1 and u_2 that each convolution term in (2.3) is well-defined. By symmetry the expression (2.3) also holds with the subscripts 1 and 2 interchanged.

We now turn to proving Theorem 1(1) where we have $p_j \geq 1$ and $\partial_t u_j \geq \frac{\sigma_j}{4\pi}\Delta u_j$ for each $j = 1, 2$. Then,

$$\frac{\partial_t u_j}{u_j} \geq \frac{\sigma_j}{4\pi}\frac{\operatorname{div}(\nabla u_j)}{u_j} = \frac{\sigma_j}{4\pi}(|v_j|^2 + \operatorname{div}(v_j))$$

and therefore, by (2.1),

$$4\pi\frac{\partial_t u}{u^{(p-2)/p}} \geq p u^{1/p}\Big(\tfrac{\sigma_1}{p_1} u_1^{1/p_1}|v_1|^2 * u_2^{1/p_2} + \tfrac{\sigma_2}{p_2} u_1^{1/p_1} * u_2^{1/p_2}|v_2|^2 + \tfrac{\sigma_1}{p_1} u_1^{1/p_1}\operatorname{div}(v_1) * u_2^{1/p_2} + \tfrac{\sigma_2}{p_2} u_1^{1/p_1} * u_2^{1/p_2}\operatorname{div}(v_2)\Big).$$

Furthermore,

$$\frac{\Delta u}{u^{(p-2)/p}} = p(p-1)|\nabla(u^{1/p})|^2 + p u^{1/p}\Delta(u^{1/p})$$

and therefore,

$$-\frac{\sigma_1 p_1}{p}\frac{\Delta u}{u^{(p-2)/p}} = -(p-1)\sigma_1 p_1|\nabla(u^{1/p})|^2 + \sigma_1(p-p_1)u^{1/p}\Delta(u^{1/p}) - p\sigma_1 u^{1/p}\Delta(u^{1/p}).$$

Hence, by (2.3),

$$-\frac{\sigma_1 p_1}{p}\frac{\Delta u}{u^{(p-2)/p}} = -(p-1)\sigma_1 p_1|\nabla(u^{1/p})|^2 + \tfrac{\sigma_1(p-p_1)}{p_1 p_2} u^{1/p}(u_1^{1/p_1} v_1 * u_2^{1/p_2} v_2) - \tfrac{p\sigma_1}{p_1^2} u^{1/p}(u_1^{1/p_1}|v_1|^2 * u_2^{1/p_2}) - \tfrac{p\sigma_1}{p_1} u^{1/p}(u_1^{1/p_1}\operatorname{div}(v_1) * u_2^{1/p_2}).$$

By symmetry, it follows that

$$-\frac{\sigma_2 p_2}{p}\frac{\Delta u}{u^{(p-2)/p}} = -(p-1)\sigma_2 p_2|\nabla(u^{1/p})|^2 + \tfrac{\sigma_1(p-p_2)}{p_1 p_2} u^{1/p}(u_1^{1/p_1} v_1 * u_2^{1/p_2} v_2) - \tfrac{p\sigma_2}{p_2^2} u^{1/p}(u_1^{1/p_1} * u_2^{1/p_2}|v_2|^2) - \tfrac{p\sigma_2}{p_2} u^{1/p}(u_1^{1/p_1} * u_2^{1/p_2}\operatorname{div}(v_2)).$$

Thus,

$$\frac{1}{u^{(p-2)/p}}\left[4\pi\partial_t u - \tfrac{1}{p}\left(\sigma_1 p_1 + \sigma_2 p_2\right)\Delta u\right]$$
$$\geq \tfrac{p\sigma_1}{p_1}(1-\tfrac{1}{p_1})u^{1/p}(u_1^{1/p_1}|v_1|^2 * u_2^{1/p_2}) + \tfrac{p\sigma_2}{p_2}(1-\tfrac{1}{p_2})u^{1/p}(u_1^{1/p_1} * u_2^{1/p_2}|v_2|^2) +$$
$$\tfrac{1}{p_1p_2}(\sigma_1(p-p_1)+\sigma_2(p-p_2))u^{1/p}(u_1^{1/p_1}v_1 * u_2^{1/p_2}v_2) - (p-1)(\sigma_1p_1+\sigma_2p_2)|\nabla(u^{1/p})|^2.$$

By Lemma 8, it suffices to verify that

$$p\sigma_1(p_1-1)+p\sigma_2(p_2-1)+2\left(\tfrac{p\sigma_1}{p_1}(1-\tfrac{1}{p_1})\right)^{1/2}\left(\tfrac{p\sigma_2}{p_2}(1-\tfrac{1}{p_2})\right)^{1/2} = (p-1)(\sigma_1p_1+\sigma_2p_2) \tag{2.4}$$

and

$$2\left(\tfrac{p\sigma_1}{p_1}(1-\tfrac{1}{p_1})\right)^{1/2}\left(\tfrac{p\sigma_2}{p_2}(1-\tfrac{1}{p_2})\right)^{1/2} = \frac{1}{p_1p_2}(\sigma_1(p-p_1)+\sigma_2(p-p_2)). \tag{2.5}$$

However, (2.4) and (2.5) are elementary consequences of the hypotheses (1.2) and (1.3). This completes the proof of Theorem 1(1).

Now suppose that we have $p_j \leq 1$ and $\partial_t u_j \leq \frac{\sigma_j}{4\pi}\Delta u_j$ for each $j = 1, 2$. By a very similar argument it follows that

$$\frac{1}{u^{(p-2)/p}}\left[-4\pi\partial_t u + \tfrac{1}{p}\left(\sigma_1 p_1 + \sigma_2 p_2\right)\Delta u\right]$$
$$\geq \tfrac{p\sigma_1}{p_1}(\tfrac{1}{p_1}-1)u^{1/p}(u_1^{1/p_1}|v_1|^2 * u_2^{1/p_2}) + \tfrac{p\sigma_2}{p_2}(\tfrac{1}{p_2}-1)u^{1/p}(u_1^{1/p_1} * u_2^{1/p_2}|v_2|^2)$$
$$-\tfrac{1}{p_1p_2}(\sigma_1(p-p_1)+\sigma_2(p-p_2))u^{1/p}(u_1^{1/p_1}v_1 * u_2^{1/p_2}v_2) + (p-1)(\sigma_1p_1+\sigma_2p_2)|\nabla(u^{1/p})|^2,$$

which is nonnegative by (1.2), (1.3) and Lemma 8. This completes the proof of Theorem 1(2).

2.2. **Proof of Corollary 2.** Firstly, we verify that u_1 and u_2 are sufficiently regular to apply Theorem 1. For $j = 1, 2$ suppose $u_j(0, \cdot) = d\mu_j$ and that $d\mu_j$ is supported in the euclidean ball of radius M centred at the origin. From the explicit formula for the solution

$$u_j(t,x) = \frac{1}{t^{d/2}}\int_{\mathbb{R}^d} e^{-\pi|x-y|^2/t}\, d\mu_j(y)$$

it is clear that $u_j(t,x) \leq t^{-d/2}e^{-\pi|x|^2/4t}$ for $|x| > 2M$ and all $t > 0$. It follows that u_j^{1/p_j} is rapidly decreasing in space locally uniformly in time. Furthermore, by interchanging differentiation and integration, it is easy to see that

$$|\nabla u_j(t,x)| \leq 2\pi t^{-d/2}(|x|+M)u_j(t,x). \tag{2.6}$$

Consequently, $\nabla(u_j^{1/p_j})$ and $u_j^{1/p_j}|\nabla \log u_j|^2$ are rapidly decreasing in space locally uniformly in time. Similar considerations show that the quantities $\partial_t(u_j^{1/p_j})$ and $\Delta(u_j^{1/p_j})$ are rapidly decreasing in space locally uniformly in time.

To complete the proof of Corollary 2 it suffices, by the divergence theorem, to show that for each $t > 0$, $\int_{R\mathbb{S}^{d-1}} |\nabla u(t,\cdot)|$ tends to zero as R tends to infinity. To see this, note that

$$\nabla u = p\nabla(u^{1/p})u^{1-1/p} = p(u_1^{1/p_1}v_1 * u_2^{1/p_2})u^{1-1/p}$$

where, as before, $v_1 = \frac{\nabla u_1}{u_1}$. However, if r is chosen such that $1 > 1/r > 1-p$ then, by Hölder's inequality,

$$|\nabla u| \leq p(u_1^{\varepsilon_1 r'}|v_1|^{r'} * u_2^{\varepsilon_2})^{1/r'} u^{1-1/p+1/rp}$$

where ε_j satisfies $(1/p_j - \varepsilon_j)r = 1/p_j$ for $j = 1, 2$. Any nonnegative power of u_1 or u_2 is rapidly decreasing in space and, by (2.6), $|v_1|$ has at most linear growth in space. Hence, for each $t > 0$, $|\nabla u(t, \cdot)|$ is rapidly decreasing in space which is clearly sufficient to see that $\int_{R\mathbb{S}^{d-1}} |\nabla u(t, \cdot)|$ tends to zero as R tends to infinity.

2.3. **Proof of Theorem 6.** The verification of the closure of the regularity properties follows in the same way as in Theorem 1. We also remark that, as before, these ingredients are sufficient to justify all convergence issues related to the integrals which appear in the proof below.

Let $\beta := d(1/p_1 + 1/p_2 - 1 - 1/p)/2$ and for $j = 1, 2$ let v_j denote the time dependent vector field on $\mathbb{R}^d$ given by $v_j := \frac{\nabla u_j}{u_j}$. Thus,

$$\begin{aligned} 4\pi \frac{\partial_t u}{t^{\beta p}(u_1^{1/p_1} * u_2^{1/p_2})^{p-2}} &\geq p(u_1^{1/p_1} * u_2^{1/p_2}) \Big[\tfrac{\sigma_1}{p_1} u_1^{1/p_1}|v_1|^2 * u_2^{1/p_2} + \tfrac{\sigma_2}{p_2} u_1^{1/p_1} * u_2^{1/p_2}|v_2|^2 \\ &\quad + \tfrac{\sigma_1}{p_1} u_1^{1/p_1}\mathrm{div}(v_1) * u_2^{1/p_2} + \tfrac{\sigma_2}{p_2} u_1^{1/p_1} * u_2^{1/p_2}\mathrm{div}(v_2)\Big] \\ &\quad + \tfrac{4\pi\beta p}{t}(u_1^{1/p_1} * u_2^{1/p_2})^2. \end{aligned}$$

The proof of (1.16) proceeds in a similar way to the proof of Theorem 1(1). The difference is that we use some of the divergence terms on the right hand side of the above inequality to kill off the factor $\frac{4\pi\beta p}{t}(u_1^{1/p_1} * u_2^{1/p_2})^2$. In particular, by (1.15),

$$\begin{aligned} &(1-\alpha_1)\tfrac{\sigma_1}{p_1}(u_1^{1/p_1}\mathrm{div}(v_1) * u_2^{1/p_2}) + (1-\alpha_2)\tfrac{\sigma_2}{p_2}(u_1^{1/p_1} * u_2^{1/p_2}\mathrm{div}(v_2)) + \tfrac{4\pi\beta}{t}(u_1^{1/p_1} * u_2^{1/p_2}) \\ &\quad \geq \tfrac{2d\pi}{t}(\tfrac{\alpha_1}{p_1} + \tfrac{\alpha_2}{p_2} - 1 - \tfrac{1}{p})(u_1^{1/p_1} * u_2^{1/p_2}) \end{aligned}$$

and the right hand side of the above inequality vanishes by (1.12). Now, guided by the proof of Theorem 1(1), it follows that

$$\begin{aligned} &\frac{1}{t^{\beta p}(u_1^{1/p_1} * u_2^{1/p_2})^{p-2}}\Big[4\pi\partial_t u - \tfrac{1}{p}(\sigma_1 p_1 + \sigma_2 p_2)\Delta u\Big] \\ &\geq \tfrac{\sigma_1 p}{p_1}(1 - \tfrac{\alpha_1}{p_1})(u_1^{1/p_1} * u_2^{1/p_2})(u_1^{1/p_1}|v_1|^2 * u_2^{1/p_2}) + \tfrac{\sigma_2 p}{p_2}(1 - \tfrac{\alpha_2}{p_2})(u_1^{1/p_1} * u_2^{1/p_2})(u_1^{1/p_1} * u_2^{1/p_2}|v_2|^2) \\ &\quad + \tfrac{1}{p_1 p_2}(\sigma_1(p\alpha_1 - p_1) + \sigma_2(p\alpha_2 - p_2))(u_1^{1/p_1} * u_2^{1/p_2})(u_1^{1/p_1} v_1 * u_2^{1/p_2} v_2) \\ &\quad - (p-1)(\sigma_1 p_1 + \sigma_2 p_2)|\nabla(u_1^{1/p_1} * u_2^{1/p_2})|^2. \end{aligned}$$

By (1.12), (1.13) and Lemma 8 the right hand side of the above inequality is nonnegative. This completes the proof of (1.16).

To prove (1.17) we let v be the time dependent vector field on $\mathbb{R}^d$ given by $v := \frac{\nabla u}{u}$. Thus,

$$\mathrm{div}(v) = \frac{p}{(u_1^{1/p_1} * u_2^{1/p_2})^2}\Big[\mathrm{div}(\nabla(u_1^{1/p_1} * u_2^{1/p_2}))(u_1^{1/p_1} * u_2^{1/p_2}) - |\nabla(u_1^{1/p_1} * u_2^{1/p_2})|^2\Big].$$

For $j = 1, 2$ let

$$\lambda_j := \left(\frac{1 - \frac{\alpha_j}{p_j}}{2 - \frac{\alpha_1}{p_1} - \frac{\alpha_2}{p_2}} \right)^2 \tag{2.7}$$

and write

$$\begin{aligned}
\mathrm{div}(\nabla(u_1^{1/p_1} * u_2^{1/p_2})) &= \tfrac{1}{2p_1}\mathrm{div}(u_1^{1/p_1} v_1 * u_2^{1/p_2}) + \tfrac{1}{2p_2}\mathrm{div}(u_1^{1/p_1} * u_2^{1/p_2} v_2) \\
&= \tfrac{\lambda_1}{p_1} u_1^{1/p_1}\mathrm{div}(v_1) * u_2^{1/p_2} + \tfrac{\lambda_2}{p_2} u_1^{1/p_1} * u_2^{1/p_2}\mathrm{div}(v_2) \\
&\quad + \tfrac{\lambda_1}{p_1^2} u_1^{1/p_1}|v_1|^2 * u_2^{1/p_2} + \tfrac{\lambda_2}{p_2^2} u_1^{1/p_1} * u_2^{1/p_2}|v_2|^2 \\
&\quad + \tfrac{1}{p_1 p_2}(1 - \lambda_1 - \lambda_2)(u_1^{1/p_1} v_1 * u_2^{1/p_2} v_2).
\end{aligned}$$

We shall assume that neither σ_1 nor σ_2 is equal to zero; a simple modification of the argument will handle the degenerate cases. By (1.15),

$$\tfrac{\lambda_1}{p_1} u_1^{1/p_1}\mathrm{div}(v_1) * u_2^{1/p_2} + \tfrac{\lambda_2}{p_2} u_1^{1/p_1} * u_2^{1/p_2}\mathrm{div}(v_2) \geq -(\tfrac{\lambda_1}{\sigma_1 p_1} + \tfrac{\lambda_2}{\sigma_2 p_2})\tfrac{2d\pi}{t}(u_1^{1/p_1} * u_2^{1/p_2}).$$

Moreover, by (1.13) and (2.7), it is easy to check that

$$\sigma p(\tfrac{\lambda_1}{\sigma_1 p_1} + \tfrac{\lambda_2}{\sigma_2 p_2}) = \lambda_1^{1/2} + \lambda_2^{1/2} = 1$$

and therefore,

$$\begin{aligned}
\mathrm{div}(v) \geq -\tfrac{2d\pi}{\sigma t} + \tfrac{p}{(u_1^{1/p_1} * u_2^{1/p_2})^2} \times \\
\Big[\tfrac{\lambda_1}{p_1^2}(u_1^{1/p_1} * u_2^{1/p_2})(u_1^{1/p_1}|v_1|^2 * u_2^{1/p_2}) + \tfrac{\lambda_2}{p_2^2}(u_1^{1/p_1} * u_2^{1/p_2})(u_1^{1/p_1} * u_2^{1/p_2}|v_2|^2) \\
+ \tfrac{1}{p_1 p_2}(1 - \lambda_1 - \lambda_2)(u_1^{1/p_1} * u_2^{1/p_2})(u_1^{1/p_1} v_1 * u_2^{1/p_2} v_2) - |\nabla(u_1^{1/p_1} * u_2^{1/p_2})|^2\Big]
\end{aligned}$$

Since $\lambda_1 + \lambda_2 + 2\lambda_1^{1/2}\lambda_2^{1/2} = (\lambda_1^{1/2} + \lambda_2^{1/2})^2 = 1$ it follows immediately from Lemma 8 that the term in square brackets in the above inequality is nonnegative. This completes the proof of Theorem 6.

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